The Foundations: Logic and Proofs

CSC-2259 Discrete Structures

Propositional Logic

Proposition is a declarative statement that is either true of false

•Baton Rouge is the capital of Louisiana	True
•Toronto is the capital of Canada	False
•1+1=2	True
•2+2=3	False

Statements which are not propositions:

•What time is it?

•x+1 = 2

$p =$ today is Thursday

Negation: $\neg p =$ today is not Thursday

truth table

p	$\neg p$
T	F
F	T

$p =$ today is Thursday

$q =$ it is raining today

Conjunction:

$p \wedge q =$ today is Thursday and it is raining today

truth table

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

$p =$ today is Thursday

$q =$ today is Friday

Disjunction:

$p \vee q =$ today is Thursday or today is Friday

truth table

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

$p =$ today is Thursday

$q =$ today is Friday

Exclusive-or: one or the other but not both

$p \oplus q =$ today is Thursday or today is Friday (but not both)

truth table

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

(hypothesis) $p =$ Maria learns discrete math

(conclusion) $q =$ Maria will find a good job

Conditional statement:

$p \rightarrow q =$ if Maria learns discrete math then she will find a good job

if p then q
p implies q
q follows from p
p only if q
p is sufficient for q

truth table

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

$$\left.\begin{array}{r} \text{Conditional statement: } p \rightarrow q \\ \\ \text{Contrapositive: } \neg q \rightarrow \neg p \end{array}\right\} \text{equivalent (same truth table)}$$

$$\left.\begin{array}{r} \text{Converse: } q \rightarrow p \\ \\ \text{Inverse: } \neg p \rightarrow \neg q \end{array}\right\} \text{equivalent}$$

$p =$ you can take the flight

$q =$ you buy a ticket

Biconditional statement:

$p \leftrightarrow q =$ you can take the flight if and only if you buy a ticket

p if and only if q

p iff q

If p then q and conversely

p is necessary and sufficient for q

truth table

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Compound propositions

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$p \vee \neg q \rightarrow p \wedge q$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

Precedence of operators

$\neg \quad \wedge \quad \vee \quad \rightarrow \quad \leftrightarrow$

higher lower

Translating English into propositions

$p =$ "you cannot ride the roller coaster if you are under 4 feet tall unless you are older than 16 years old"

$q =$ you can ride the roller coaster

$r =$ you are under 4 feet tall

$s =$ you are older than 16 years old

$$p = \quad r \wedge \neg s \;\rightarrow\; \neg q$$

Logic and Bit Operations

Boolean variables

		OR	AND	XOR
x	y	$x \vee y$	$x \wedge y$	$x \oplus y$
0	0	0	0	0
0	1	1	0	1
1	0	1	0	1
1	1	1	1	0

$T = 1$

$F = 0$

Bit string

0110110110
1100011101

1110111111 bitwise OR
0100010100 bitwise AND
1010101011 bitwise XOR

Propositional Equivalences

Compound proposition

Tautology: always true

Contradiction: always false

p	$\neg p$	$p \vee \neg p$ (tautology)	$p \wedge \neg p$ (contradiction)
T	F	T	F
F	T	T	F

Contingency: not a tautology and not a contradiction

Logically equivalent compound propositions: $p \equiv q$

$p \leftrightarrow q$ is a tautology

Have same truth table

Example: $\neg x \vee y \equiv x \rightarrow y$

x	y	$\neg x$	$\neg x \vee y$	$x \rightarrow y$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

De Morgan's laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Identity laws

$$p \wedge T \equiv p$$

$$p \vee F \equiv p$$

Domination laws

$$p \vee T \equiv T$$

$$p \wedge F \equiv F$$

Idempotent laws

$$p \vee p \equiv p$$

$$p \wedge p \equiv p$$

Negation laws

$$p \vee \neg p \equiv T$$

$$p \wedge \neg p \equiv F$$

Double Negation law

$$\neg(\neg p) \equiv p$$

Commutative laws

$$p \vee q \equiv q \vee p$$

$$p \wedge q \equiv q \wedge p$$

Associative laws

$$(p \vee q) \vee r \equiv p \vee (q \vee r)$$

$$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$$

Distributive laws

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$

Absorption laws

$$p \vee (p \wedge q) \equiv p$$

$$p \wedge (p \vee q) \equiv p$$

Conditional Statements

$$p \to q \equiv \neg p \vee q$$

$$p \to q \equiv \neg q \to \neg p$$

$$\neg(p \to q) \equiv p \wedge \neg q$$

$$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$$

$$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$$

$$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$$

$$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$$

Biconditional Statements

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$$

$$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$$

$$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$$

$$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$$

Construct new logical equivalences

$$\neg(p \to q) \equiv \neg(\neg p \vee q) \qquad (\text{since } \neg x \vee y \equiv x \to y)$$

$$\equiv \neg(\neg p) \wedge \neg p \qquad \text{(De Morgan's laws)}$$

$$\equiv p \wedge \neg q \qquad \text{(Double negation law)}$$

Predicates and Quantifiers

$$A(x):\ \overbrace{\text{Computer } x}^{\text{variable}}\ \overbrace{\text{is under attack by an intruder}}^{\text{predicate}}$$

$$B(x):\ \text{Computer } x \text{ is functioning properly}$$

Propositional functions

$$P(x):\ x > 3$$

$$Q(x, y):\ x = y + 3$$

$$R(x, y, z):\ x + y = z$$

Predicate logic

$$\text{Computers} = \{CS1, CS2, MATH1\}$$

$A(x)$: Computer x is under attack by an intruder

$$A(CS1) = T$$
$$A(CS2) = F$$
$$A(MATH1) = T$$

$B(x)$: Computer x is functioning properly

$$B(CS1) = F$$
$$B(CS2) = T$$
$$B(MATH1) = F$$

Predicate logic

predicate	2-ary predicate	3-ary predicate
$P(x):\ x > 3$	$Q(x,y):\ x = y + 3$	$R(x,y,z):\ x + y = z$
$P(1) = F$	$Q(4,1) = T$	$R(1,1,2) = T$
$P(4) = T$	$Q(3,2) = F$	$R(2,1,1) = F$

n-ary predicate

$$P(x_1, x_2, \ldots, x_n)$$

Universal quantifier: $\forall x\, P(x)$

for all x it holds $P(x)$

$P(x): x+1 > x$

(for every element in domain)

$\forall x\, P(x)$ is true for every real number x

$Q(x): x^2 > 0$

(for every element in domain)

$\forall x\, Q(x)$ is not true for every real number x

Counterexample: $Q(0) = F$

Existential quantifier: $\exists x\, P(x)$

there is x such that $P(x)$

$P(x): x > 3$

$\exists x\, P(x)$ is true because $P(4) = T$

$Q(x): x+1 = 1 \wedge x > 0$

$\exists x\, Q(x)$ is not true

For finite domain $\{x_1, x_2, \ldots, x_n\}$

$$\forall x\, P(x) \equiv P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n)$$

$$\exists x\, P(x) \equiv P(x_1) \vee P(x_2) \vee \cdots \vee P(x_n)$$

Quantifiers with restricted domain

$$\forall x < 0\, (x^2 > 0)$$

$$\forall y \neq 0\, (y^3 \neq 0)$$

$$\exists z > 0\, (z^2 = 2)$$

Precedence of operators

$$\begin{array}{cccccc} \forall & \neg & \wedge & \vee & \rightarrow & \leftrightarrow \\ \exists & & & & & \end{array}$$

higher lower

$$\exists x(\underset{\text{Bound variable}}{x} + \underset{\text{free variable}}{y} = 1)$$

$$\exists x\underbrace{(P(x) \wedge Q(x))}_{\text{Scope of } x} \vee \forall x \underbrace{R(x)}_{\text{Scope of } x}$$

Logical equivalences with quantifiers

$$\forall x(P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x)$$

$$\exists x(P(x) \vee Q(x)) \equiv \exists x P(x) \vee \exists x Q(x)$$

$\forall x(P(x) \vee Q(x)) \equiv \forall x P(x) \vee \forall x Q(x)$? False

$\exists x(P(x) \wedge Q(x)) \equiv \exists x P(x) \wedge \exists x Q(x)$? False

De Morgan's Laws for Quantifiers

$$\neg \forall x P(x) \equiv \exists x \neg P(x)$$

$$\neg \exists x P(x) \equiv \forall x \neg P(x)$$

Example

$$\begin{aligned} \neg \forall x (P(x) \rightarrow Q(x)) &\equiv \exists x \neg (P(x) \rightarrow Q(x)) \\ &\equiv \exists x (P(x) \wedge \neg Q(x)) \end{aligned}$$

Recall that: $\neg (p \rightarrow q) \equiv p \wedge \neg q$

Translating English into Logical Expressions

$$P(x) = x \text{ is a hummingbird}$$
$$Q(x) = x \text{ is large bird}$$
$$R(x) = x \text{ lives on honey}$$
$$S(x) = x \text{ is richly colored}$$

"All hummingbirds are richly colored" $\forall x(P(x) \rightarrow S(x))$

"No large birds live on honey" $\neg \exists x(Q(x) \wedge R(x))$

"Birds that do not live on honey are dull in color" $\forall x(\neg R(x) \rightarrow \neg S(x))$

"Hummingbirds are small" $\forall x(P(x) \rightarrow \neg Q(x))$

Nested Quantifiers

Additive inverse

$$\forall x \exists y (x + y = 0)$$

Commutative law for addition

$$\forall x \forall y (x + y = y + x)$$

Associative law for addition

$$\forall x \forall y \forall z (x + (y + z) = (x + y) + z)$$

Order of quantifiers

$$\forall x \forall y P(x, y) \equiv \forall y \forall x P(x, y)$$

$$\exists x \exists y P(x, y) \equiv \exists y \exists x P(x, y)$$

We cannot always change the order of quantifiers

$$\forall x \exists y P(x, y) \equiv \exists y \forall x P(x, y) \, ?$$

true	false
$\forall x \exists y (x + y = 0)$	$\exists y \forall x (x + y = 0)$

$$\exists y \forall x P(x, y) \rightarrow \forall x \exists y P(x, y)$$

But not the converse

Translating Math Statements

"The sum of two positive integers is always positive"

$$\forall x \forall y ((x > 0) \wedge (y > 0) \rightarrow (x + y > 0))$$

"Every real number except zero has a multiplicative inverse"

$$\forall x ((x \neq 0) \rightarrow \exists y (xy = 1))$$

$$\lim_{x \to a} f(x) = L$$

For every real number $\varepsilon > 0$ there exists a real number $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$

$$\forall \varepsilon > 0 \, \exists \delta > 0 \, \forall x \, (0 < |x - a| < \delta \rightarrow |f(x) - L| < \varepsilon)$$

Translating into English

$C(x):$ student x has a computer

$F(x,y):$ students x and y are friends

$$\forall x(C(x) \vee \exists y(C(y) \wedge F(x,y)))$$

"Every student has a computer or has a friend who has a computer"

$F(x,y):$ students x and y are friends

$$\exists x \forall y \forall z((F(x,y) \wedge F(x,z) \wedge (y \neq z)) \rightarrow \neg F(y,z))$$

"There is a student none of whose friends are also friends with each other"

Translating English into Logical Expressions

"If a person is female and is a parent, then this person is someone's mother"

$$\forall x((F(x) \wedge P(x)) \rightarrow \exists y M(x,y))$$

female

parent

mother of y

"Everyone has exactly one best friend"

$$\forall x \exists y (B(x,y) \wedge \forall z((z \neq y) \rightarrow \neg B(x,z)))$$

Best friends

Negating nested quantifiers

$$\lim_{x\to a} f(x) \neq L$$

$$\neg\forall\varepsilon>0\ \exists\delta>0\ \forall x\ (0<|x-a|<\delta\ \rightarrow\ |f(x)-L|<\varepsilon)$$
$$\equiv\exists\varepsilon>0\ \neg\exists\delta>0\ \forall x\ (0<|x-a|<\delta\ \rightarrow\ |f(x)-L|<\varepsilon)$$
$$\equiv\exists\varepsilon>0\ \forall\delta>0\ \neg\forall x\ (0<|x-a|<\delta\ \rightarrow\ |f(x)-L|<\varepsilon)$$
$$\equiv\exists\varepsilon>0\ \forall\delta>0\ \exists x\ \neg(0<|x-a|<\delta\ \rightarrow\ |f(x)-L|<\varepsilon)$$
$$\equiv\exists\varepsilon>0\ \forall\delta>0\ \exists x\ (0<|x-a|<\delta\ \wedge\neg(|f(x)-L|<\varepsilon))$$
$$\equiv\exists\varepsilon>0\ \forall\delta>0\ \exists x\ (0<|x-a|<\delta\ \wedge\ |f(x)-L|\geq\varepsilon)$$

Recall: $\neg(p\rightarrow q)\equiv p\wedge\neg q$

Rules of Inference

If you have a current password, then you can log onto the network $p\rightarrow q$

You have a current password p

Therefore, you can log onto the network q

Modus Ponens

$$\begin{array}{l} p\rightarrow q \\ p \\ \hline \therefore q \end{array}$$

Valid argument:
if premises are true
then conclusion is true

If $\sqrt{2} > 1$, then $\left(\sqrt{2}\right)^2 > (1)^2$ $\qquad p \rightarrow q$

We know that $\sqrt{2} > 1$ $\qquad p$ (true)

Therefore,

$$\left(\sqrt{2}\right)^2 = 2 > 1 = (1)^2 \qquad q \text{ (true)}$$

Valid argument,
true conclusion

$$\begin{array}{l} p \rightarrow q \\ p \\ \hline \therefore q \end{array}$$

If $\sqrt{2} > \frac{3}{2}$, then $\left(\sqrt{2}\right)^2 > \left(\frac{3}{2}\right)^2$ $\qquad p \rightarrow q$

We know that $\sqrt{2} > \frac{3}{2}$ $\qquad p$ (false)

Therefore,

$$\left(\sqrt{2}\right)^2 = 2 > \frac{9}{4} = \left(\frac{3}{2}\right)^2 \qquad q \text{ (false)}$$

Valid argument,
false conclusion

$$\begin{array}{l} p \rightarrow q \\ p \\ \hline \therefore q \end{array}$$

Modus Ponens

$$\begin{array}{l} p \to q \\ \underline{p \qquad\quad} \\ \therefore q \end{array}$$

$$((p \to q) \land p) \to q$$

If $p \to q$ and p then q

Rules of Inference

Modus Ponens

$$\begin{array}{l} p \to q \\ \underline{p \qquad\quad} \\ \therefore q \end{array}$$

Modus Tollens

$$\begin{array}{l} p \to q \\ \underline{\neg q \qquad\quad} \\ \therefore \neg p \end{array}$$

Hypothetical Syllogism

$$\begin{array}{l} p \to q \\ \underline{q \to r \qquad\quad} \\ \therefore p \to r \end{array}$$

Disjunctive Syllogism

$$\begin{array}{l} p \lor q \\ \underline{\neg p \qquad\quad} \\ \therefore q \end{array}$$

Rules of Inference

Addition

$$\frac{p}{\therefore p \vee q}$$

Simplification

$$\frac{p \wedge q}{\therefore p}$$

Conjunction

$$\frac{\begin{array}{l} p \\ q \end{array}}{\therefore p \wedge q}$$

Resolution

$$\frac{\begin{array}{l} p \vee q \\ \neg p \vee r \end{array}}{\therefore q \vee r}$$

It is below freezing now p

Therefore,
p it is either below freezing
q or raining now $p \vee q$

Addition

$$\frac{p}{\therefore p \vee q}$$

p It is below freezing
q and raining now

$$p \wedge q$$

Therefore,
it is below freezing now

$$p$$

Simplification

$$\frac{p \wedge q}{\therefore p}$$

p If it rains today
q then we will not have a barbecue today

$$p \rightarrow q$$

q If we do not have a barbecue today
r then we will have a barbecue tomorrow

$$q \rightarrow r$$

Therefore,
p if it rains today
r then we will have a barbecue tomorrow

$$p \rightarrow r$$

Hypothetical Syllogism

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$$

$\neg p$ it is not snowing
q or Jasmine is skiing

$$\neg p \vee q$$

p It is snowing
r or Bart is playing hockey

$$p \vee r$$

Therefore,
Jasmine is skiing
or Bart is playing hockey

$$q \vee r$$

Resolution

$$\begin{array}{l} p \vee q \\ \neg p \vee r \\ \hline \therefore q \vee r \end{array}$$

Hypothesis:

$\neg p$ It is not sunny this afternoon
q and it is colder than yesterday

$$\neg p \wedge q$$

r We will go swimming
p only if it is sunny

$$r \rightarrow p$$

$\neg r$ If we do not go swimming,
s then we will take a canoe trip

$$\neg r \rightarrow s$$

s If we take a canoe trip,
t then we will be home by sunset

$$s \rightarrow t$$

Conclusion:

t We will be home by sunset

1.	$\neg p \wedge q$	Hypothesis
2.	$\neg p$	Simplification from 1
3.	$r \to p$	Hypothesis
4.	$\neg r$	Modus tollens from 2,3
5.	$\neg r \to s$	Hypothesis
6.	s	Modus ponens from 4,5
7.	$s \to t$	Hypothesis
8.	t	Modus ponens from 6,7

Fallacy of affirming the conclusion

$$\begin{array}{l} p \to q \\ q \\ \hline \therefore p \end{array}$$

p If you do every problem in this book
q then you will learn discrete mathematics — $p \to q$

You learned discrete mathematics — q

Therefore,
you did every problem in this book — p

Fallacy of denying the hypothesis

$$\begin{array}{l} p \to q \\ \neg p \\ \hline \therefore \neg q \end{array}$$

p If you do every problem in this book $p \to q$
q then you will learn discrete mathematics

You didn't do every problem in this book $\neg p$

Therefore,
you didn't learn discrete mathematics $\neg q$

Rules of inference for quantifiers

Universal Instantiation

$$\begin{array}{l} \forall x P(x) \\ \hline \therefore P(c) \text{ for any } c \end{array}$$

Universal Generalization

$$\begin{array}{l} P(c) \text{ for arbitrary } c \\ \hline \therefore \forall x P(x) \end{array}$$

Existential Instantiation

$$\begin{array}{l} \exists x P(x) \\ \hline \therefore P(c) \text{ for some } c \end{array}$$

Existential Generalization

$$\begin{array}{l} P(c) \text{ for some } c \\ \hline \therefore \exists x P(x) \end{array}$$

Premises:

$C(x)$ A student in this class
$\neg B(x)$ has not read the book

$$\exists x(C(x) \wedge \neg B(x))$$

$C(x)$ Everyone in this class
$P(x)$ passed the first exam

$$\forall x(C(x) \rightarrow P(x))$$

Conclusion:

$P(x)$ Someone who passed the first exam
$\neg B(x)$ has not read the book

$$\exists x(P(x) \wedge \neg B(x))$$

1.	$\exists x(C(x) \wedge \neg B(x))$	Premise
2.	$C(a) \wedge \neg B(a)$	Existential instantiation from 1
3.	$C(a)$	Simplification from 2
4.	$\forall x(C(x) \rightarrow P(x))$	Premise
5.	$C(a) \rightarrow P(a)$	Universal instantiation from 1
6.	$P(a)$	Modus Ponens from 3,5
7.	$\neg B(a)$	Simplification from 2
8.	$P(a) \wedge \neg B(a)$	Conjunction from 6,7
9.	$\exists x(P(x) \wedge \neg B(x))$	Existential generalization from 8

Universal Modus Ponens

$$\forall x(P(x) \to Q(x))$$
$$P(a), \text{for some particular } a \text{ in domain}$$
$$\therefore Q(a)$$

	For all positive integers x,	
$P(x)$	if $\underline{x > 4}$	$\forall x(P(x) \to Q(x))$
$Q(x)$	then $\underline{x^2 < 2^x}$	
	$100 > 4$	$P(100)$
	Therefore, $100^2 < 2^{100}$	$Q(100)$

Proofs

Theorem: the main result that we want to prove

Lemma: intermediate result used in theorem proof

Axiom: basic truth

Corollary: immediate consequence of theorem

Conjecture: something to be proven

Typically, we want to prove statements

$$\forall x(P(x) \rightarrow Q(x))$$

Proof technique:

show that for some arbitrary c

$$P(c) \rightarrow Q(c)$$

and apply universal generalization

Direct proof: $P(c) \rightarrow Q(c)$

Proof by contraposition: $\neg Q(c) \rightarrow \neg P(c)$

Proof by contradiction: $\neg P(c) \rightarrow (r \wedge \neg r)$

If we want to prove $P(c)$

Definition: integer n is even $\leftrightarrow \exists k \; n = 2k$

integer n is odd $\leftrightarrow \exists k \; n = 2k + 1$

An integer is either even or odd

Theorem: if n is an even integer, $P(n)$
then n^2 is even $Q(n)$

Proof: (direct proof $P(n) \rightarrow Q(n)$)

$$n \text{ is even} \rightarrow \exists k \; n = 2k$$

$$n^2 = (2k)^2 = 4k^2 = 2(2k^2)$$

Therefore, n^2 is even

End of proof

Theorem: if n is an odd integer, $P(n)$
then n^2 is odd $Q(n)$

Proof: (direct proof $P(n) \to Q(n)$)

$$n \text{ is odd} \to \exists k \; n = 2k+1$$

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$$

Therefore, n^2 is odd

End of proof

Theorem: if n^2 is an even integer, $P(n)$
then n is even $Q(n)$

Proof: (proof by contraposition $\neg Q(n) \to \neg P(n)$)

$$\neg Q(n) \to \neg P(n)$$

$$n \text{ is odd} \to n^2 \text{ is odd}$$ (see last proof)

Therefore: $P(n) \to Q(n)$

End of proof

Theorem: if n^2 is an odd integer, then n is odd $\quad P(n) \quad Q(n)$

Proof: (proof by contraposition $\neg Q(n) \rightarrow \neg P(n)$)

$$\neg Q(n) \rightarrow \neg P(n)$$

$$n \text{ is even} \rightarrow n^2 \text{ is even}$$

Therefore: $P(n) \rightarrow Q(n)$

End of proof

Theorem: $\sqrt{2}$ is irrational $\quad P$

Proof: (proof by contradiction $\neg P \rightarrow (r \wedge \neg r)$)

$\neg P$: Assume $\sqrt{2}$ is rational

$$\sqrt{2} = \frac{m}{n}$$

r : m and n have no common divisor greater than 1

Therefore: $\neg P \rightarrow r$

$$2 = \frac{m^2}{n^2} \Rightarrow m^2 = 2n^2 \Rightarrow m = 2k_1 \ (m \text{ is even})$$

$$2n^2 = m^2 = 4k_1^{\ 2} \Rightarrow n^2 = 2k_1^{\ 2} \Rightarrow n = 2k_2 \ (n \text{ is even})$$

$\neg r:$ $\quad \dfrac{m}{n} = \dfrac{2k_1}{2k_2}$ common divisor is 2

Therefore: $\neg P \rightarrow \neg r$

Therefore:

$$\neg P \rightarrow r$$

$$\neg P \rightarrow \neg r$$

$$\therefore (\neg P \rightarrow r) \wedge (\neg P \rightarrow \neg r)$$ Conjunction

$$\equiv \neg P \rightarrow (r \wedge \neg r)$$

contradiction

Therefore:

$$\neg P \to (r \wedge \neg r)$$

$$\neg(r \wedge \neg r)$$

$$\therefore \neg(\neg P)$$ Modus Tollens

$$\equiv P$$

End of proof

Counterexamples

False statement:

"Every positive integer is the sum of the squares of two integers"

$$\forall x > 0\ \exists y \exists z (x = y^2 + z^2)$$

Counterexample: $x = 3$

$$3 \neq 1^2 + 1^2 = 2$$

$$3 \neq 1^2 + 2^2 = 1 + 4 = 5$$

Any other combination gives sum larger than 3

Proof by cases

We want to prove $p \rightarrow q$

We know $p = p_1 \vee p_2 \vee \cdots \vee p_n$

Instead, we can prove each case

$$
\begin{aligned}
& p \rightarrow q \\
& \equiv p_1 \vee p_2 \vee \cdots \vee p_n \rightarrow q \\
& \equiv \underbrace{(p_1 \rightarrow q)}_{\text{Case 1}} \wedge \underbrace{(p_2 \rightarrow q)}_{\text{Case 2}} \wedge \cdots \wedge \underbrace{(p_n \rightarrow q)}_{\text{Case n}}
\end{aligned}
$$

Theorem: If n is integer, then $n^2 \geq n$

Proof: n is integer $\equiv \overbrace{(\mathrm{n}=0)}^{\text{Case 1}} \vee \overbrace{(\mathrm{n} \geq 1)}^{\text{Case 2}} \vee \overbrace{(\mathrm{n} \leq -1)}^{\text{Case 3}}$

Case 1: $\mathrm{n}=0$ $\qquad \mathrm{n}^2 = 0^2 = 0 = n$

Case 2: $\mathrm{n} \geq 1$ $\qquad n^2 = n \cdot n \geq n \cdot 1 = n$

Case 3: $\mathrm{n} \leq -1$ $\qquad n^2 > 0 > n$

End of proof

Existence Proofs

Theorem: There is a positive integer that can be written as the sum of cubes in two different ways

Proof: (constructive existence proof)

$$1729 = 10^3 + 9^3 = 12^3 + 1^3$$

End of proof

Theorem: There exist irrational numbers x, y such that x^y is rational

Proof: (non-constructive existence proof)

We know: $\sqrt{2}$ is irrational

If $\sqrt{2}^{\sqrt{2}}$ is rational $\Rightarrow$ $x = \sqrt{2}, \quad y = \sqrt{2}$

If $\sqrt{2}^{\sqrt{2}}$ is irrational $\Rightarrow$ $x = \sqrt{2}^{\sqrt{2}}, \quad y = \sqrt{2}$

$$x^y = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\sqrt{2}} = \sqrt{2}^2 = 2 = \frac{2}{1}$$ rational

End of proof