Induction and Recursion

CSC-2259 Discrete Structures

Induction

Induction is a very useful proof technique

In computer science, induction is used
to prove properties of algorithms

Induction and recursion are closely related

- Recursion is a description method for algorithms
- Induction is a proof method suitable for recursive algorithms

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Use induction to prove that
a proposition $P(n)$ is true:

Inductive Basis: Prove that $P(1)$ is true

Inductive Hypothesis: Assume $P(k)$ is true
(for any positive integer k)

Inductive Step: Prove that $P(k+1)$ is true

Inductive Hypothesis: Assume $P(k)$ is true
(for any positive integer k)

Inductive Step: Prove that $P(k+1)$ is true

In other words in inductive step we prove:

$$P(k) \rightarrow P(k+1)$$

for every positive integer k

Inductive basis

$P(1)$

True

Inductive Step

$P(k) \rightarrow P(k+1)$

True

Proposition true for all positive integers

$P(1) \rightarrow P(2) \rightarrow P(3) \rightarrow P(4) \rightarrow \cdots$

Induction as a rule of inference:

$$[P(1) \wedge \forall k(P(k) \rightarrow P(k+1))] \rightarrow \forall n P(n)$$

Theorem: $P(n): \quad 1+2+3+\cdots+n = \dfrac{n(n+1)}{2}$

Proof:

Inductive Basis: $P(1): \quad 1 = \dfrac{1(1+1)}{2}$

Inductive Hypothesis: assume that it holds

$$P(k): \quad 1+2+\cdots k = \frac{k(k+1)}{2}$$

Inductive Step: We will prove

$$P(k+1): \quad 1+2+\cdots k+(k+1) = \frac{(k+1)((k+1)+1)}{2}$$

Inductive Step:

(inductive hypothesis)

$$P(k+1): \quad 1+2+\cdots k+(k+1)$$

$$= \frac{k(k+1)}{2} + (k+1)$$

$$= \frac{k(k+1)+2(k+1)}{2}$$

$$= \frac{(k+1)((k+1)+1)}{2}$$

End of Proof

Harmonic numbers

$$H_j = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{j}$$

$$j = 1, 2, 3, \ldots$$

Example: $H_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12}$

Theorem: $H_{2^n} \geq 1 + \frac{n}{2}$ $\quad n \geq 0$

Proof:

Inductive Basis: $n = 0$

$$H_{2^n} = H_{2^0} = H_1 = 1 = 1 + \frac{0}{2} = 1 + \frac{n}{2}$$

Inductive Hypothesis: $n = k$

Suppose it holds: $H_{2^k} \geq 1 + \frac{k}{2}$

Inductive Step: $n = k + 1$

We will show: $H_{2^{k+1}} \geq 1 + \frac{k+1}{2}$

$$\begin{aligned}
H_{2^{k+1}} &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^k} + \frac{1}{2^k + 1} + \cdots + \frac{1}{2^{k+1}} \\
&= H_{2^k} + \frac{1}{2^k + 1} + \cdots + \frac{1}{2^{k+1}} \\
&\geq \left(1 + \frac{k}{2}\right) + \frac{1}{2^k + 1} + \cdots + \frac{1}{2^{k+1}} \\
&\geq \left(1 + \frac{k}{2}\right) + 2^k \cdot \frac{1}{2^{k+1}} \\
&= \left(1 + \frac{k}{2}\right) + \frac{1}{2} \\
&= 1 + \frac{k+1}{2}
\end{aligned}$$

from inductive hypothesis

End of Proof

Theorem: $H_{2^n} \le 1+n$ $\qquad n \ge 0$

Proof:

Inductive Basis: $n=0$

$$H_{2^n} = H_{2^0} = H_1 = 1 = 1+0 = 1+n$$

Inductive Hypothesis: $n=k$

Suppose it holds: $H_{2^k} \le 1+k$

Inductive Step: $n=k+1$

We will show: $H_{2^{k+1}} \le 1+(k+1)$

$$\begin{aligned}
H_{2^{k+1}} &= 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^k}+\frac{1}{2^k+1}+\cdots+\frac{1}{2^{k+1}} \\
&= H_{2^k}+\frac{1}{2^k+1}+\cdots+\frac{1}{2^{k+1}} \\
&\le (1+k)+\frac{1}{2^k+1}+\cdots+\frac{1}{2^{k+1}} \\
&\le (1+k)+2^k\cdot\frac{1}{2^k+1} \\
&\le (1+k)+1 \\
&= 1+(k+1)
\end{aligned}$$

from inductive hypothesis

End of Proof

We have shown: $1+\frac{n}{2} \le H_{2^n} \le 1+n$

It holds that: $H_{2^{\lfloor \log k \rfloor}} \le H_k \le H_{2^{\lceil \log k \rceil}}$

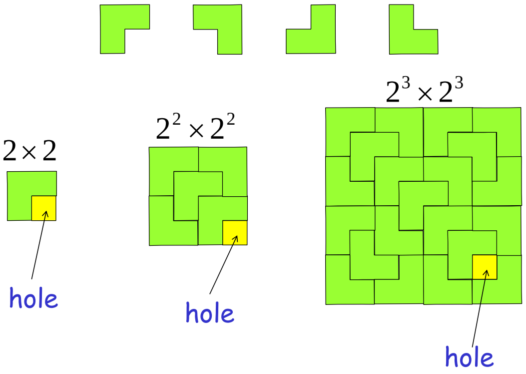
$$\Downarrow$$

$$1+\frac{\lfloor \log k \rfloor}{2} \le H_k \le 1+\lceil \log k \rceil$$

$$\Downarrow$$

$$H_k = \Theta(\log k)$$

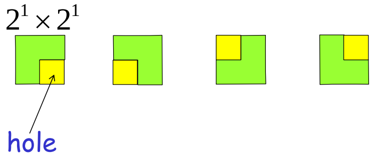
Triominos



Theorem: Every $2^n\times 2^n, n\geq 1$ checkerboard with one square removed can be tiled with triominoes

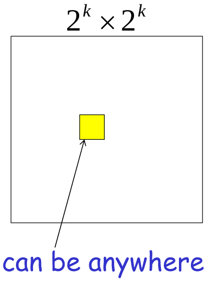
Proof:

Inductive Basis: $n=1$

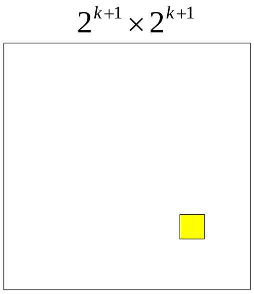


Inductive Hypothesis: $n=k$

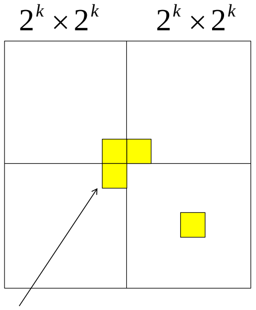
Assume that a $2^k\times 2^k$ checkerboard can be tiled with the hole anywhere



Inductive Step: $n=k+1$

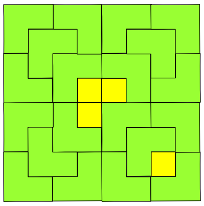


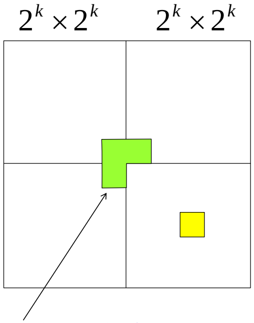
By inductive hypothesis $2^k \times 2^k$ squares with a hole can be tiled



add three artificial holes

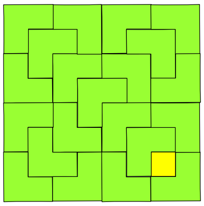
2^3 x 2^3 case:





Replace the three holes with a triomino
Now, the whole area can be tiled

2^3 x 2^3 case:



End of Proof

Strong Induction

To prove $P(n)$:

Inductive Basis: Prove that $P(1)$ is true

Inductive Hypothesis:

Assume $P(1) \wedge P(2) \wedge \cdots \wedge P(k)$ is true

Inductive Step: Prove that $P(k+1)$ is true

Theorem: Every integer $n \geq 2$
is a product of primes
(at least one prime in the product)

Proof: (Strong Induction)

Inductive Basis: $n = 2$

Number 2 is a prime

Inductive Hypothesis: $2 \leq n \leq k$

Suppose that every integer between 2 and k is a product of primes

Inductive Step: $n = k+1$

If $k+1$ is prime then the proof is finished

If $k+1$ is not a prime then it is composite:

$$k+1 = a \cdot b \qquad 2 \leq a, b \leq k$$

$$k+1 = a \cdot b \qquad 2 \leq a, b \leq k$$

By the inductive hypothesis:

$$\left.\begin{array}{c} i, j \geq 1 \\ a = p_1 p_2 \cdots p_i \\ b = q_1 q_2 \cdots q_j \\ \text{primes} \end{array}\right\} \Rightarrow k+1 = a \cdot b = p_1 \cdots p_i q_1 \cdots q_j$$

primes

End of Proof

Theorem: Every postage amount $n \geq 12$ can be generated by using 4-cent and 5-cent stamps

Proof: (Strong Induction)

Inductive Basis: We examine four cases (because of the inductive step)

$n = 12 = 4 + 4 + 4$ $\qquad$ $n = 14 = 5 + 5 + 4$

$n = 13 = 4 + 4 + 5$ $\qquad$ $n = 15 = 5 + 5 + 5$

Inductive Hypothesis: $12 \leq n \leq k$

Assume that every postage amount between 12 and k can be generated by using 4-cent and 5-cent stamps

$$n = a \cdot 4 + b \cdot 5$$

Inductive Step: $n = k + 1$

If $12 \leq k \leq 14$ then the inductive step follows directly from inductive basis

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Consider: $k \geq 15$ $\qquad$ $k + 1 = (k - 3) + 4$

$$12 \leq (k - 3) \leq k$$

⇓ Inductive hypothesis

$$(k - 3) = a' \cdot 4 + b' \cdot 5$$

⇓

$$k + 1 = (k - 3) + 4 = (a' + 1) \cdot 4 + b' \cdot 5$$

End of Proof

Recursion

Recursion is used to describe functions, sets, algorithms

Example: Factorial function $f(n) = n!$

Recursive Basis: $f(0) = 1$

Recursive Step: $f(n+1) = (n+1) \cdot f(n)$

Recursive algorithm for factorial

```
factorial( $n$ ) {
   if $n = 1$ then      //recursive basis
      return $1$
   else                 //recursive step
      return $n \cdot \text{factorial}(n\text{-}1)$
}
```

Fibonacci numbers

$$f_0, f_1, f_2, f_3, \ldots$$

Recursive Basis: $f_0 = 0, \quad f_1 = 1$

Recursive Step: $f_n = f_{n-1} + f_{n-2}$

$$n = 2,3,4,\ldots$$

$$f_0 = 0$$
$$f_1 = 1$$
$$f_2 = f_1 + f_0 = 1 + 0 = 1$$
$$f_3 = f_2 + f_1 = 1 + 1 = 2$$
$$f_4 = f_3 + f_2 = 2 + 1 = 3$$
$$f_5 = f_4 + f_3 = 3 + 2 = 5$$
$$f_6 = f_5 + f_4 = 5 + 3 = 8$$
$$f_7 = f_6 + f_5 = 8 + 5 = 13$$
$$\vdots$$

Recursive algorithm for Fibonacci function

```
fibonacci( $n$ ) {
   if $n \in \{0,1\}$ then      //recursive basis
      return $n$
   else                         //recursive step
      return $\text{fibonacci}(n\text{-}1) + \text{fibonacci}(n\text{-}2)$
}
```

Iterative algorithm for Fibonacci function

```
fibonacci($n$) {
  if $n = 0$ then $y \leftarrow 0$
  else {
    $x \leftarrow 0$
    $y \leftarrow 1$
    for $i \leftarrow 1$ to $n-1$ do {
      $z \leftarrow x + y$
      $x \leftarrow y$
      $y \leftarrow z$
    }
  return $y$
}
```

Theorem: $f_n > \delta^{n-2}$ for $n \geq 3$

$$\delta = \frac{1+\sqrt{5}}{2}$$ (golden ratio)

Proof: Proof by (strong) induction

Inductive Basis: $n = 3$ $n = 4$

$$f_3 = 2 > \delta$$

$$f_4 = 3 > \delta^2$$

Inductive Hypothesis: $3 \leq n \leq k$

Suppose it holds $f_n > \delta^{n-2}$

Inductive Step: $n = k+1$

We will prove $f_{k+1} > \delta^{(k-1)}$ for $4 \leq k$

δ is the solution to equation $x^2 - x - 1 = 0$

$$\Downarrow$$

$$\delta^2 = \delta + 1$$

$$\Downarrow$$

$$\delta^{k-1} = \delta^2 \cdot \delta^{k-3} = (\delta+1)\delta^{k-3} = \delta^{k-2} + \delta^{k-3}$$

$$\Downarrow$$

$$f_{k+1} = \underbrace{f_k + f_{k-1} \geq \delta^{k-2} + \delta^{k-3}}_{\text{induction hypothesis}} = \delta^{k-1}$$

End of Proof

Greatest common divisor

Recursive Basis: $\gcd(a,0) = a$

Recursive Step: $\gcd(a,b) = \gcd(b, a \bmod b)$

$a > b$

Recursive algorithm for greatest common divisor

```
gcd( $a,b$ ) {            //assume a>b
  if $b = 0$ then           //recursive basis
    return $a$
  else                      //recursive step
    return $\gcd(b, a \bmod b)$
}
```

Lamés Theorem:

The Euclidian algorithm for $\gcd(a,b)$, $a \geq b$ uses at most $5 \cdot \log_{10} b$ divisions (iterations)

Proof:

We show that there is a Fibonacci relation in the divisions of the algorithm

remainder

divisions	$a = r_0$		$b = r_1$	
r_0 / r_1	r_0	$=$	$r_1 q_1 + r_2$	$0 < r_2 < r_1$
r_1 / r_2	r_1	$=$	$r_2 q_2 + r_3$	$0 < r_3 < r_2$
$\vdots$		$\vdots$		$\vdots$
r_{n-2} / r_{n-1}	r_{n-2}	$=$	$r_{n-1} q_{n-1} + r_n$	$0 < r_n < r_{n-1}$
r_{n-1} / r_n	r_{n-1}	$=$	$r_n q_n + 0$	

first zero

result

$$\gcd(a,b) = \gcd(r_0, r_1) = \gcd(r_1, r_2) = \gcd(r_2, r_3) \cdots$$
$$\cdots = \gcd(r_{n-2}, r_{n-1}) = \gcd(r_{n-1}, r_n) = \gcd(r_n, 0) = r_n$$

$$\begin{array}{rcccl} r_0 &=& r_1 q_1 + r_2 &\Rightarrow& r_0 \ge r_1 + r_2 \\ r_1 &=& r_2 q_2 + r_3 &\Rightarrow& r_1 \ge r_2 + r_3 \\ &\vdots& && \\ r_{n-2} &=& r_{n-1} q_{n-1} + r_n &\Rightarrow& r_{n-2} \ge r_{n-1} + r_n \\ r_{n-1} &=& r_n q_n + 0 &\Rightarrow& \boxed{r_{n-1} \ge 2r_n} \end{array}$$

This holds since $r_n < r_{n-1}$
and q_n is integer

This holds since $r_n = \gcd(a,b) \ge 1$

$$\begin{array}{l} \boxed{r_n \ge 1} \\ r_{n-1} \ge 2r_n \\ r_{n-2} \ge r_{n-1} + r_n \\ r_{n-3} \ge r_{n-2} + r_{n-1} \\ \vdots \\ r_2 \ge r_3 + r_4 \\ r_1 \ge r_2 + r_3 \end{array} \Rightarrow \begin{array}{l} r_n \ge 1 = f_2 \\ r_{n-1} \ge 2r_n \ge 2 f_2 = 2 = f_3 \\ r_{n-2} \ge r_{n-1} + r_n \ge f_3 + f_2 = f_4 \\ r_{n-3} \ge r_{n-2} + r_{n-1} \ge f_4 + f_3 = f_5 \\ \vdots \\ r_2 \ge r_3 + r_4 \ge f_{n-1} + f_{n-2} = f_n \\ r_1 \ge r_1 + r_2 \ge f_n + f_{n-1} = f_{n+1} \end{array}$$

$$\left.\begin{array}{l} b = r_1 \ge f_{n+1} \\ \\ f_{n+1} > \delta^{n-1} \end{array}\right\} \Rightarrow b > \delta^{n-1}$$

$$\delta = \frac{1+\sqrt{5}}{2}$$

$$\Downarrow$$

$$\log_{10} b > (n-1)\log_{10}\delta$$

$$\Downarrow$$

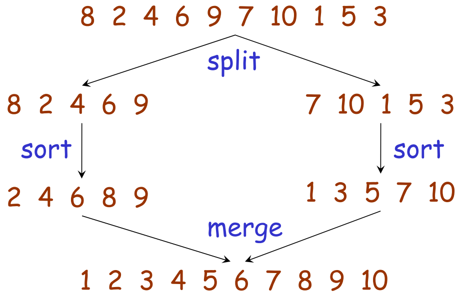
$$n < \frac{\log_{10} b}{\log_{10}\delta} + 1 < 5 \cdot \log_{10} b + 1$$

$$\Downarrow$$

$$n \le 5 \cdot \log_{10} b$$

End of Proof

Algorithm Mergesort



8 2 4 6 9 7 10 1 5 3

split

8 2 4 6 9 7 10 1 5 3

sort sort

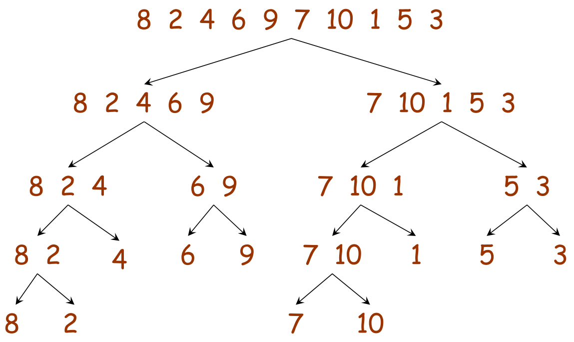
2 4 6 8 9 1 3 5 7 10

merge

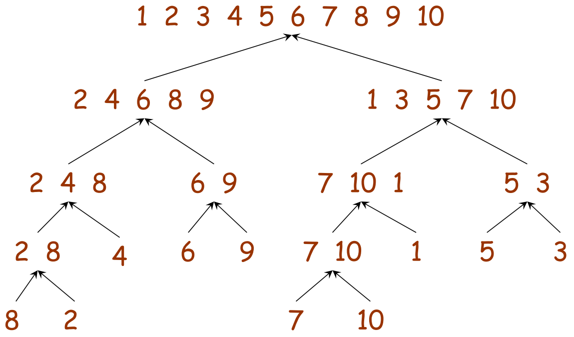
1 2 3 4 5 6 7 8 9 10

```
sort( $a_1, a_2, \ldots, a_n$ ) {
    if $n > 1$ then {
        $m = \lfloor n/2 \rfloor$
        $A \leftarrow \mathrm{sort}(a_1, a_2, \ldots, a_m)$
        $B \leftarrow \mathrm{sort}(a_m, a_{m+1}, \ldots, a_n)$
        return $\mathrm{merge}(A, B)$
    }
    else return $a_1$
}
```

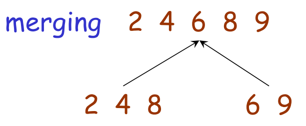
Input values of recursive calls



Input and output values of merging

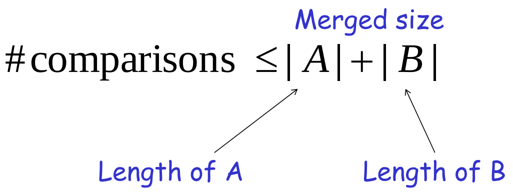


```
merge( $A, B$ ) {  //two sorted lists
    $L \leftarrow \emptyset$
    while $A \neq \emptyset$ and $B \neq \emptyset$ do {
        Remove smaller first element of $A, B$
        from its list and insert it to $L$
    }
    if $A \neq \emptyset$ or $B \neq \emptyset$ then {
        append remaining elements to $L$
    }
    return $L$
}
```



A	B	L	Comparison
2 4 8	6 9	2	2<6
4 8	6 9	2 4	4<6
8	6 9	2 4 6	6<8
8	9	3 4 6 8	8<9
	9	2 4 6 8 9	

The total number of comparisons to merge two lists A, B is at most:



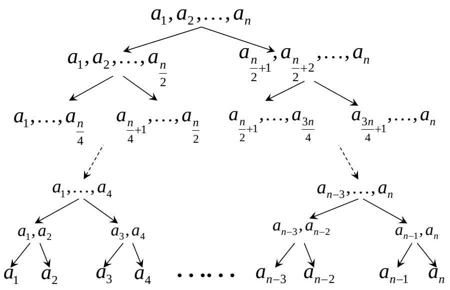
Merged size

$$\#\text{comparisons} \ \leq |A| + |B|$$

Length of A

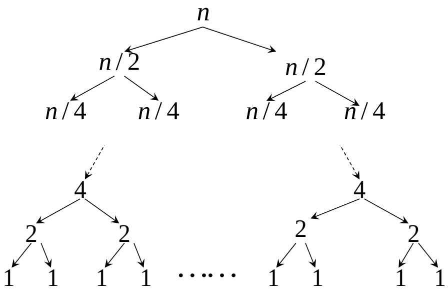
Length of B

Recursive invocation tree



Assume
$n = 2^k$

Recursive invocation tree



Elements per list

$n = 2^{\log n - 0}$

$n/2 = 2^{\log n - 1}$

$n/4 = 2^{\log n - 2}$

$\vdots$

$4 = 2^{\log n - (\log n - 2)}$

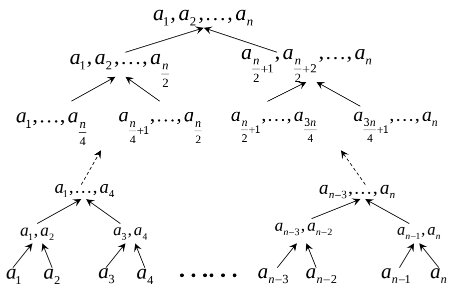
$2 = 2^{\log n - (\log n - 1)}$

$1 = 2^{\log n - \log n}$

Assume
$n = 2^k$

#levels of tree $= 1 + \log n$

merging tree



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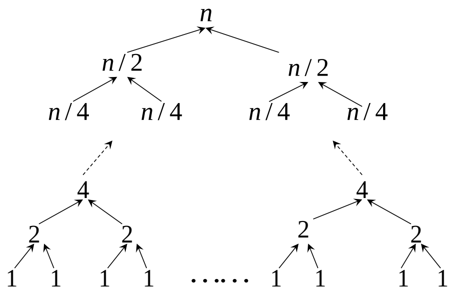
merging tree



Elements per list

n

$n/2$

$n/4$

$\vdots$

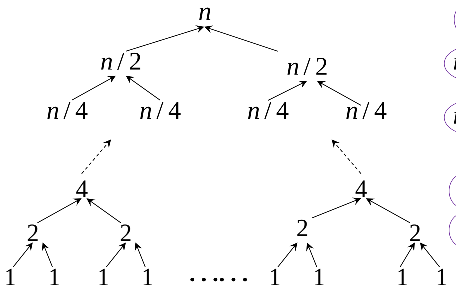
4

2

1

---

merging tree



Comparisons per level

$n \cdot 1 = n$

$n/2 \cdot 2 = n$

$n/4 \cdot 4 = n$

$\vdots$

$4 \cdot n/4 = n$

$2 \cdot n/2 = n$

Merges per level

Elements per merge

Total cost: $n \cdot (\#\text{levels} - 1) = n \log n$

If $n = 2^k$ the number of comparisons is at most $n \log n$

If $n \neq 2^k$ the number of comparisons is at most $m \log m = O(n \log n)$

$$m = 2^{\lceil \log n \rceil} < 2n$$

Time complexity of merge sort: $O(n \log n)$