Counting

CSC-2259 Discrete Structures

Basic Counting Principles

Product Rule:

Suppose a procedure consists of 2 tasks

n_1 ways to perform 1st task

n_2 ways to perform 2nd task

$$\Downarrow$$

$n_1 \cdot n_2$ ways to perform procedure

Example: 2 employees 10 offices

How many ways to assign employees to offices?

1st employee has 10 office choices

2nd employee has 9 office choices

Total office assignment ways: 10x9 = 90

Example: How many different variable names with 2 symbols?

XY (e.g. A1, A2, AA)

1st symbol
letter
26 choices

2nd symbol
alphanumeric
26+10 = 36 choices

Total variable name choices: 26x36 = 936

Generalized Product Rule:

Suppose a procedure consists of k tasks

n_1 ways to perform 1st task

n_2 ways to perform 2nd task

$\vdots$

n_k ways to perform k th task

$\Downarrow$

$n_1 n_2 \cdots n_k$ ways to perform procedure

Example: How many different variable names with exactly $k \geq 1$ symbols?

$XY_1 \cdots Y_{k-1}$ (e.g. D1B…6)

1st symbol
letter
26 choices

Remaining symbols
alphanumeric
36 choices for each

Total choices: $26 \cdot \underbrace{36 \cdots 36}_{k-1} = 26 \cdot (36)^{k-1}$

Sum Rule:

Suppose a procedure can be performed with either of 2 different methods

n_1 ways to perform 1st method

n_2 ways to perform 2nd method

$\Downarrow$

$n_1 + n_2$ ways to perform procedure

Example: Number of variable names with 1 or 2 symbols

Variables with 1 symbol: 26

Variables with 2 symbols: 936

Total number of variables: 26+936=962

Principle of Inclusion-Exclusion:

Suppose a procedure can be performed with either of 2 different methods

n_1 ways to perform 1st method

n_2 ways to perform 2nd method

c common ways in both methods

$n_1 + n_2 - c$ ways to perform procedure

Example:
Number of binary strings of length 8 that either start with 1 or end with 0

Strings that start with 1: $1x_1x_2\cdots x_7$ 128 choices

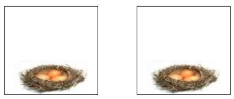
Strings that end with 0: $y_1y_2\cdots y_7 0$ 128 choices

Common strings: $1z_1\cdots z_7 0$ 64 choices

Total strings: 128+128-64=192

Pigeonhole Principle

3 pigeons

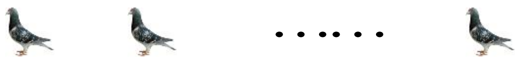


2 pigeonholes

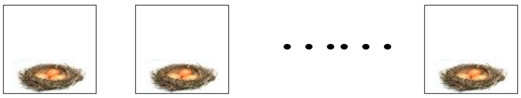
One pigeonhole contains 2 pigeons

3 pigeons

2 pigeonholes



k+1 pigeons



k pigeonholes

At least one pigeonhole contains 2 pigeons

k+1 pigeons

k pigeonholes

Pigeonhole Principle:

If k+1 objects are placed into k boxes, then at least one box contains 2 objects

Examples:

- Among 367 people at least 2 have the same birthday (366 different birthdays)
- Among 27 English words at least 2 start with same letter (26 different letters)

Generalized Pigeonhole Principle:

If N objects are placed into k boxes, then at least one box contains $\left\lceil \frac{N}{k} \right\rceil$ objects

Proof:

If each box contains less than $\left\lceil \frac{N}{k} \right\rceil$ objects:

$$\#\text{objects} \leq k\left(\left\lceil \frac{N}{k} \right\rceil - 1\right) < k\left(\left(\frac{N}{k} + 1\right) - 1\right) = N$$

contradiction

End of Proof

Example:

Among 100 people, at least $\left\lceil \frac{100}{12} \right\rceil = 9$ have birthday in same month

$N = 100$ people (objects)

$k = 12$ months (boxes)

Example:

How many students do we need to have so that at least six receive same grade (A,B,C,D,F)?

$N = ?$ students (objects)

$k = 5$ grades (boxes)

$\left\lceil \frac{N}{k} \right\rceil \geq 6$ at least six students receive same grade

Smallest integer N with $\left\lceil \frac{N}{k} \right\rceil \geq r$

is smallest integer with $\frac{N}{k} > r - 1$

$$\frac{N}{k} > r - 1 \Rightarrow N > k(r-1) \Rightarrow N = k(r-1)+1$$

$$\left\lceil \frac{N}{k} \right\rceil \geq r \Rightarrow N = k(r-1)+1$$

For our example:

$k = 5$, $r = 6$ $\Rightarrow$ $N = 5(6-1)+1 = 26$ students

We need at least 26 students

An elegant example:

In any sequence of n^2+1 numbers
there is a sorted subsequence of length $n+1$
(ascending or descending)

$n=3$

$n+1=4$ $\quad$ $n^2+1=10$ numbers

Ascending subsequence: 8, 11, 9, 1, 4, 6, 12, 10, 5, 7

Descending subsequence: 8, 11, 9, 1, 4, 6, 12, 10, 5, 7

Theorem:

In any sequence of n^2+1 numbers
there is a sorted subsequence of length $n+1$
(ascending or descending)

Proof: Sequence $a_1, a_2, a_3, \ldots, a_{n^2+1}$

(x_i, y_i)

Length of longest ascending subsequence starting from a_i

Length of longest descending subsequence starting from a_i

For example: $(x_1, y_1) = (3,3)$

Longest ascending subsequence from $a_1 = 8$

8, 11, 9, 1, 4, 6, 12, 10, 5, 7

$x_1 = 3$

Longest descending subsequence from $a_1 = 8$

8, 11, 9, 1, 4, 6, 12, 10, 5, 7

$y_1 = 3$

For example: $(x_2, y_2) = (2,4)$

Longest ascending subsequence from $a_2 = 11$

8, 11, 9, 1, 4, 6, 12, 10, 5, 7

$x_2 = 2$

Longest descending subsequence from $a_2 = 11$

8, 11, 9, 1, 4, 6, 12, 10, 5, 7

$y_2 = 4$

We want to prove that
there is a (x_i, y_i) with:

$$x_i \geq n+1 \quad \text{or} \quad y_i \geq n+1$$

Assume (for sake of contradiction) that
for every (x_i, y_i):

$$1 \leq x_i \leq n \ \text{ and } \ 1 \leq y_i \leq n$$

Number of unique pairs of form (x_i, y_i)
with $1 \leq x_i \leq n$ and $1 \leq y_i \leq n$:

$$(x_i, y_i)$$

n choices n choices

$n \cdot n = n^2$ unique pairs

For example: $(1,1), (1,2), (2,1), (1,3), (3,1), \ldots, (n,n)$

$n \cdot n = n^2$ unique pairs of form (x_i, y_i)

Since $a_1, a_2, a_3, \ldots, a_{n^2+1}$ has n^2+1 elements
there are exactly n^2+1 pairs of form (x_i, y_i)

$$\Downarrow$$

From pigeonhole principle, there are two
equal pairs $(x_j, y_j) = (x_k, y_k), \quad j < k$

$$x_j = x_k \ \text{ and } \ y_j = y_k$$

Case $a_j \leq a_k$:

$$(x_j, y_j) = (x_k, y_k), \quad j < k \quad \Rightarrow \quad x_j = x_k$$

$$a_1, a_2, a_3, \ldots, \underbrace{a_j, \ldots, \overbrace{a_k, \ldots, a_{k_2}, \ldots, a_{k_{x_k}}}^{\text{Ascending subsequence with } x_k \text{ elements}}}_{\text{Ascending subsequence with } x_k + 1 = x_j + 1 > x_j \text{ elements}}, \ldots, a_{n^2+1}$$

Contradiction, since longest ascending subsequence
from a_j has length x_j

Case $a_j > a_k$:

$(x_j, y_j) = (x_k, y_k), \quad j < k \Longrightarrow y_j = y_k$

Descending subsequence with y_k elements

$$a_1, a_2, a_3, \ldots, \underbrace{a_j, \ldots, \overbrace{a_k, \ldots, a_{k_2}, \ldots, a_{k_{y_k}}}^{}}_{}, \ldots, a_{n^2+1}$$

Descending subsequence with $y_k + 1 = y_j + 1 > y_j$ elements

Contradiction, since longest descending subsequence from a_j has length y_j

Therefore, it is not true the assumption that for every (x_i, y_i): $1 \le x_i \le n$ and $1 \le y_i \le n$

Therefore, there is a (x_i, y_i) with:

$x_i \ge n+1$ or $y_i \ge n+1$

End of Proof

Permutations

Permutation: An ordered arrangement of objects

Example: Objects: a,b,c

Permutations: a,b,c a,c,b
b,a,c b,c,a
c,a,b c,b,a

r-permutation: An ordered arrangement of r objects

Example: Objects: a,b,c,d

2-permutations: a,b a,c a,d
b,a b,c b,d
c,a c,b c,d
d,a d,b d,c

How many ways to arrange 5 students in line?

1st position in line: 5 student choices

2nd position in line: 4 student choices

3rd position in line: 3 student choices

4th position in line: 2 student choices

5th position in line: 1 student choices

Total permutations: 5x4x3x2x1=120

How many ways to arrange 3 students in line out of a group of 5 students?

1st position in line: 5 student choices

2nd position in line: 4 student choices

3rd position in line: 3 student choices

Total 3-permutations: 5x4x3=60

Given n objects the number of r-permutations is denoted

$$P(n,r)$$

Examples:

$$P(5,5)=120$$
$$P(5,3)=60$$
$$P(4,2)=12$$
$$P(3,3)=6$$

Theorem: $P(n,r)=\dfrac{n!}{(n-r)!}$ $\quad 0 \le r \le n$

Proof:

$$P(n,r)=n\cdot(n-1)\cdot(n-2)\cdots(n-(r-1))$$

1st position object choices

2nd position object choices

3rd position object choices

rth position object choices

$$P(n,r) = n \cdot (n-1) \cdot (n-2) \cdots (n-(r-1))$$

$$= \frac{n \cdot (n-1) \cdot (n-2) \cdots (n-(r-1)) \cdot (n-r) \cdot (n-(r+1)) \cdots 2 \cdot 1}{(n-r) \cdot (n-(r+1)) \cdots 2 \cdot 1}$$

$$= \frac{n!}{(n-r)!}$$

Multiply and divide with same product

End of Proof

Example: How many different ways to order gold, silver, and bronze medalists out of 8 athletes?

$$P(8,3) = \frac{8!}{(8-3)!} = \frac{8!}{5!} = 8 \cdot 7 \cdot 6 = 336$$

Combinations

r-combination: An unordered arrangement of r objects

Example: Objects: a,b,c,d

2-combinations: a,b a,c a,d b,c b,d c,d

3-combinations: a,b,c a,b,d a,c,d b,c,d

Given n objects the number of r-combinations is denoted

$$C(n,r) \quad \text{or} \quad \binom{n}{r}$$

Also known as binomial coefficient

Examples: $C(4,2) = 6$

$C(4,3) = 4$

Combinations can be used to find permutations

3-combinations $C(4,3)$

a,b,c	a,b,d	a,c,d	b,c,d

Objects: a,b,c,d

Combinations can be used to find permutations

3-combinations $C(4,3)$

3-permutations $P(3,3)$

a,b,c	a,b,d	a,c,d	b,c,d
a,c,b	a,d,b	a,d,c	b,d,c
b,a,c	b,a,d	c,a,d	c,b,d
b,c,a	b,d,a	c,d,a	c,d,b
c,a,b	d,a,b	d,a,c	d,b,c
c,b,a	d,b,a	d,c,a	d,c,b

Objects: a,b,c,d

Combinations can be used to find permutations

Total permutations: $P(4,3) = C(4,3) \cdot P(3,3)$

3-combinations $C(4,3)$

3-permutations $P(3,3)$

a,b,c	a,b,d	a,c,d	b,c,d
a,c,b	a,d,b	a,d,c	b,d,c
b,a,c	b,a,d	c,a,d	c,b,d
b,c,a	b,d,a	c,d,a	c,d,b
c,a,b	d,a,b	d,a,c	d,b,c
c,b,a	d,b,a	d,c,a	d,c,b

Objects: a,b,c,d

Theorem: $C(n,r) = \dfrac{n!}{r!(n-r)!}$ $\quad 0 \le r \le n$

Proof: $P(n,r) = C(n,r) \cdot P(r,r)$

$\Downarrow$

$$C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{\frac{n!}{(n-r)!}}{\frac{r!}{(r-r)!}} = \frac{n!}{r!(n-r)!}$$

End of Proof

Example: Different ways to choose 5 cards out of 52 cards

$$C(52,5) = \frac{52!}{5!(47)!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 26 \cdot 17 \cdot 10 \cdot 49 \cdot 12 = 2{,}598{,}960$$

Observation: $C(n,r) = C(n,n-r)$

$$C(n,r) = \frac{n!}{r!(n-r)!} = \frac{n!}{(n-(n-r))!(n-r)!} = C(n,n-r)$$

Example: $C(52,5) = C(52,47)$

Binomial Coefficients

$$C(n,r) = \binom{n}{r}$$

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

$$\begin{aligned}(x+y)^3 &= (x+y)(x+y)(x+y) \\ &= xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy \\ &= x^3 + 3x^2y + 3xy^2 + y^3\end{aligned}$$

$$(x+y)^3 = \binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}y^3$$

Possible ways to obtain product of 3 terms of x and 0 terms of y

Possible ways to obtain product of 2 terms of x and 1 terms of y

Possible ways to obtain product of 0 terms of x and 3 terms of y

$$(x+y)^n = \overbrace{(x+y)(x+y)(x+y)\cdots(x+y)}^{n \text{ times}}$$

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

Possible ways to obtain product of n terms of x and 0 terms of y

Possible ways to obtain product of $n-1$ terms of x and 1 terms of y

Possible ways to obtain product of 0 terms of x and n terms of y

Observation: $2^n = \sum_{j=0}^{n} \binom{n}{j}$

$$2^n = (1+1)^n = \sum_{j=0}^{n} \binom{n}{j} 1^{n-j} 1^j = \sum_{j=0}^{n} \binom{n}{j}$$

Observation: $3^n = \sum_{j=0}^{n} 2^j \binom{n}{j}$

$$3^n = (1+2)^n = \sum_{j=0}^{n} \binom{n}{j} 1^{n-j} 2^j = \sum_{j=0}^{n} \binom{n}{j} 2^j$$

Observation: $0 = \sum_{j=0}^{n} (-1)^j \binom{n}{j}$

$$0 = 0^n = (1+(-1))^n = \sum_{j=0}^{n} \binom{n}{j} 1^{n-j} (-1)^j = \sum_{j=0}^{n} \binom{n}{j} (-1)^j$$

$$0=\sum_{j=0}^{n}(-1)^j\binom{n}{j}$$

$$0=\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\binom{n}{3}+\binom{n}{4}-\binom{n}{5}+\cdots$$

$$\underbrace{\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+\cdots}_{\text{even}}=\underbrace{\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\cdots}_{\text{odd}}$$

Pascal's Identity: $\binom{n+1}{k}=\binom{n}{k-1}+\binom{n}{k}$

$n \geq k > 0$

Proof: T set with $n+1$ elements

$a \in T$ element of T

number of subsets of T
with size k

$$\binom{n+1}{k}=|X|+|Y|$$

subsets that contain a ($|X|$)

subsets that do not contain a ($|Y|$)

X :subsets of T with size k that contain a

Each $s \in X$ has form: $s=\{a,\underbrace{t_1,\ldots,t_{k-1}}_{}\}$

$k-1$ elements from $T-\{a\}$

Total ways for constructing $s \in X$:

$$|X|=\binom{|T-\{a\}|}{k-1}=\binom{n}{k-1}$$

Y :subsets of T with size k
that do not contain a

Each $s \in Y$ has form: $s = \underbrace{\{t_1, \ldots, t_k\}}_{k \text{ elements from } T-\{a\}}$

Total ways for constructing $s \in Y$:

$$|Y| = \binom{|T-\{a\}|}{k} = \binom{n}{k}$$

$$\binom{n+1}{k} = |X| + |Y| = \binom{n}{k-1} + \binom{n}{k}$$

End of Proof

Pascal's Triangle

$$\begin{array}{ccccccccccccc}
 & & & & & & \binom{0}{0} & & & & & & \\
 & & & & & \binom{1}{0} & & \binom{1}{1} & & & & & \\
 & & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & & \\
 & & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & & \\
 & & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & & \\
 & \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} & \\
\binom{6}{0} & & \binom{6}{1} & & \binom{6}{2} & & \binom{6}{3} & & \binom{6}{4} & & \binom{6}{5} & & \binom{6}{6}
\end{array}$$

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

$$\binom{6}{4} = \binom{5}{3} + \binom{5}{4} \qquad 15 = 10 + 5$$

Generalized Permutations and Combinations

Permutations with repetition for: a,b,c

aaa	aab	aba	abb	aac	aca	acc	abc	acb
bbb	bba	bab	baa	bbc	bcb	bcc	bac	bca
ccc	cca	cac	caa	ccb	cbc	cbb	cab	cba

3 ways to chose each symbol

Total permutations with repetition: 3x3x3 = 27

Permutations with repetition:

#ways to arrange in line r objects chosen from a collection of n objects:

$$n^r$$

Example: Strings of length $r=5$ from English alphabet: $n^r = 26^5$

aaaaa, aaaab, aaaba, aaabb, aabab,...

Combinations with repetition for: a,b,c

aaa aab ~~aba~~ abb aac ~~aca~~ acc abc ~~acb~~
bbb ~~bba~~ ~~bab~~ ~~baa~~ bbc ~~bcb~~ bcc ~~bac~~ ~~bca~~
ccc ~~cca~~ ~~cac~~ ~~caa~~ ~~ccb~~ ~~cbc~~ ~~cbb~~ ~~cab~~ ~~cba~~

After removing redundant permutations

aaa aab abb aac acc abc
bbb bbc bcc
ccc

Total combinations with repetition: 10

Encoding for combinations with repetitions:

a | b | c

* * *

1st letter choice — 2nd letter choice — 3rd letter choice

abc = a|b|c = *|*|*

acc = a||cc = *||**

aab = aa|b|= **|*|

bcc =|b|cc = |*|**

All possible combinations with repetitions for objects a,b,c:

aaa: ***||
aab: **|*|
aac: **||*
abb: *|**|
abc: *|*|*
acc: *||**
bbb: |***|
bbc: |**|*
bcc: |*|**
ccc: ||***

Equivalent to finding all possible ways to arrange *** and ||

All possible ways to arrange *** and || :

equivalent to all possible ways to select 3 objects out of 5:

5 total positions in a string
3 positions are dedicated for *

$$\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5!}{3!\cdot 2!} = \frac{4\cdot 5}{2} = 10$$

2-combinations with repetition for: a,b,c

aa ab ac bb bc cc Total = 6

Each combination can be encoded with ** and || :

ab = a|b| = *|*|

ac = a||c = *||*

All possible 2-combinations with repetitions for objects a,b,c:

aa: **||
ab: *|*|
ac: *||*
bb: |**|
bc: |*|*
cc: ||**

Equivalent to finding all possible ways to arrange ** and ||

All possible ways to arrange ** and || :

equivalent to all possible ways to select 2 objects out of 4:

4 total positions in a string
2 positions are dedicated for *

$$\binom{4}{2} = \frac{4!}{2!(4-2)!} = \frac{4!}{2!\cdot 2!} = \frac{4\cdot 3}{2} = 6$$

4-combinations with repetition for: a,b,c

aaaa aaab aaac
aabb aabc aacc
abbb abbc abcc accc
bbbb bbbc bbcc bccc cccc

Total = 15

Each combination can be encoded with **** and || :

aabb = aa|bb| = **|**|

abcc = a|b|cc = *|*|**

All possible ways to arrange **** and || :

equivalent to all possible ways to select 4 objects out of 6:

6 total positions in a string
4 positions are dedicated for *

$$\binom{6}{4} = \frac{6!}{4!(6-4)!} = \frac{6!}{3!\cdot 2!} = \frac{6\cdot 5}{2} = 15$$

r-combinations with repetition:

Number of ways to select r objects out of n:

$$\binom{n+r-1}{r}$$

Proof:

Each of the r objects corresponds to a *

The original n objects create $n-1$ of |

The | separate the n original objects

$$\text{obj 1}\underbrace{|\text{obj 2}|\text{obj 3}|\ldots|\text{obj(n-1)}|}_{n-1}\text{obj n}$$

The * represent the r selected objects

*|***|*||*|...|*

1st selected object

2nd selected object

rth selected object

Each r-combination can be encoded with a unique string formed with $\underbrace{**\ldots*}_{r}$ and $\underbrace{||\ldots|}_{n-1}$:

$$\underbrace{*|***|*||*|\ldots|*}$$

String length: $r+(n-1)$

Example 3-combinations of 5 objects a,b,c,d,e:

abc = a|b|c|| = *|*|*||

cde = ||c|d|e = ||*|*|*

All possible strings made of $\underbrace{**\ldots*}_{r}$ and $\underbrace{||\ldots|}_{n-1}$:

Equivalent to all possible ways to select r objects out of $r+(n-1)$:

$r+(n-1)$ total positions in a string

r positions are dedicated for *

$$\binom{r+(n-1)}{r}=\binom{n+r-1}{r}$$

End of Proof

Example:

How many ways to select r=6 cookies from a collection of n=4 different kinds of cookies?

Equivalent to combinations with repetition:

$$\binom{n+r-1}{r}=\binom{6+4-1}{6}=\binom{9}{6}=\frac{9!}{6!(9-6)!}=\frac{9\cdot 8\cdot 7}{1\cdot 2\cdot 3}=84$$

Example: How many integer solutions does the equation have?

$$x_1+x_2+x_3=11 \qquad x_1,x_2,x_3\geq 0$$

$6+1+4=11 \qquad x_1=6,\ x_2=1,\ x_3=4$

$0+3+8=11 \qquad x_1=0,\ x_2=8,\ x_3=4$

$$4+2+5=11$$

$$\underbrace{1_{x_1}+1_{x_1}+1_{x_1}+1_{x_1}}_{x_1=4}+\underbrace{1_{x_2}+1_{x_2}}_{x_2=2}+\underbrace{1_{x_3}+1_{x_3}+1_{x_3}+1_{x_3}+1_{x_3}}_{x_3=5}=11$$

Equivalent to selecting **r=11** items (ones) from **n=3** kinds (variables)

$$\binom{n+r-1}{r}=\binom{3+11-1}{11}=\binom{13}{11}=\frac{13\cdot 12}{1\cdot 2}=78$$

Permutations of indistinguishable objects

SUCCESS

How many different strings are made by reordering the letters in the string?

SUSCCES, USCSECS, CESUSSC,...

SUCCESS SUCCESS SUCCESS SUCCESS

$\binom{7}{3}$	$\binom{4}{2}$	$\binom{2}{1}$	$\binom{1}{1}$
available positions for 3 S	available positions for 2 C	available positions for 1 U	available positions for 1 E

Total possible strings:

$$\binom{7}{3}\binom{4}{2}\binom{2}{1}\binom{1}{1}=\frac{7!}{3!\cdot 4!}\cdot\frac{4!}{2!\cdot 2!}\cdot\frac{2!}{1!\cdot 1!}\cdot\frac{1!}{1!\cdot 0!}=\frac{7!}{3!\cdot 2!\cdot 1!\cdot 1!}=420$$

Permutations of indistinguishable objects:

n_1 indistinguishable objects of type 1

n_2 indistinguishable objects of type 2

$\vdots$

n_k indistinguishable objects of type k

$$n=n_1+n_2+\cdots+n_k$$

Total permutations for the n objects:

$$\frac{n!}{n_1!n_2!\cdots n_k!}$$

Proof:

$X_1 X_2 \cdots X_n$ $\quad$ $X_1 X_2 \cdots X_n$ $\quad$ $X_1 X_2 \cdots X_n$

$$\binom{n}{n_1} \qquad \binom{n-n_1}{n_2} \qquad \cdots \qquad \binom{n-n_1-\cdots-n_{k-1}}{n_k}$$

Available positions for n_1 objects of type 1

Available positions for n_2 objects of type 2

Available positions for n_k objects of type k

Total possible permutations:

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-\cdots-n_{k-1}}{n_k}$$

$$=\frac{n!}{n_1!\cancel{(n-n_1)!}}\cdot\frac{\cancel{(n-n_1)!}}{n_2!\cancel{(n-n_1-n_2)!}}\cdots\frac{\cancel{(n-n_1-\cdots-n_{k-1})!}}{n_k!\cancel{(n-n_1-\cdots-n_k)!}}$$

$(n-n_1-n_2-\cdots-n_k=0)$

$$=\frac{n!}{n_1!}\cdot\frac{1}{n_2!}\cdots\frac{1}{n_k!}$$

End of Proof

Distributing objects into boxes

5 distinguishable objects: a, b, c, d, e

3 distinguishable boxes:

Box1: holds 2 objects
Box2: holds 1 object
Box3: holds 2 objects

How many ways to distribute the objects into the boxes? (position inside box doesn't matter)

a e	b	c d
Box 1	Box 2	Box 3

Problem is equivalent to finding all permutations with indistinguishable objects

$n=5$ distinct positions: a b c d e

<u>a</u> b c d <u>e</u> $\quad$ a <u>b</u> c d e $\quad$ a b <u>c</u> <u>d</u> e

$n_1=2$ positions for Box 1 $\quad$ $n_2=1$ positions for Box 2 $\quad$ $n_3=2$ positions for Box 3

a e	b	c d

Total arrangements of objects into boxes:

$$\frac{n!}{n_1!n_2!n_3!} = \frac{5!}{2!\,1!\,2!} = \frac{3 \cdot 4 \cdot 5}{2} = 30$$

Same as permutations
of indistinguishable objects

In general:

n distinguishable objects

k distinguishable boxes:

Box1: holds n_1 objects
Box2: holds n_2 objects
...
Boxk: holds n_k objects

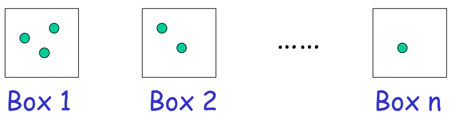
$$n = n_1 + n_2 + \cdots + n_k$$

Total arrangements: $\dfrac{n!}{n_1!n_2!\cdots n_k!}$

Distributing indistinguishable objects
into distinguishable boxes

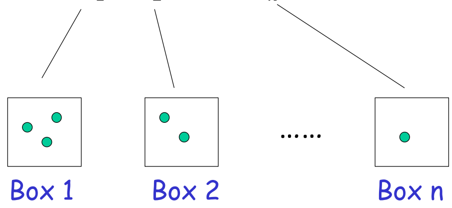
r indistinguishable objects

n distinguishable boxes:



Problem is same with finding the number
of solutions to equation:

$$x_1 + x_2 + \cdots + x_n = r$$



$x_i \geq 0$: number of objects in Box i

$$x_1 + x_2 + \cdots + x_n = r \qquad x_i \geq 0$$

Total number of solutions:

$$\binom{n+r-1}{r}$$

Is equal to number of ways to distribute
the indistinguishable objects into the boxes