Discrete Probability

CSC-2259 Discrete Structures

Introduction to Discrete Probability

Unbiased die

Sample Space: $S = \{1,2,3,4,5,6\}$

All possible outcomes

Event: any subset of sample space

$E_1 = \{3\}$ $\qquad$ $E_2 = \{2,5\}$

Experiment: procedure that yields events

Throw die

Probability of event E:

$$p(E) = \frac{\text{size of event set}}{\text{size of sample space}} = \frac{|E|}{|S|}$$

Note that: $0 \leq p(E) \leq 1$

since $0 \leq |E| \leq |S|$

What is the probability that
a die brings 3?

Event Space: $E = \{3\}$

Sample Space: $S = \{1,2,3,4,5,6\}$

Probability: $p(E) = \frac{|E|}{|S|} = \frac{1}{6}$

What is the probability that
a die brings 2 or 5?

Event Space: $E = \{2,5\}$

Sample Space: $S = \{1,2,3,4,5,6\}$

Probability: $p(E) = \frac{|E|}{|S|} = \frac{2}{6}$

Two unbiased dice

Sample Space: 36 possible outcomes

$$S = \{(1,1), (1,2), (1,3), \ldots, (6,6)\}$$

First die

Second die

Ordered pair

What is the probability that
two dice bring (1,1)?

Event Space: $E = \{(1,1)\}$

Sample Space: $S = \{(1,1), (1,2), (1,3), \ldots, (6,6)\}$

Probability: $p(E) = \frac{|E|}{|S|} = \frac{1}{36}$

What is the probability that
two dice bring same numbers?

Event Space: $E = \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}$

Sample Space: $S = \{(1,1),(1,2),(1,3),\ldots,(6,6)\}$

Probability: $p(E) = \dfrac{|E|}{|S|} = \dfrac{6}{36}$

Game with unordered numbers

Game authority selects
a set of 6 winning numbers out of 40

Number choices: 1,2,3,...,40
i.e. winning numbers: 4,7,16,25,33,39

Player picks a set of 6 numbers
(order is irrelevant)

i.e. player numbers: 8,13,16,23,33,40

What is the probability that a player wins?

Winning event:

$E = \{\{4,7,16,25,33,39\}\} \qquad |E| = 1$

a single set with the 6 winning numbers

Sample space:

$S = \{\text{all subsets with 6 numbers out of 40}\}$

$= \{\{1,2,3,4,5,6\},\{1,2,3,4,5,7\},\{1,2,3,4,5,8\},\ldots\}$

$$|S| = \binom{40}{6} = 3{,}838{,}380$$

Probability that player wins:

$$P(E) = \frac{|E|}{|S|} = \frac{1}{\binom{40}{6}} = \frac{1}{3{,}838{,}380}$$

A card game

Deck has 52 cards

13 kinds of cards (2,3,4,5,6,7,8,9,10,a,k,q,j),
each kind has 4 suits (h,d,c,s)

Player is given hand with 4 cards

What is the probability that the cards of the player are all of the same kind?

Event: $E = \{\{2_h, 2_d, 2_c, 2_s\}, \{3_h, 3_d, 3_c, 3_s\}, \ldots, \{j_h, j_d, j_c, j_s\}\}$

$|E| = 13$ each set of 4 cards is of same kind

Sample Space:

$$S = \{\text{all possible sets of 4 cards out of 52}\}$$
$$= \{\{2_h, 2_d, 2_c, 2_s\}, \{2_h, 2_d, 2_c, 3_h\}, \{2_h, 2_d, 2_c, 3_d\}, \ldots\}$$

$$|S| = \binom{52}{4} = \frac{52!}{4!\,48!} = \frac{52 \cdot 51 \cdot 50 \cdot 49}{4 \cdot 3 \cdot 2} = 270{,}725$$

Probability that hand has 4 same kind cards:

$$P(E) = \frac{|E|}{|S|} = \frac{13}{\binom{52}{4}} = \frac{13}{270{,}725}$$

Game with ordered numbers

Game authority selects from a bin 5 balls in some order labeled with numbers 1…50

Number choices: 1,2,3,…,50
i.e. winning numbers: 37,4,16,33,9

Player picks a set of 5 numbers (order is important)

i.e. player numbers: 40,16,13,25,33

What is the probability that a player wins?

Sampling without replacement:
After a ball is selected
it is not returned to bin

Sample space size: 5-permutations of 50 balls

$$|S| = P(50,5) = \frac{50!}{(50-5)!} = \frac{50}{45!} = 50 \cdot 49 \cdot 48 \cdot 47 \cdot 46 = 245{,}251{,}200$$

Probability of success: $P(E) = \dfrac{|E|}{|S|} = \dfrac{1}{245{,}251{,}200}$

Sampling with replacement:
After a ball is selected
it is returned to bin

Sample space size: 5-permutations of 50 balls
with repetition

$$|S| = 50^5 = 312{,}500{,}000$$

Probability of success: $P(E) = \dfrac{|E|}{|S|} = \dfrac{1}{312{,}500{,}000}$

Probability of Inverse: $p(\overline{E}) = 1 - p(E)$

Proof: $\overline{E} = S - E$

$$p(\overline{E}) = \frac{|S-E|}{|S|} = \frac{|S|-|E|}{|S|} = 1 - \frac{|E|}{|S|} = 1 - p(E)$$

End of Proof

Example: What is the probability that
a binary string of 8 bits contains
at least one 0?

$E = \{01111111, 10111111, \ldots, 00111111, \ldots, 00000000\}$

$\overline{E} = \{11111111\}$

$$p(E) = 1 - p(\overline{E}) = 1 - \frac{|\overline{E}|}{|S|} = 1 - \frac{1}{2^8}$$

Probability of Union: $E_1, E_2 \subseteq S$

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$$

Proof: $|E_1 \cup E_2| = |E_1| + |E_2| - |E_1 \cap E_2|$

$$p(E_1 \cup E_2) = \frac{|E_1 \cup E_2|}{|S|} = \frac{|E_1| + |E_1| - |E_1 \cap E_2|}{|S|}$$

$$= \frac{|E_1|}{|S|} + \frac{|E_2|}{|S|} - \frac{|E_1 \cap E_2|}{|S|}$$

$$= p(E_1) + p(E_2) - p(E_1 \cap E_2)$$

End of Proof

Example: What is the probability that a binary string of 8 bits starts with 0 or ends with 11?

Strings that start with 0:

$E_1 = \{00000000, 00000001, \ldots, 01111111\}$

$|E_1| = 2^7$ (all binary strings with 7 bits 0xxxxxxx)

Strings that end with 11:

$E_2 = \{00000011, 00000111, \ldots, 11111111\}$

$|E_2| = 2^6$ (all binary strings with 6 bits xxxxxx11)

Strings that start with 0 and end with 11:

$E_1 \cap E_2 = \{000000011, 00000111, \ldots, 01111111\}$

$|E_1 \cap E_2| = 2^5$ (all binary strings with 5 bits 0xxxxx11)

Strings that start with 0 or end with 11:

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$$

$$= \frac{|E_1|}{|S|} + \frac{|E_2|}{|S|} - \frac{|E_1 \cap E_2|}{|S|}$$

$$= \frac{2^7}{2^8} + \frac{2^6}{2^8} - \frac{2^5}{2^8} = \frac{1}{2} + \frac{1}{4} - \frac{1}{8} = \frac{5}{8}$$

Probability Theory

Sample space: $S = \{x_1, x_2, \ldots, x_n\}$

Probability distribution function p:

$$0 \leq p(x_i) \leq 1$$

$$\sum_{x=1}^{n} p(x_i) = 1$$

Notice that it can be: $p(x_i) \neq p(x_j)$

Example: Biased Coin

Heads (H) with probability 2/3

Tails (T) with probability 1/3

Sample space: $S = \{H, T\}$

$p(H) = \frac{2}{3}$ $\quad p(T) = \frac{1}{3}$ $\quad p(H) + p(T) = \frac{2}{3} + \frac{1}{3} = 1$

Uniform probability distribution:

$$p(x_i) = \frac{1}{n}$$

Sample space: $S = \{x_1, x_2, \ldots, x_n\}$

Example: Unbiased Coin

Heads (H) or Tails (T) with probability 1/2

$S = \{H, T\}$ $\quad p(H) = \frac{1}{2}$ $\quad p(T) = \frac{1}{2}$

Probability of event E:

$$E = \{x_1, x_2, \ldots, x_k\} \subseteq S$$

$$p(E) = \sum_{i=1}^{k} p(x_i)$$

For uniform probability distribution: $p(E) = \frac{|E|}{|S|}$

Example: Biased die $S = \{1,2,3,4,5,6\}$

$p(1) = p(2) = p(3) = p(4) = p(5) = \frac{1}{7}$ $\quad p(6) = \frac{2}{7}$

What is the probability that the die outcome is 2 or 6? $E = \{2,6\}$

$$p(E) = p(2) + p(6) = \frac{1}{7} + \frac{2}{7} = \frac{3}{7}$$

Combinations of Events:

Complement: $p(\overline{E}) = 1 - p(E)$

Union: $p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$

Union of disjoint events: $p\left(\bigcup_i E_i\right) = \sum_i p(E_i)$

Conditional Probability

Three tosses of an unbiased coin

Condition: first coin is Tails

Question: What is the probability that there is an odd number of Tails, given that first coin is Tails?

Sample space:

$S = \{TTT, TTH, THT, THH, HTT, HTH, HHT, HHH\}$

Restricted sample space given condition:

$F = \{TTT, TTH, THT, THH\}$

first coin is Tails

Event without condition:

$E = \{TTT, THH, HTH, HHT\}$

Odd number of Tails

Event with condition:

$E_F = E \cap F = \{TTT, THH\}$

first coin is Tails

$F = \{TTT, TTH, THT, THH\}$

$E_F = E \cap F = \{TTT, THH\}$

Given condition,
the sample space changes to F

$$p(E_F) = \frac{|E \cap F|}{|F|} = \frac{|E \cap F| / |S|}{|F| / |S|} = \frac{p(E \cap F)}{p(F)} = \frac{2/8}{4/8} = 0.5$$

(the coin is unbiased)

Notation of event with condition:

$$E_F = E \mid F$$

event E given F

$$p(E_F) = p(E \mid F) = \frac{p(E \cap F)}{p(F)}$$

Conditional probability definition:

(for arbitrary probability distribution)

Given sample space S with
events E and F (where $p(F) > 0$)
the conditional probability of E given F is:

$$p(E \mid F) = \frac{p(E \cap F)}{p(F)}$$

Example: What is probability that a family of two children has two boys given that one child is a boy

Assume equal probability to have boy or girl

Sample space: $S = \{BB, BG, GB, GG\}$

Condition: $F = \{BB, BG, GB\}$
one child is a boy

Event: $E = \{BB\}$
both children are boys

Conditional probability of event:

$$p(E \mid F) = \frac{p(E \cap F)}{p(F)} = \frac{p(\{BB\})}{p(\{BB, BG, GB\})} = \frac{1/4}{3/4} = \frac{1}{3}$$

Independent Events

Events E_1 and E_2 are independent iff:

$$p(E_1 \cap E_2) = p(E_1)\,p(E_2)$$

Equivalent definition (if $p(E_2) > 0$):

$$p(E_1 \mid E_2) = p(E_1)$$

Example: 4 bit uniformly random strings
E_1: a string begins with 1
E_2: a string contains even 1

$$E_1 = \{1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111\}$$
$$E_2 = \{0000, 0011, 0101, 0110, 1001, 1010, 1100, 1111\}$$
$$E_1 \cap E_2 = \{1111, 1100, 1010, 1001\}$$

$|E_1| = |E_2| = 8$ $\qquad p(E_1) = p(E_2) = \frac{8}{16} = \frac{1}{2}$

$|E_1 \cap E_2| = 4$ $\qquad \boxed{p(E_1 \cap E_2) = \frac{4}{16} = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = p(E_1)\,p(E_2)}$

Events E_1 and E_2 are independent

Bernoulli trial: Experiment with two outcomes: success or failure

Success probability: p

Failure probability: $q = 1 - p$

Example: Biased Coin


Success = Heads
$$p = p(H) = \frac{2}{3}$$

Failure = Tails
$$q = p(T) = \frac{1}{3}$$

Independent Bernoulli trials:

the outcomes of successive Bernoulli trials do not depend on each other

Example: Successive coin tosses

Throw the biased coin 5 times

What is the probability to have 3 heads?

Heads probability: $p = \frac{2}{3}$ (success)

Tails probability: $q = \frac{1}{3}$ (failure)

HHHTT

HTHHT

HTHTH

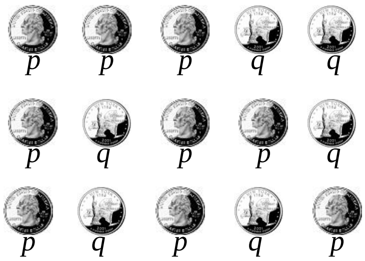
THHTH

⋮

Total numbers of ways to arrange in sequence 5 coins with 3 heads: $\binom{5}{3}$

Probability that any particular sequence has 3 heads and 2 tails is specified positions: $p^3 q^2$

For example:



HHHTT $pppqq = p^3 q^2$

HTHHT $pqppq = p^3 q^2$

HTHTH $pqpqp = p^3 q^2$

Probability of having 3 heads:

$$p^3q^2 + p^3q^2 + \cdots + p^3q^2 = \binom{5}{3} p^3q^2$$

1st sequence success (3 heads)

2nd sequence success (3 heads)

$\binom{5}{3}$ st sequence success (3 heads)

Throw the biased coin 5 times

Probability to have exactly 3 heads:

$$\binom{5}{3} p^3q^2 = \frac{5!}{3!2!}\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^2 \approx 0.0086$$

Probability to have 3 heads and 2 tails in specified sequence positions

All possible ways to arrange in sequence 5 coins with 3 heads

Theorem: Probability to have k successes in n independent Bernoulli trials:

$$\binom{n}{k} p^k q^{n-k}$$

Also known as binomial probability distribution:

$$b(k; n, p) = \binom{n}{k} p^k q^{n-k}$$

Proof:

$$\binom{n}{k} \underbrace{p^k q^{n-k}}$$

Total number of sequences with k successes and $n-k$ failures

Probability that a sequence has k successes and $n-k$ failures in specified positions

Example: SFSFFS…SSF

End of Proof

Example: Random uniform binary strings
probability for 0 bit is 0.9
probability for 1 bit is 0.1

What is probability of 8 bit 0s out of 10 bits?

i.e. 0100001000

$$p = 0.9 \qquad q = 0.1 \qquad k = 8 \qquad n = 10$$

$$b(k; n, p) = \binom{n}{k} p^k q^{n-k} = \binom{10}{8} (0.9)^8 (0.1)^2 = 0.1937102445$$

Birthday Problem

Birthday collision: two people have birthday in same day

Problem:

How many people should be in a room so that the probability of birthday collision is at least ½?

Assumption: equal probability to be born in any day

366 days in a year

If the number of people is 367 or more then birthday collision is guaranteed by pigeonhole principle

Assume that we have $n \leq 366$ people

We will compute

p_n :probability that n people have all different birthdays

It will helps us to get

$1 - p_n$:probability that there is a birthday collision among n people

Sample space: Cartesian product

$$S = \{1,2,\ldots,366\} \times \{1,2,\ldots,366\} \times \cdots \times \{1,2,\ldots,366\}$$

1st person's Birthday choices

2nd person's Birthday choices

nth person's Birthday choices

$$S = \{(1,1,\ldots,1),(2,1,\ldots,1),\ldots(366,366,\ldots,366)\}$$

Sample space size: $|S| = 366 \cdot 366 \cdots 366 = 366^n$

Event set: each person's birthday is different

$$E = \{(1,2,\ldots,366),(366,1,\ldots,365),\ldots,(366,365,\ldots,1)\}$$

1st person's birthday

2nd person's birthday

nth person's birthday

#choices #choices #choices

Sample size:

$$|E| = P(366,n) = \frac{366!}{(366-n)!} = 366 \cdot 365 \cdot 364 \cdots (366-n+1)$$

Probability of no birthday collision

$$p_n = \frac{|E|}{|S|} = \frac{366 \cdot 365 \cdot 364 \cdots (366-n+1)}{366^n}$$

Probability of birthday collision: $1 - p_n$

$n = 22$ $\quad 1 - p_n \approx 0.475$

$n = 23$ $\quad 1 - p_n \approx 0.506$

Probability of birthday collision: $1 - p_n$

$n = 23$ $\quad 1 - p_n \approx 0.506$

Therefore: $n \geq 23$ people have probability at least ½ of birthday collision

The birthday problem analysis can be used to determine appropriate hash table sizes that minimize collisions

Hash function collision: $h(k_1) = h(k_2)$

Randomized algorithms:
algorithms with randomized choices (Example: quicksort)

Las Vegas algorithms:
randomized algorithms whose output is always correct (i.e. quicksort)

Monte Carlo algorithms:
randomized algorithms whose output is correct with some probability (may produce wrong output)

A Monte Carlo algorithm

```
Primality_Test( n,k ) {
   for(i = 1 to k) {
      b ← random_number(1,...,n)
      if (Miller_Test(n,b ) == failure)
         return(false)   // n is not prime
   }
   return(true) // most likely n is prime
}
```


Miller_Test(n,b) {
$n-1 = 2^s t$
$s,t \geq 0, \quad s \leq \log n, \quad t$ is odd
for ($j = 0$ to $s-1$) {
if ($b^t \equiv 1 (\bmod n)$ or $b^{2^j t} \equiv -1 (\bmod n)$)
return(success)
}
return(failure)
}

A prime number n passes the Miller test for every $1 \leq b \leq n$

A composite number n passes the Miller test in range $1 \leq b \leq n$ for fewer than $\frac{n}{4}$ numbers

false positive with probability: $\frac{1}{4}$

If the primality test algorithm returns false then the number is not prime for sure

If the algorithm returns true then the answer is correct (number is prime) with high probability:

$$1 - \left(\frac{1}{4}\right)^k = 1 - \frac{1}{n} \approx 1$$

for $k = \log_4 n$

Bayes' Theorem

$p(E) \neq 0 \qquad p(F) \neq 0$

$$p(F \mid E) = \frac{p(E \mid F)p(F)}{p(E \mid F)p(F) + p(E \mid \overline{F})p(\overline{F})}$$

Applications: Machine Learning
Spam Filters

Bayes' Theorem Proof:

$$\left.\begin{aligned} p(F \mid E) &= \frac{p(E \cap F)}{p(E)} \\ p(E \mid F) &= \frac{p(E \cap F)}{p(F)} \end{aligned}\right\} \Rightarrow \quad \begin{aligned} p(E \cap F) &= p(F \mid E)p(E) \\ p(E \cap F) &= p(E \mid F)p(F) \end{aligned}$$

$$\Downarrow$$

$$p(F \mid E)p(E) = p(E \mid F)p(F)$$

$$\Downarrow$$

$$p(F \mid E) = \frac{p(E \mid F)p(F)}{p(E)}$$

$$E=(E\cap F)\cup(E\cap\overline{F})$$

$$(E\cap F)\cap(E\cap\overline{F})=\varnothing$$

$$\Rightarrow p(E)=p(E\cap F)+p(E\cap\overline{F})$$

$$\underbrace{\begin{gathered}p(E\cap F)=p(E\mid F)p(F)\\ p(E\cap\overline{F})=p(E\mid\overline{F})p(\overline{F})\end{gathered}}$$

$$\Downarrow$$

$$p(E)=p(E\mid F)p(F)+p(E\mid\overline{F})p(\overline{F})$$

$$p(F\mid E)=\frac{p(E\mid F)p(F)}{p(E)}$$

$$p(E)=p(E\mid F)p(F)+p(E\mid\overline{F})p(\overline{F})$$

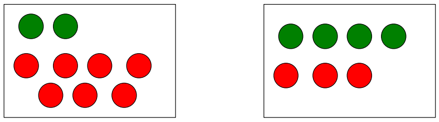
$$\Downarrow$$

$$p(F\mid E)=\frac{p(E\mid F)p(F)}{p(E\mid F)p(F)+p(E\mid\overline{F})p(\overline{F})}$$

End of Proof

Example: Select random box
then select random ball in box

Box 1 Box 2



Question:
If a red ball is selected,
what is the probability it was taken from box 1?

E: select red ball

$\overline{E}$: select green ball

F: select box 1

$\overline{F}$: select box 2

Question probability: $P(F\mid E)=?$

Question:
If a red ball is selected,
what is the probability it was taken from box 1?

Bayes' Theorem:

$$p(F \mid E) = \frac{p(E \mid F)\,p(F)}{p(E \mid F)\,p(F) + p(E \mid \overline{F})\,p(\overline{F})}$$

We only need to compute:

$p(F) \qquad p(\overline{F}) \qquad p(E \mid F) \qquad p(E \mid \overline{F})$

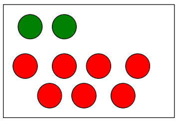
$E:$ select red ball

$\overline{E}:$ select green ball

$F:$ select box 1

$\overline{F}:$ select box 2

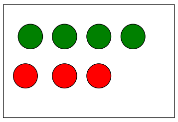
Box 1



$p(F) = 1/2 = 0.5$

Probability to select box 1

Box 2



$p(\overline{F}) = 1/2 = 0.5$

Probability to select box 2

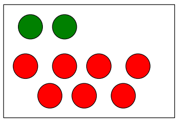
$E:$ select red ball

$\overline{E}:$ select green ball

$F:$ select box 1

$\overline{F}:$ select box 2

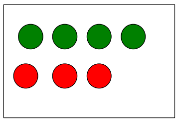
Box 1



$p(E \mid F) = 7/9 = 0.777...$

Probability to select red ball from box 1

Box 2



$p(E \mid \overline{F}) = 3/7 = 0.428....$

Probability to select red ball from box 2

$$p(F \mid E) = \frac{p(E \mid F)\,p(F)}{p(E \mid F)\,p(F) + p(E \mid \overline{F})\,p(\overline{F})}$$

$p(F) = 1/2 = 0.5 \qquad p(\overline{F}) = 1/2 = 0.5$

$p(E \mid F) = 7/9 = 0.777... \qquad p(E \mid \overline{F}) = 3/7 = 0.428....$

$$p(F \mid E) = \frac{0.777 \times 0.5}{0.777 \times 0.5 + 0.428 \times 0.5} = \frac{0.777}{0.777 + 0.428} = 0.644$$

Final result

What if we had more boxes?

Generalized Bayes' Theorem:

$$p(F_j \mid E) = \frac{p(E \mid F_j)\,p(F_j)}{\sum_{i=1}^{n} p(E \mid F_i)\,p(F_i)}$$

Sample space $S = F_1 \cup F_2 \cup \cdots \cup F_n$

mutually exclusive events

Spam Filters

Training set: Spam (bad) emails B

Good emails G

A user classifies each email
in training set as good or bad

Find words that occur in B and G

$n_B(w)$

number of spam emails
that contain word w

$n_G(w)$

number of good emails
that contain word w

$$p(w) = \frac{n_B(w)}{|B|}$$

Probability that
a spam email
contains w

$$q(w) = \frac{n_G(w)}{|G|}$$

Probability that
a good email
contains w

A new email X arrives

S: Event that X is spam

E: Event that X contains word w

What is the probability that X is spam
given that it contains word w?

$$P(S \mid E) = ?$$

Reject if this probability is at least 0.9

$$p(S \mid E) = \frac{p(E \mid S)\,p(S)}{p(E \mid S)\,p(S) + p(E \mid \overline{S})\,p(\overline{S})}$$

We only need to compute:

$p(S)$	$p(\overline{S})$	$p(E \mid S)$	$p(E \mid \overline{S})$
0.5	0.5	$p(w) = \frac{n_B(w)}{\lvert B \rvert}$	$q(w) = \frac{n_G(w)}{\lvert G \rvert}$

simplified assumption

Computed from training set

Example:

Training set for word "Rolex":

"Rolex" occurs in 250 of 2000 spam emails

"Rolex" occurs in 5 of 1000 good emails

If new email contains word "Rolex"
what is the probability that it is a spam?

"Rolex" occurs in 250 of 2000 spam emails

$$n_B(Rolex) = 250$$

$$p(Rolex) = \frac{n_B(Rolex)}{\lvert B \rvert} = \frac{250}{2000} = 0.125$$

"Rolex" occurs in 5 of 1000 good emails

$$n_G(Rolex) = 5$$

$$q(Rolex) = \frac{n_G(Rolex)}{\lvert G \rvert} = \frac{5}{1000} = 0.005$$

If new email X contains word "Rolex"
what is the probability that it is a spam?

$S:$ Event that X is spam

$E:$ Event that X contains word "Rolex"

$$P(S \mid E) = ?$$

$$p(S \mid E) = \frac{p(E \mid S)\,p(S)}{p(E \mid S)\,p(S) + p(E \mid \overline{S})\,p(\overline{S})}$$

We only need to compute:

$p(S)$ — 0.5

$p(\overline{S})$ — 0.5

simplified assumption

$p(E \mid S)$ — $p(Rolex) = 0.125$

$p(E \mid \overline{S})$ — $q(Rolex) = 0.005$

Computed from training set

$$p(S \mid E) = \frac{0.125 \cdot 0.5}{0.125 \cdot 0.5 + 0.005 \cdot 0.5} = \frac{0.125}{0.13} = 0.961\ldots$$

New email is considered to be spam because:

$p(S \mid E) = 0.961 > 0.9$ spam threshold

Better spam filters use two words:

$$p(S \mid E_1 \cap E_2) = \frac{p(E_1 \mid S)\,p(E_2 \mid S)}{p(E_1 \mid S)\,p(E_2 \mid S) + p(E_1 \mid \overline{S})\,p(E_2 \mid \overline{S})}$$

Assumption: E_1 and E_2 are independent

the two words appear independent of each other