Nonlinear Control Systems Class Notes

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Textbook and Slides Information

These undergraduate senior year elective Nonlinear Control Control Systems course slides are based mainly on the textbook:

J.J.E. Slotine and W. Li, Applied Nonlinear Control, Prentice Hall, 1991.

These are intended for the classroom use of the lecturer only; not for sale under any circumstances.

INTRODUCTION

Definition 1.1

A system is a group of interacting or interrelated elements that act according to a set of rules to form a unified whole.

A system can be conceptualized as a mechanism that converts its inputs into corresponding outputs.

Definition 1.2

A system T is said to be linear if
(1) input u produces output $y \rightarrow$ input ku produces output ky, for every input u and every scalar k ; AND
(2) input u_1 and u_2 produce outputs y_1 and y_2 respectively $\rightarrow$ input $u_1 + u_2$ produces output $y_1 + y_2$, for every u_1 and u_2.

Example 1.1

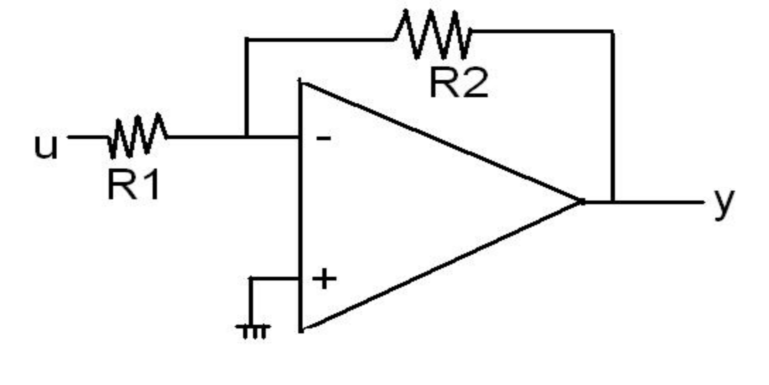


Figure 1: A typical system with input u and output y

Example 1.2

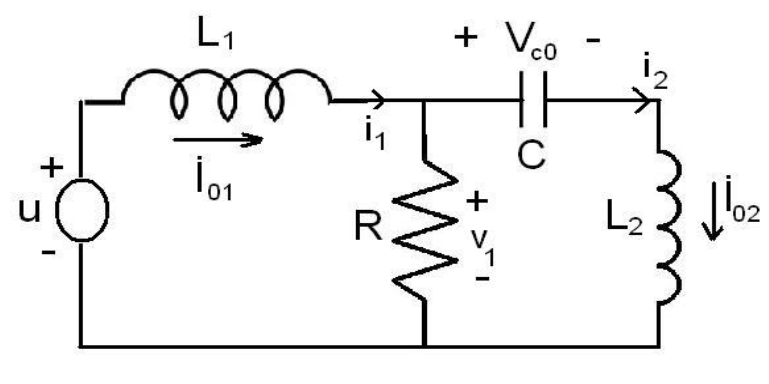


Figure 2: A typical system with multi-inputs $\begin{bmatrix} u \\ \dot{i}_{01} \\ \dot{i}_{02} \\ v_{c0} \end{bmatrix}$ and multi-outputs $\begin{bmatrix} \dot{i}_1 \\ \dot{i}_2 \\ v_1 \end{bmatrix}$

System Outputs

The outputs of a system depend on the specific nature of the system and its purpose.The outputs can be physical, informational, or a combination of both. Here are some examples across different domains:

Physical Systems:

A manufacturing system: Outputs could be finished products or components.

An HVAC system: Outputs could be heated or cooled air.

A transportation system: Outputs could be vehicles reaching their destinations.

Information Systems:

Computer software: Outputs could be processed data, reports, or visualizations.

Communication systems: Outputs could be transmitted messages or data packets.

Search engines: Outputs could be search results or relevant information.

Biological Systems:
Human body: Outputs could be movement, speech, or biochemical signals.
Ecosystems: Outputs could be the growth of plants, population dynamics, or nutrient cycles.
Decision-Making Systems:
Artificial intelligence systems: Outputs could be predictions, recommendations, or decisions.
Financial systems: Outputs could be investment strategies, risk assessments, or financial reports.

Inputs in Macroeconomics

In macroeconomics, the following are some of the **inputs**:

Labor: The human effort, skills, and knowledge employed in the production of goods and services.

Capital: The physical capital goods, such as machinery, buildings, and equipment, used in production.

Land: Natural resources, such as land, water, minerals, and other raw materials, used in production.

Entrepreneurship: The ability to organize and coordinate the other factors of production to create goods and services.

Interest rate: Input or output

In macroeconomics, the interest rate is generally considered an output rather than an input. The interest rate is the price of borrowing or lending money and is determined by the interaction of supply and demand in the financial markets.

The interest rate is influenced by various factors in the economy, including monetary policy set by the central bank, inflation expectations, investment demand, and the overall level of economic activity. These factors affect the demand for and supply of loanable funds in the economy, which in turn determines the prevailing interest rate.

The course covers the study of **dynamic systems**, which are characterized by differential or difference equations.
A dynamic system is a system that changes over time, and its behavior can be influenced by both internal and external factors. It is a system where the output at any given time depends not only on the current input but also on the past inputs and the system's internal state.

Recall that:
A differential equation is an equation that relates an unknown function to its derivatives.
Solutions to differential equations are functions, whereas solutions to algebraic equations are numbers.

Definition 1.3

A linear ordinary differential equation of order n, in the dependent variable y and the independent variable x, is an equation that is in, or can be expressed in, the form

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_{n-2}(x)\frac{d^2 y}{dx^2} + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = b(x)$$

where a_0 is not identically zero.

Functions of x: $x^2, \sin(x), x+1, 5, 0$

Not functions of x: $y, 3y, y^2, \frac{dy}{dx}, (\frac{dy}{dx})^2, x+y, xy$

Definition 1.4

A nonlinear ordinary differential equation is an ordinary differential equation that is not linear.

Example 1.3

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0 \ldots \text{ Linear} \tag{1}$$

$$\frac{d^4y}{dx^4} + x^2\frac{d^3y}{dx^3} + x^3\frac{dy}{dx} = xe^x \ldots \text{ Linear} \tag{2}$$

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y^2 = 0 \ldots \text{ Nonlinear} \tag{3}$$

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6yy = 0 \ldots \text{ Nonlinear}$$

$$\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 + 6y = 0 \ldots \text{ Nonlinear} \tag{4}$$

$$\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^2\frac{dy}{dx} + 6y = 0 \ldots \text{ Nonlinear}$$

$$\frac{d^2y}{dx^2} + 5y\frac{dy}{dx} + 6y = 0 \ldots \text{ Nonlinear} \tag{5}$$

$$\frac{d^2y}{dx^2} + 5y\frac{dy}{dx} + 6y = 0 \ldots \text{Nonlinear}$$

Definition 1.5

The normal form (aka state-space form) of a linear system of n differential equations in n unknown functions $x_1, x_2, \dots, x_n$, is in the following form:

$$\left.\begin{array}{c} \frac{dx_1}{dt} = f_1(x_1, x_2, \dots, x_n, t) \\ \frac{dx_2}{dt} = f_2(x_1, x_2, \dots, x_n, t) \\ \vdots \\ \frac{dx_n}{dt} = f_n(x_1, x_2, \dots, x_n, t) \end{array}\right\} \tag{6}$$

A compact representation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$$

where

$$\mathbf{x} \triangleq \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{f}(\mathbf{x}, t) \triangleq \begin{bmatrix} f_1(\mathbf{x}, t) \\ f_2(\mathbf{x}, t) \\ \vdots \\ f_n(\mathbf{x}, t) \end{bmatrix}$$

Definition 1.6

The normal form (aka state-space form), in the general case of a linear system of n differential equations in n unknown functions $x_1, x_2, \ldots, x_n$, is in the following form:

$$\left.\begin{aligned}
\frac{dx_1}{dt} &= a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + b_1(t) \\
\frac{dx_2}{dt} &= a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + b_2(t) \\
&\ \vdots \\
\frac{dx_n}{dt} &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + b_n(t)
\end{aligned}\right\} \tag{7}$$

A compact representation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \ldots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \ldots & a_{2n}(t) \\ \vdots & \vdots & \ldots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \ldots & a_{nn}(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_n(t) \end{bmatrix}$$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}$$

A single n-th order linear differential equation can be converted into this form. Consider

$$\frac{d^n x}{dt^n} + a_1(t)\frac{d^{n-1}x}{dt^{n-1}} + a_2(t)\frac{d^{n-2}x}{dt^{n-2}} + \cdots + a_{n-2}(t)\frac{d^2x}{dt^2} + a_{n-1}(t)\frac{dx}{dt} + a_n(t)x = F(t) \quad (8)$$

$$\frac{d^n x}{dt^n} + a_1(t)\underbrace{\frac{d^{n-1}x}{dt^{n-1}}}_{x_n} + a_2(t)\underbrace{\frac{d^{n-2}x}{dt^{n-2}}}_{x_{n-1}} + \cdots + a_{n-2}(t)\underbrace{\frac{d^2x}{dt^2}}_{x_3} + a_{n-1}(t)\underbrace{\frac{dx}{dt}}_{x_2} + a_n(t)\underbrace{x}_{x_1} = F(t)$$

$$\frac{d^n x}{dt^n} + a_1(t)\underbrace{\frac{d^{n-1}x}{dt^{n-1}}}_{x_n} + a_2(t)\underbrace{\frac{d^{n-2}x}{dt^{n-2}}}_{x_{n-1}} + \cdots + a_{n-2}(t)\underbrace{\frac{d^2x}{dt^2}}_{x_3}$$

$$+a_{n-1}(t)\underbrace{\frac{dx}{dt}}_{x_2} + a_n(t)\underbrace{x}_{x_1} = F(t)$$

Using these definitions, the normal form (i.e., state space form) equivalent of (8) is

$$\left.\begin{array}{rcl} \frac{dx_1}{dt} & = & x_2 \\ \frac{dx_2}{dt} & = & x_3 \\ \vdots & & \vdots \\ \frac{dx_{n-1}}{dt} & = & x_n \\ \frac{dx_n}{dt} & = & -a_n(t)x_1 - a_{n-1}(t)x_2 - \cdots - a_1(t)x_n + F(t) \end{array}\right\}$$

Consider the first order differential equation

$$\frac{dy}{dx} + \frac{2x+1}{x} y = e^{-2x}, \quad y(1) = 2. \tag{9}$$

We want to find a solution in the interval $[1,5]$. Form two m-files: Let their names be mymain.m and myequation.m.
mymain.m:

```
[t,x]=ode23('myequation',[1,5],2);
plot(t,x,'o')
```

myequation.m:

```
function ydot=myequation(x,y)
ydot=-((2*x+1)/x)*y +exp(-2*x);
```

The graphics is concatenation of o characters due to the 'o' option in the plot command.

Save mymain.m and myequation.m files in the work folder of MATLAB.

In the workplace of MATLAB, type mymain and press enter key. The graphics obtained is depicted below:

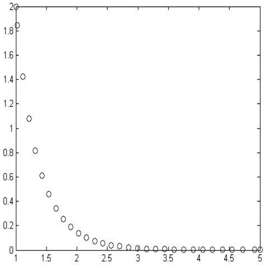


Figure 3: Numerical solution of the 1st order differential equation

The differential equation (9) is linear and its analytical solution is $y(x) = \frac{x}{2}e^{-2x} + \frac{14.27}{x}e^{-2x}$. For the purpose of comparison with the numerical solution we can plot this over the previous graphics by using the following codes in the workplace of MATLAB (Figure 4):

```
hold on
x=1:0.1:5
y=exp(-2*x).*x/2+14.27*exp(-2*x)./x
plot(x,y)
```

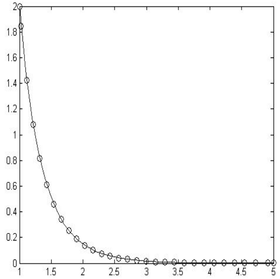


Figure 4: Analytical solution of the 1st order differential equation

MATLAB Operators .* and ./

In the previous slide we used a less frequently encountered the arithmetic operators .* and ./
The following example illustrates their functions:

```
>> a=[1 2 3];
>> b=[4 5 6];
>> a.*b

ans =

     4    10    18

>> a./b

ans =

    0.2500    0.4000    0.5000
```

Now let us modify the files mymain.m and myequation.m to solve the following second order differential equation in the interval $[0, 5]$

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 4y = \sin(t), \quad y(0) = 3; \ \dot{y}(0) = 9. \tag{10}$$

This can be written in the normal form as:

$$\left.\begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & -4x_1 - 5x_2 + \sin(t) \end{array}\right\} \ x_1(0) = 3; \ x_2(0) = 9. \tag{11}$$

Corresponding m-files are formed as shown below:
mymain.m:

```
[t,x]=ode23('myequation',[0,5],[3,9]);
plot(t,x(:,1),'o',t,x(:,2),'*')
```

myequation.m:

```
function xdot=myequation(t,x)
xdot=[x(2); -4*x(1)-5*x(2)+sin(t)];
```

The codes above yields the following graphics:

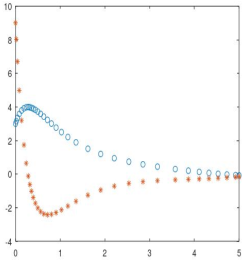


Figure 5: Numerical solution of the 2nd order differential equation

Example 1.4

A third order single differential equation

$$x^{(3)} + 3\ddot{x} + 6x = \sin(t)$$

can be written in the normal form as

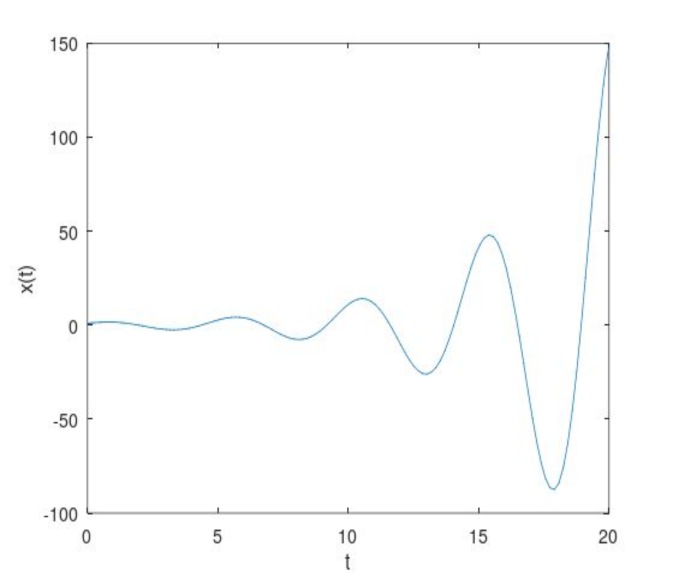
$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -6x_1 - 3x_3 + \sin(t) \end{aligned}$$

Continued from the previous page

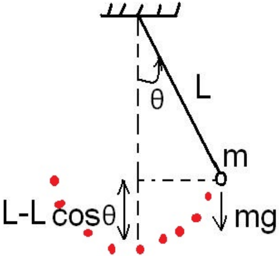
```
function myNormal
  [t y] = ode45(@myEquations, [0 20], [1; 1; 1]);

  plot(t, y(:,1))
  xlabel('t')
  ylabel('x(t)')
  return

  function xdot = myEquations(t,y)
    X = y(1);
    Y = y(2);
    Z = y(3);
    xdot = [Y;Z; -6*X-3*Z+sin(t)];
return
```

Continued from the previous page

Running the function myNormal.m the following graphics is generated:



The simple plane pendulum



Assumptions

Pendulum bob of mass *m* attached to a rigid rod with length *L* having negligible mass. Pendulum is confined to swing in a plane.

Modelling

Speed of the pendulum bob: $v = L\dot{\theta}$
Kinetic energy of the bob: $T = \frac{1}{2}mv^2 = \frac{1}{2}mL^2\dot{\theta}^2$
Potential energy: $V = mg(L - L\cos\theta)$
Total energy: $E = \frac{1}{2}mL^2\dot{\theta}^2 + mg(L - L\cos\theta)$
Use conservation of energy: $\frac{dE}{dt} = 0 \rightarrow mL^2\dot{\theta}\ddot{\theta} + mgL\dot{\theta}\sin\theta = 0$. This is pendulum's equation of motion (EOM) $\theta = n\pi,\ n \in \mathbb{Z}$ are solutions corresponding to the pendulum hanging straight down without swinging, or just balancing straight up. Let us investigate the other solutions. First, simplify the EOM:

$$\ddot{\theta} + \frac{g}{L}\sin\theta = 0$$

This differential equation is not easy to solve exactly, so we explore its properties in some other way.

$$\ddot{\theta} + \frac{g}{L}\sin\theta = 0$$

This differential equation is not easy to solve exactly, so we explore its properties in some other way. Note that, when θ is very small we use the approximation $\sin\theta \approx \theta$. This converts EOM to a linear differential equation:

$$\ddot{\theta} + \frac{g}{L}\theta = 0$$

This has a closed form solution $\theta(t) = A\sin\sqrt{\frac{g}{L}}t + B\cos\sqrt{\frac{g}{L}}t$.

Taylor's expansion of $\sin x$ about 0:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

At $x = 0.1$: $\quad 0.099833 = 0.1 - 1.6667 \times 10^{-04} + 8.3333 \times 10^{-08} - \cdots$
At $x = 0.05$: $\quad 0.049979 = 0.05 - 2.0833 \times 10^{-05} + 2.6042 \times 10^{-09} - \cdots$

When θ is not restricted to small angles we can still express the solution graphically. The appropriate graphics would be $\theta \leftrightarrow \dot{\theta}$ graphics. Throughout this course we shall see that dependent variable versus dependent variable graphics are more useful compared to dependent variable versus independent variable ones (i.e., $\theta \leftrightarrow \dot{\theta}$ is more useful compared to $t \leftrightarrow \theta$).

Let us employ the following identity:

$$\ddot{\theta} = \frac{d\dot{\theta}}{dt} = \frac{d\dot{\theta}}{d\theta}\frac{d\theta}{dt} = \dot{\theta}\frac{d\dot{\theta}}{d\theta}$$

Equation of motion becomes:

$$\dot{\theta}\frac{d\dot{\theta}}{d\theta} + \frac{g}{L}\sin\theta = 0$$

$$\rightarrow \dot{\theta}d\dot{\theta} + \frac{g}{L}\sin\theta d\theta = 0$$

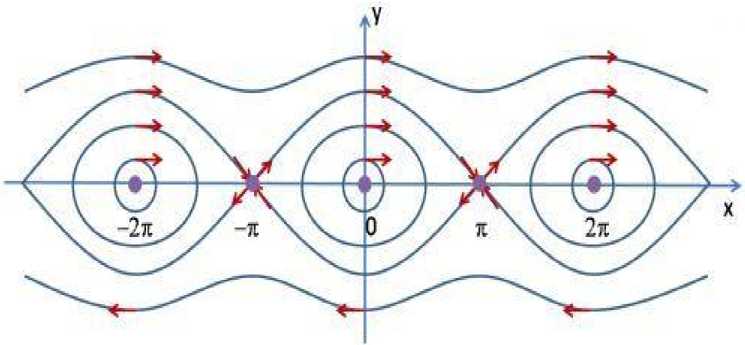
$$\rightarrow \int \dot{\theta}d\dot{\theta} + \int \frac{g}{L}\sin\theta d\theta = \int 0 d\theta$$

$$\rightarrow \frac{\dot{\theta}^2}{2} + \frac{g}{L}(-\cos\theta) = c_1$$

$$\rightarrow \frac{\dot{\theta}^2}{2} + \frac{g}{L}(-\cos\theta) = c_1$$

$$\dot{\theta}^2 - 2\frac{g}{L}\cos\theta = c \tag{12}$$

Set up a cartesian phase plane having θ and $\dot{\theta}$ as its axes, and plot the one parameter family of curves generated by (12) for different values of c. For this example, $(\theta, \dot{\theta})$ is called the state of the system.



The curves depicted in Figure 6 are known as the phase paths or trajectories corresponding to the EOM. Complete figure is is called the phase diagram or **phase portrait** of the system. Direction of the arrows is settled by observing that when $\dot{\theta}$ is positive θ must be increasing with time, and when $\dot{\theta}$ is negative θ must be decreasing with time. $(\theta, \dot{\theta}) = (0, 0)$ and $(\theta, \dot{\theta}) = (\pi, 0)$ are distinct equilibrium states, at each EP the trajectory is a single point. Any trajectory starting at EP stays there forever.

Digression

$$X' = (x_1', x_2', x_3') \to X = (x_1'+p_1, x_2'+p_2, x_3'+p_3)$$

$$X = (x_1, x_2, x_3) \to X' = (x_1-p_1, x_2-p_2, x_3-p_3)$$

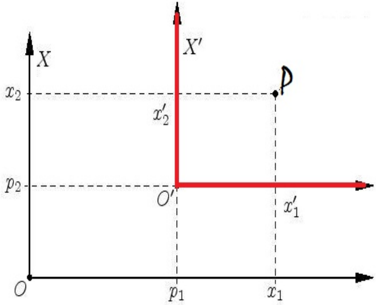


Figure 7: Effect of translated axes on the coordinates

Digression

Consider a single particle having position x and velocity $\dot{x}$. Let the LaGrangian function L be defined by

$$L \triangleq T - V$$

where T and V are particle's kinetic and potential energies respectively. More explicity, one may write it in the form

$$L = \frac{1}{2}m\dot{x}^2 - V(x)$$

The LaGrangian equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}$$

leads to

$$m\ddot{x} = -\frac{dV(x)}{dx}$$

Recognizing the righthand side term as force, it is Newton's formula $m\ddot{x} = F$.

EOD

The simple double pendulum

[R. Tedrake, Underactuated Robotics; learning, Planning and Control for Efficient and Agile Machines, Class notes, MIT, 2009.]

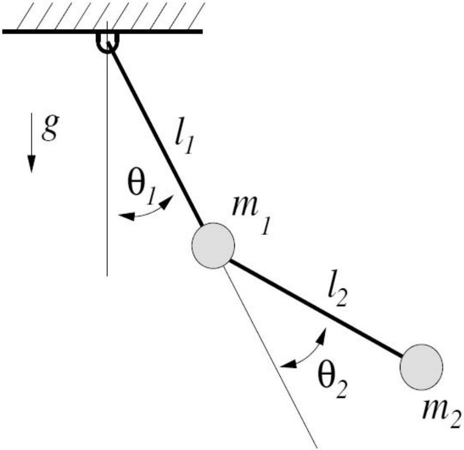


Figure 8: Simple double pendulum

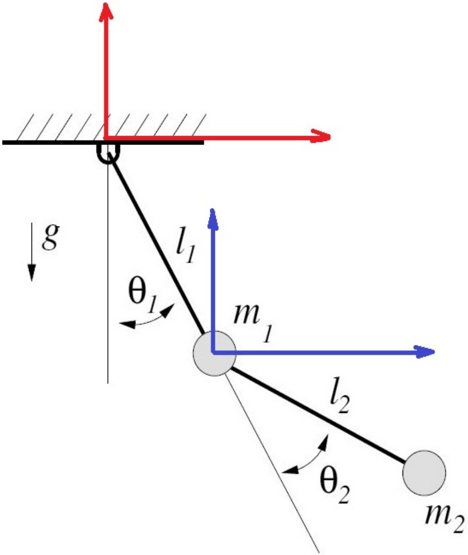


Figure 9: Simple double pendulum with coordinate axes

Consider the system in Figure 9 with torque actuation at both joints, and all of the mass concentrated in two points. Using $q = [\theta_1 \theta_2]^T =: [q_1 q_2]^T$, and x_1, x_2 to denote the locations of m_1, m_2 respectively, kinematics of this system are

$$x_1 = \begin{bmatrix} l_1 s_1 \\ -l_1 c_1 \end{bmatrix}, \; x_2 = x_1 + \begin{bmatrix} l_2 s_{1+2} \\ -l_2 c_{1+2} \end{bmatrix}$$

$$\dot{x}_1 = \begin{bmatrix} l_1 \dot{q}_1 c_1 \\ l_1 \dot{q}_1 s_1 \end{bmatrix}, \; \dot{x}_2 = \dot{x}_1 + \begin{bmatrix} l_2(\dot{q}_1 + \dot{q}_2) c_{1+2} \\ l_2(\dot{q}_1 + \dot{q}_2) s_{1+2} \end{bmatrix}$$

where s_1 is shorthand for $\sin q_1$, c_{1+2} is shorthand for $\cos(q_1 + q_2)$, and so on. From this, we can write the kinetic and potential energies(T and U respectively):

$$T = \frac{1}{2}\dot{x}_1^T m_1 \dot{x}_1 + \frac{1}{2}\dot{x}_2^T m_2 \dot{x}_2$$

$$= \frac{1}{2}(m_1 + m_2) l_1^2 \dot{q}_1^2 + \frac{1}{2} m_2 l_2^2 (\dot{q}_1 + \dot{q}_2)^2 + m_2 l_1 l_2 \dot{q}_1 (\dot{q}_1 + \dot{q}_2) c_2$$

$$U = m_1 g y_1 + m_2 g y_2 = -(m_1 + m_2) g l_1 c_1 - m_2 g l_2 c_{1+2}$$

Defining $L := T - U$, and Q_i as the generalized force for the joint q_i, the Lagrangian dynamic equations are:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i$$

Taking the partial derivatives $\frac{\partial T}{\partial q_i}, \frac{\partial T}{\partial \dot{q}_i}$ and $\frac{\partial U}{\partial q_i}$ ($\frac{\partial U}{\partial \dot{q}_i}$ terms are always zero), then $\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i}$, and plugging them into Lagrangian reveals the equation of motion:

$$(m_1 + m_2)l_1^2\ddot{q}_1 + m_2 l_2^2(\ddot{q}_1 + \ddot{q}_2) + m_2 l_1 l_2(2\ddot{q}_1 + \ddot{q}_2)c_2$$

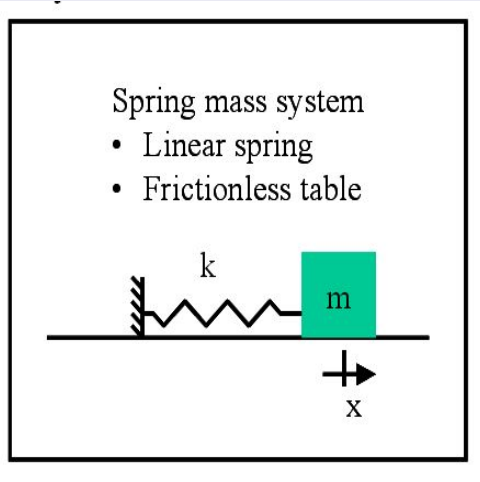
$$-m_2 l_1 l_2(2\dot{q}_1 + \dot{q}_2)\dot{q}_2 s_2 + (m_1 + m_2)l_1 g s_1 + m_2 g l_2 s_{1+2} = \tau_1$$

$$m_2 l_2^2(\ddot{q}_1 + \ddot{q}_2) + m_2 l_1 l_2 \ddot{q}_1 c_2 + m_2 l_1 l_2 \dot{q}_1^2 s_2 + m_2 g l_2 s_{1+2} = \tau_2$$

Simulation:
http://scienceworld.wolfram.com/physics/DoublePendulum.html

Spring-mass system [core.org.cn]

Example 2.1



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Lagrangian

$$L = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

Lagrange'e Equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

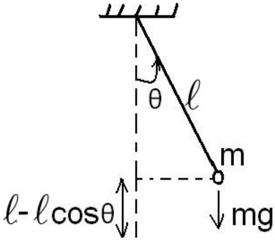
Do the derivatives

$$\frac{\partial L}{\partial \dot{q}_i} = m\dot{x},\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = m\ddot{x},\ \frac{\partial L}{\partial q_i} = -kx$$

Put it all together

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = m\ddot{x} + kx = 0$$

A background for the next example



Consider a pendulum of mass m and length ℓ with angular displacement θ from the vertical. Its kinetic and potential energies are

$$K = \frac{1}{2}m(\ell\dot{\theta})^2, \; U = mg\ell(1 - cos\theta)$$

Example 2.2

Consider a pendulum of mass m and length ℓ with angular displacement θ from the vertical. Its kinetic and potential energies are [engr.iupui.edu]:

$$K = \frac{1}{2}m(\ell\dot{\theta})^2, \; U = mg\ell(1 - cos\theta)$$

The Lagrangian is

$$L = K - U = \frac{1}{2}m(\ell\dot{\theta})^2 - mg\ell(1 - cos\theta)$$

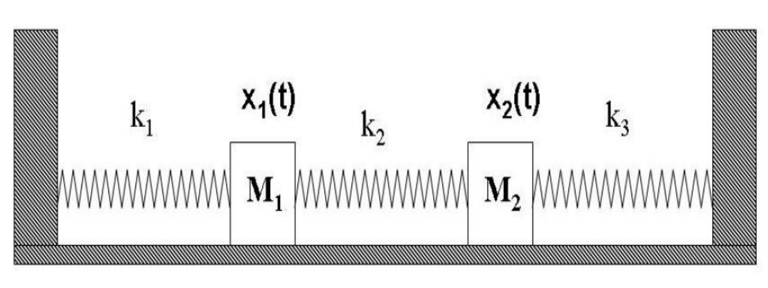
Thus

$$\frac{\partial L}{\partial \theta} = -mg\ell\sin\theta, \; \frac{\partial L}{\partial \dot{\theta}} = m\ell^2\dot{\theta}, \; \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = m\ell^2\ddot{\theta}$$

So, the EOM is

$$m\ell^2\ddot{\theta} + mg\ell\sin\theta = 0$$

Example 2.3



$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2$$

$$V = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2(x_2 - x_1)^2 + \frac{1}{2}k_3x_2^2$$

Apply Lagrange's equation to $L = T - V$:

Continued from the previous page

Apply Lagrange's equation to $L = T - V$:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = 0$$

Obtain:

$$m_1 \frac{d^2 x_1}{dt^2} = -k_1 x_1 + k_2(x_2 - x_1)$$

$$m_2 \frac{d^2 x_2}{dt^2} = -k_2(x_2 - x_1) - k_3 x_2$$

Example 2.4

A simplified model of the motion of an underwater vehicle can be written as

$$\dot{v} + |v|v = u$$

where v is the vehicle velocity and u is the thrust. Responses to the pulses of amplitude $u = 1$ and $u = 10$ are shown in the graphics.

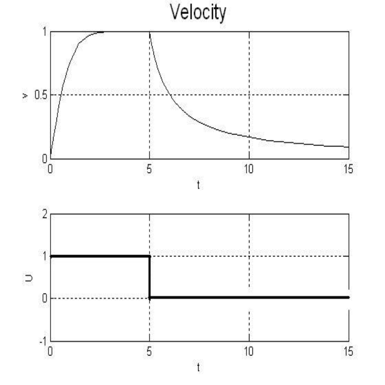


Figure 10: Underwater vehicles response to $u = 1$

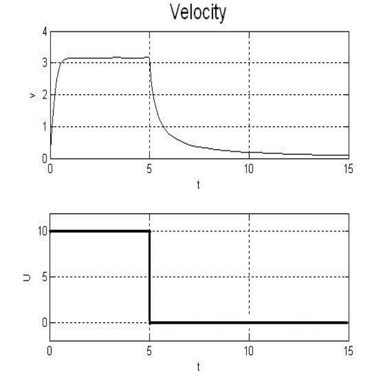


Figure 11: Underwater vehicles response to $u = 10$

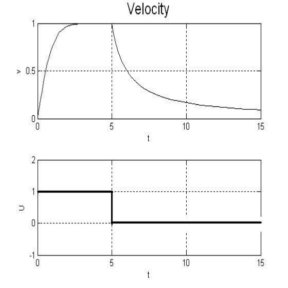
Observe that 10-folding the input amplitude did not cause response to 10-fold.

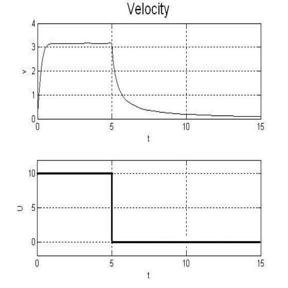
EOM of the underwater vehicle:

$$\dot{v} + |v|v = u$$

Continued from the previous page

steady state response to $u = 1$ is $v_s = 1$, since $|v|v = 1 \rightarrow v_s = \sqrt{1}$
steady state response to $u = 10$ is $v_s \approx 3.16$, since $|v|v = 10 \rightarrow v_s = \sqrt{10}$


Velocity
v
t
u
t


Velocity
v
t
u
t

Multiple equilibrium points Nonlinear systems frequently have more than one equilibrium point (an EP is a point where the system can stay forever without moving)

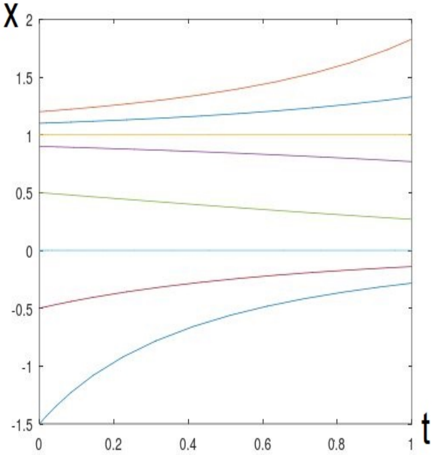
Example 2.5

$$\dot{x} = -x + x^2, \; x(0) = x_0$$

has the closed form solution

$$x(t) = \frac{x_0 e^{-t}}{1 - x_0 + x_0 e^{-t}}$$

$x(t) = \frac{x_0 e^{-t}}{1 - x_0 + x_0 e^{-t}}$



This system has two equilibrium points $x = 0$ and $x = 1$, where former is stable and latter is unstable.

A linear system can have only one isolated equilibrium point, thus it can have only one steady state operating point that attracts the state of the system irrespective of the initial state. A nonlinear system can have more than one isolated equilibrium point. The state may converge to one of several steady state operating points, depending on the initial state of the system.

Limit cycles

Nonlinear systems can display oscillations of fixed amplitude and fixed period without external excitation. These oscillations are called limit cycles. More formally,
"A limit-cycle on a plane or a two-dimensional manifold is a closed trajectory in phase space having the property that at least one other trajectory spirals into it either as time approaches infinity or as time approaches negative infinity. Such behavior is exhibited in some nonlinear systems. In the case where all the neighbouring trajectories approach the limit-cycle as time approaches infinity, it is called a stable limit-cycle. If instead all neighbouring trajectories approach it as time approaches negative infinity, it is an unstable limit-cycle. Stable limit-cycles imply self sustained oscillations. Any small perturbation from the closed trajectory would cause the system to return to the limit-cycle, making the system stick to the limit-cycle." [Wikipedia]

Example 2.6

Van der Pol equation is

$$m\ddot{x} + 2c(x^2 - 1)\dot{x} + kx = 0 \tag{13}$$

where m, c, k are positive constants. For large values of x damping coefficient is positive, and damper removes energy from the system. For small values of x the damping coefficient is negative and damper adds energy into the system.
When $x > 1$, all the coefficients of Eq. (13) become positive. This causes $x(t)$ to decrease. When $x < 1$, one coefficient of Eq. (13) becomes negative. This causes $x(t)$ to increase. therefore, the system motion neither grows unboundedly nor decay to zero. It displays sustained oscillation independent of the initial condition.

Continued from the previous page

Solutions of $ay'' + by' + c = 0$

Roots of $ar^2 + br + c = 0$	General solution
r_1, r_2 real and distinct	$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$
$r_1 = r_2 = r$	$y = c_1 e^{rx} + c_2 x e^{rx}$
r_1, r_2 complex: $\alpha \pm i\beta$	$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$

If a, b, c are real numbers with the same sign, then each root has negative real part. Corresponding general solutions approach zero. If the coefficient signs are not the same, then at least one root has a positive real part. This causes general solution to go to infinity.

Van der Pol equation

$$m\ddot{x} + 2c(x^2 - 1)\dot{x} + kx = 0 \qquad \text{(cf. 13)}$$

is expressed in the state space form as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{k}{m}x_1 - \frac{2c}{m}(x_1^2 - 1)x_2 \end{aligned} \qquad (14)$$

where $x_1 \triangleq x$ and $x_2 \triangleq \dot{x}$. For the initial condition $(x_1, x_2) = (0.1, 0)$, time graphics and phase graphics are given below.

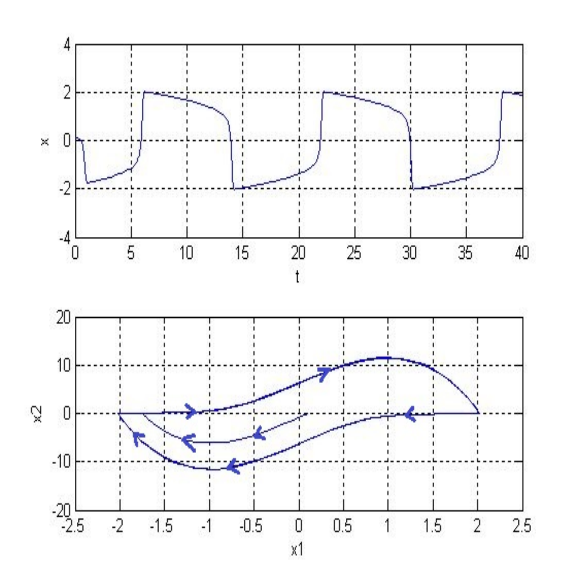


Figure 12: Van der Pol equation for $m = 1, c = 4, k = 1$

For the Van der Pol equation with $m = 1, c = 4$, and $k = 1$; and initial condition $(x_1, x_2) = (0.1, 0)$ what is the initial direction of the phase plane graphics? Recall the Van der Pol equation:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{k}{m}x_1 - \frac{2c}{m}(x_1^2 - 1)x_2 \end{aligned} \qquad \text{(cf. 14)}$$

at the given parameters and the initial condition, it equals

$$\begin{aligned} \dot{x}_1 &= 0 \\ \dot{x}_2 &= -\frac{1}{1} \cdot 0.1 - \frac{2 \cdot 4}{1}(0.1^2 - 1) \cdot 0 = -0.1 \end{aligned}$$

As shown in the graphics, initially x_2 decreases. For a different set of parameters and initial condition, time and phase graphics are shown below.

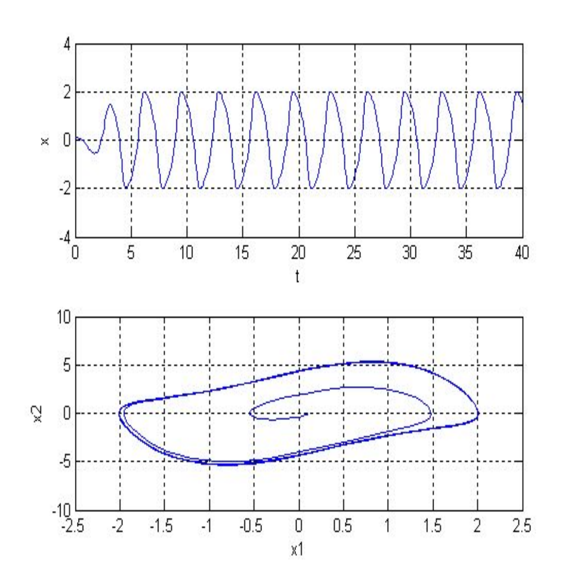


Figure 13: Van der Pol equation for $m = 1, c = 1, k = 4$

Continued from the previous page

Sustained oscillations can also be found in linear systems, in the case of marginally stable linear systems in response to a sinusoidal inputs. However, limit cycles in nonlinear systems are different from linear oscillations in a number of fundamental aspects. First the amplitude of the self sustained oscillation is independent of the initial condition while the oscillation of the marginally stable system has its amplitude determined by its initial conditions. Second, marginally stable linear systems are very sensitive to changes in system parameters.

In practical applications that involve differential equations it very often happens that the differential equation contains parameters and the value of these parameters are often only known approximately. In particular they are generally determined by measurements which are not exact. For that reason it is important to study the behavior of solutions and examine their dependence on the parameters. This study leads to the area referred to as bifurcation theory. It can happen that a slight variation in a parameter can have significant impact on the solution.
Bifurcation means the splitting of a main body into two parts. In our context, as the parameters of the nonlinear dynamic systems are changed, qualitative nature of the d. e. can change, for instance, the stability of EP can change, and so can the number of EP's. Values of these parameters at which the qualitative nature of the system's motion changes are known as critical or bifurcation values.

Example 2.7 (Bifurcations in linear systems)

Consider the system with linear differential equations model

$$\frac{d^2y}{dt^2} + k\frac{dy}{dt} + y = 0$$

There is only one bifurcation point above, $k = 0$. If k crosses this point stsbility of the system above changes.
In linear systems the bifurcations are related to changes in the stability of the system.

In NL systems, the change in parameters can cause significant changes in system behavior and can lead to complex and unpredictable dynamics. For example, a simple system with a single stable equilibrium point can undergo a bifurcation, resulting in the emergence of two or more stable equilibrium points or limit cycles. There are many different types of bif. that can occur in NL systems, each with its own characteristics and effects on system behavior. Some of them are: saddle-node bifurcation, Hopf bifurcation, pitchfork bifurcation, and transcritical bifurcation

Example 2.8

Duffing Equation

$$\ddot{x} + \alpha x + x^3 = 0$$

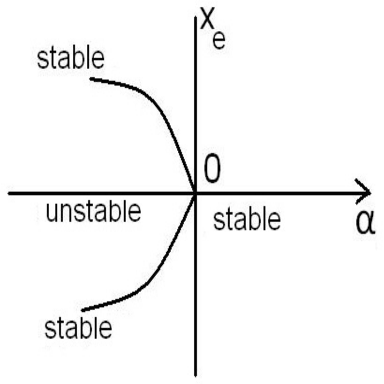
As α varies from positive to negative, one EP splits into three points (i.e., $0, \sqrt{-\alpha}, -\sqrt{-\alpha}$). This kind is known as pitchfork bifurcation.

Define $x := x_1,\ \dot{x} := x_2$
The d.e. in the state space form

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\alpha x_1 - x_1^3 \end{aligned}$$

Equilibrium points satisfy

$$\begin{aligned} 0 &= x_2 \\ 0 &= -\alpha x_1 - x_1^3 \end{aligned}$$



EP's corresponding to negative α values : $(0,0), (\sqrt{-\alpha}, 0), (-\sqrt{-\alpha}, 0)$

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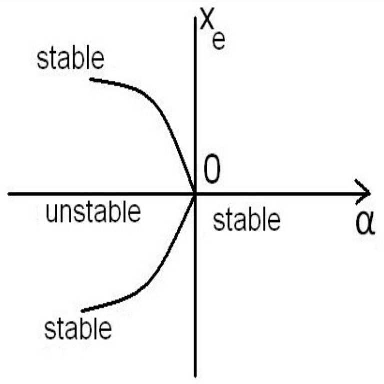
Define $x := x_1, \; \dot{x} := x_2$
The d.e. in the state space form

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\alpha x_1 - x_1^3 \end{aligned}$$

Equilibrium points satisfy

$$\begin{aligned} 0 &= x_2 \\ 0 &= -\alpha x_1 - x_1^3 \end{aligned}$$



Reading the figure: For negative α values the system has 3 EP's, and for positive α values the system has only one EP. For instance, when $\alpha = -2$ the three EP's we have are $0, \sqrt{2}, -\sqrt{2}$ where 0 is unstable EP and $\sqrt{2}, -\sqrt{2}$ are stable EP's.

Example 2.9

We next study the time intervals between successive water drops falling from a dripping tap.

"We imagine a slowly forming drop at the end of a tap as a mass M suspended from a spring. As time passes, the mass M of this drop increases before a fraction of it breaks off at some critical point. This fraction is thus lost with the remainder (the residue) staying attached to the tap. Note that the residue recoils upwards after detachment."

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The differential equation model:

$$
\begin{aligned}
\frac{dM}{dt} &= R \\
\frac{dx}{dt} &= v \\
\frac{dv}{dt} &= g - \frac{k}{M}x - \frac{b}{M}v
\end{aligned}
$$

where

M: mass of the forming drop

x: position (centre of mass) of the forming drop

v: velocity of the drop

R is the flow rate (assumed constant)

k, b and g are the constants determining the strength of the spring, the strength of the damping and the gravity respectively.

Continued from the previous page

$$\begin{aligned}\frac{dM}{dt} &= R\\ \frac{dx}{dt} &= v\\ \frac{dv}{dt} &= g - \frac{k}{M}x - \frac{b}{M}v\end{aligned}$$

Explanation the dynamics

When the position of the forming drop exceeds some fixed threshold x_c, a water drop falls away and we must thus reduce the mass M. Let us take it that the mass of the falling drop is given by

$$\Delta M = \alpha M_c v_c$$

where α is some constant and v_c and M_c are the velocity and mass of the forming drop at the threshold x_c. The mass of the drop thus suddenly changes from M_c to $M_c - \Delta M$.

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Position of the forming drop x suddenly changes to the new value

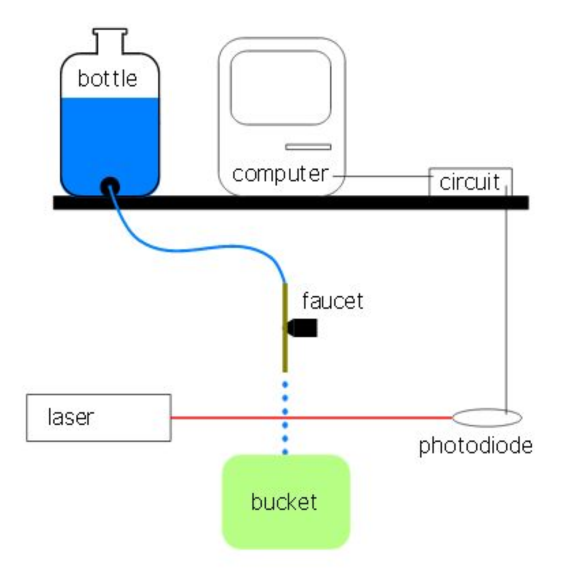
$$x = x_0 = x_c - \frac{r\Delta M}{M_c}$$

where $r = (3\Delta M/4\pi\rho)^{1/3}$ is the radius of the falling drop (ρ: density of water).

Use the values $R = 0.6g/s,\ g = 980cm/s^2, x_c = 0.25cm,\ k = 475dyn/cm,\ b = 1g/s,\ \alpha = 0.5s/cm$ and $\rho = 1g/cm^3$ and initial conditions $x = 0,\ v = 0$ and $M = 0.01$.

You should see that after some initial transient behaviour, the time intervals T between each successive drop takes on only two alternating values. We are thus observing periodic behaviour. Re-running the code with the values $R \in (0.7, 0.8)$, one observes the cases that the time intervals T doubles for some R. Also, for some $R \in (0.7, 0.8)$, chaos occurs.

An experimental set up



The state of an unstable linear system goes to infinity as time approaches infinity; a nonlinear system's state, however, can go to infinity in finite time.

Example 2.10

Consider the dynamic model

$$\dot{x} = -x^2, \;\; x(0) = -1$$

Its solution is

$$x(t) = \frac{1}{t-1}$$

It tends to infinity in a finite time, more specifically, $x(t) \to -\infty$ as $t \to 1$.

Continued from the previous page

Octave codes for the finite escape time d.e.

```
function myFE
  [t y] = ode45(@myFinEscape, [0 0.98], [-1]);
  plot(t, y)
  xlabel('t')
  ylabel('x(t)')
  grid
return

function xdot = myFinEscape(t,y)
    xdot = -y^2;
return
```

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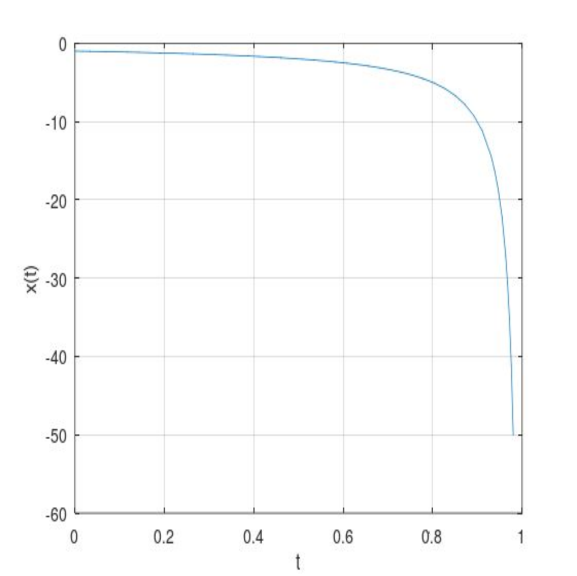

Figure 14: Finite escape time: $x(t) \to -\infty$ as $t \to 1$

Chaos

A nonlinear system can have a more complicated steady state behavior that is not equilibrium, periodic oscillation, or almost periodic oscillation. Such behavior is usually referred to as chaos.
"Chaos theory studies the behavior of dynamical systems that are highly sensitive to initial conditions, an effect which is popularly referred to as the butterfly effect. Small differences in initial conditions (such as those due to rounding errors in numerical computation) yield widely diverging outcomes for chaotic systems, rendering long-term prediction impossible in general. This happens **even though these systems are deterministic, meaning that their future behavior is fully determined by their initial conditions**, with no random elements involved. In other words, **the deterministic nature of these systems does not make them predictable**. This behavior is known as deterministic chaos, or simply chaos." [Wikipedia]

Chaos in nonlinear systems refers to a phenomenon where small changes in initial conditions can lead to dramatically different outcomes or trajectories over time. Nonlinear systems are systems that do not follow a linear relationship between cause and effect. Instead, the behavior of these systems can be highly complex and unpredictable.
In chaotic systems, small differences in initial conditions can lead to large differences in outcomes, making it difficult to predict the behavior of the system over time. This is known as the butterfly effect, where a butterfly flapping its wings in one location can potentially cause a hurricane in another location weeks later.
Chaotic systems can be found in a variety of natural and human-made systems, including weather patterns, fluid dynamics, population dynamics, and financial markets. While chaotic systems may appear to be unpredictable, they often exhibit underlying patterns or structures known as fractals.
Chaos theory is a branch of mathematics that studies the behavior of nonlinear systems and seeks to understand and predict their behavior. It has important applications in a variety of fields, including physics, engineering, economics, and biology.

Example 2.11

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1.2 & -0.4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} (0.8x_1 - x_1^3)$$

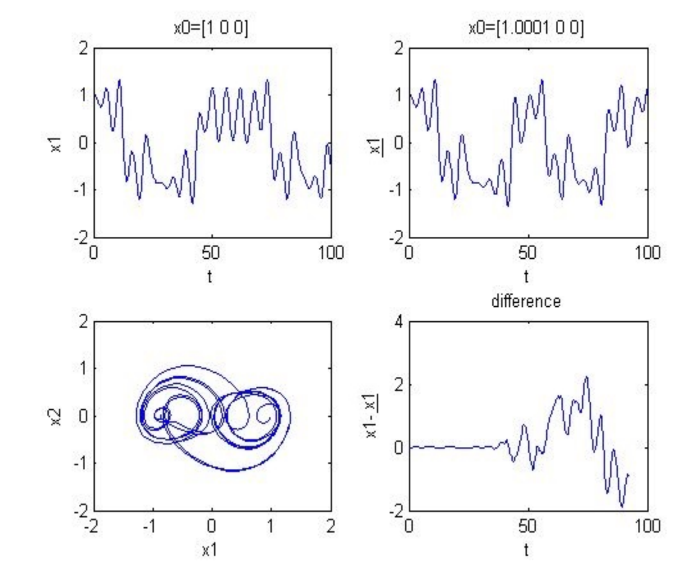

Figure 15: Chaotic behavior

Example 2.12

Consider the ordinary differential equation set that models waves in non-equilibrium substances [M. Rabinovich and A. Fabrikant, 1979]

$$\begin{aligned} \dot{x} &= y(z - 1 + x^2) + \gamma x \\ \dot{y} &= x(3z + 1 - x^2) + \gamma y \\ \dot{z} &= -2z(\alpha + xy) \end{aligned}$$

where α and γ are constants. Setting $\alpha = 1.2$ and $\gamma = 0.87$ results in chatic output. however, $\alpha = 1.5$ and $\gamma = 0.55$ results in periodic output.

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```
function mychaos
  [t y] = ode45(@myEqns, [0 60], [-1; 0; 0.5]);

  plot(-y(:,1), -y(:,2))
  xlabel('X')
  ylabel('Y')
  return

  function xdot = myEqns(t,y)
    alpha = 1.1;
    gamma = 0.87;
    X = y(1);
    Y = y(2);
    Z = y(3);
    xdot = [Y*(Z - 1 + X^2) + gamma*X; ...
            X*(3*Z + 1 - X^2) + gamma*Y; ...
            -2*Z*(alpha + X*Y)];
return
```

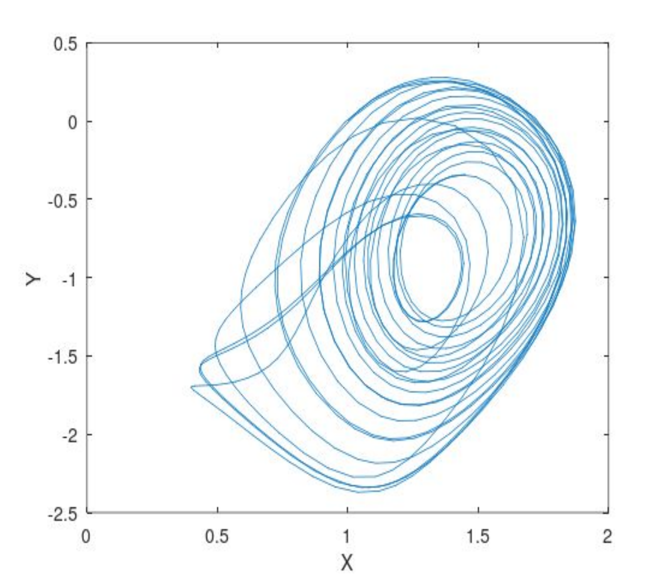

Figure 16: Chaotic behavior

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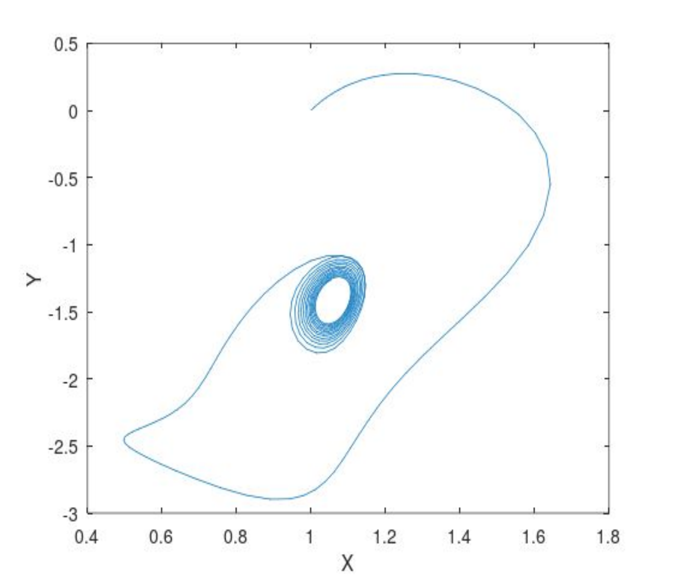

Figure 17: Periodic behavior

Subharmonic, harmonic, or almost periodic oscillations

A stable linear system under a **periodic input** produces an output of the same frequency. A nonlinear system under periodic excitation can oscillate with frequencies that are **submultiples**, or **multiples** of the input frequency. It may even generate an **almost periodic** oscillation.

A **harmonic** is a signal whose frequency is an integral multiple of the frequency of some reference signal.

Subharmonic is an oscillation with a frequency equal to an integral submultiple of some other reference frequency.

We consider complex functions of a real variable. Such a function is **periodic** if there is a period T such that for all t, $f(t+T) = f(t)$.

A number τ is an ϵ-period for a function f if, for all t, $|f(t) - f(t+\tau)| \le \epsilon$. A function is then **almost periodic** if, for each positive ϵ there is a real number ℓ such that each real interval of length ℓ contains an ϵ-period.

Phase plane analysis is a graphical method for studying second order systems. The basic idea of the method is to **generate**, in the state space of a second order system, **motion trajectories** corresponding to various initial conditions, and then to examine the qualitative features of the trajectories.

Phase plane analysis is not restricted to small or smooth nonlinearities, but applies equally well to **strong** nonlinearities and to **hard** nonlinearities.

Hard nonlinearities have nonsmooth i-o relationships.They do not allow linear approximations. Examples are: Saturation, backlash, deadzone, hysteresis, friction, switching.

Strong nonlinearities are x^3, x^4, x^5-like terms, which have rapid change in the output in response to small changes in the input.

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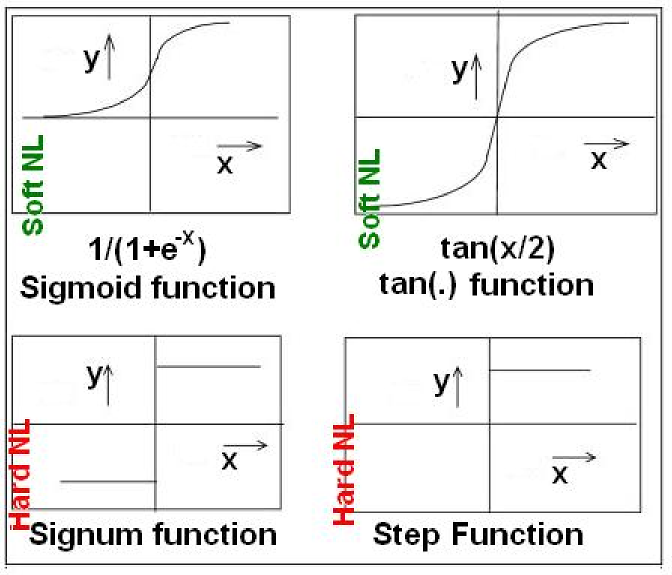


Figure 18: Hard and soft nonlinearities

Some practical control systems can be approximated as second order systems, and the phase plane method can be used for their analysis. The phase plane analysis deals with the graphical study of second order autonomous (i.e., which do not depend explicitly on the independent variable) systems described by

$$\left.\begin{array}{rcl} \dot{x}_1 &=& f_1(x_1, x_2) \\ \dot{x}_2 &=& f_2(x_1, x_2) \end{array}\right\} \tag{15}$$

where x_1, x_2 are states of the system, and f_1, f_2 are nonlinear functions of the states. Geometrically, the state space of this system is a plane having x_1 and x_2 as coordinates. This plane is called the phase plane.

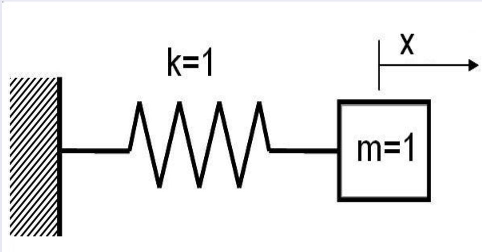
Given a set of initial conditions $\mathbf{x}(0) = \mathbf{x}_0 \triangleq \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$, Eq.(15) defines a solution $\mathbf{x}(t) \triangleq \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$. With time t varied from zero to infinity, the solution $\mathbf{x}(t)$ can be represented geometrically as a curve in the phase plane, such a curve is called a **phase plane trajectory**. A family of phase plane trajectories corresponding to various initial conditions is called a **phase portrait** of a system.

Example 3.1

The governing equation of the mass spring system in Figure 16 is a familiar 2nd order linear differential equation

$$\ddot{x} + x = 0 \tag{16}$$

Let the unstreched and uncompressed position of the mass be $x = 0$ and let the mass have initial position x_0, and initial velocity 0. Then the solution of the equation is $x(t) = x_0 \cos t, \;\; \dot{x}(t) = -x_0 \sin t$. Eliminating time t from the above equations, we obtain the equation of the trajectories $x^2 + \dot{x}^2 = x_0^2$. This represents a circle in the phase plane.



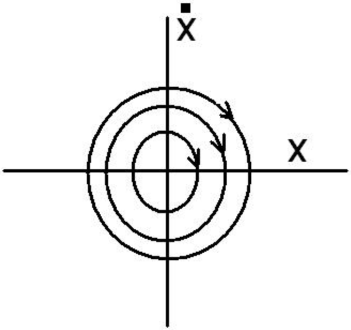


Figure 19: Phase portrait of the mass-spring system

In the example above, we see that the system trajectories neither converge to the origin nor diverge to infinity. They simply circle around the origin, indicating the marginal nature of the system's stability.

A major class of 2nd order systems can be described by the differential equations of the form

$$\ddot{x} + f(x, \dot{x}) = 0 \tag{17}$$

In the state space form, this dynamics can be represented as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -f(x_1, x_2) \end{aligned} \tag{18}$$

with $x_1 := x$ and $x_2 := \dot{x}$. For this class of 2nd order systems, we have closed form formulas for the symmetry properties of the phase portrait.

Singular points

A singular point is an equilibrium point in the phase plane. Since an equilibrium point is defined as a point where the system states can stay forever; this implies that $\dot{x} = 0$, and using (15),

$$f_1(x_1, x_2) = 0, \quad f_2(x_1, x_2) = 0 \tag{19}$$

The values of the equilibrium states can be solved from (19).
For a linear system, there is usually only one singular point or infinitely many. However, a nonlinear system often has more than one isolated singular point.

Example 3.2

Consider the system

$$\ddot{x} + 0.6\dot{x} + 3x + x^2 = 0$$

This has two singular points. One at (0,0) and the other is at (-3,0). The trajectories move toward (0,0) but moving away from (-3,0). So these two singular points are different in nature.

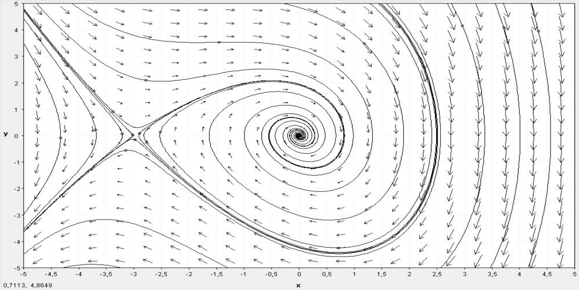
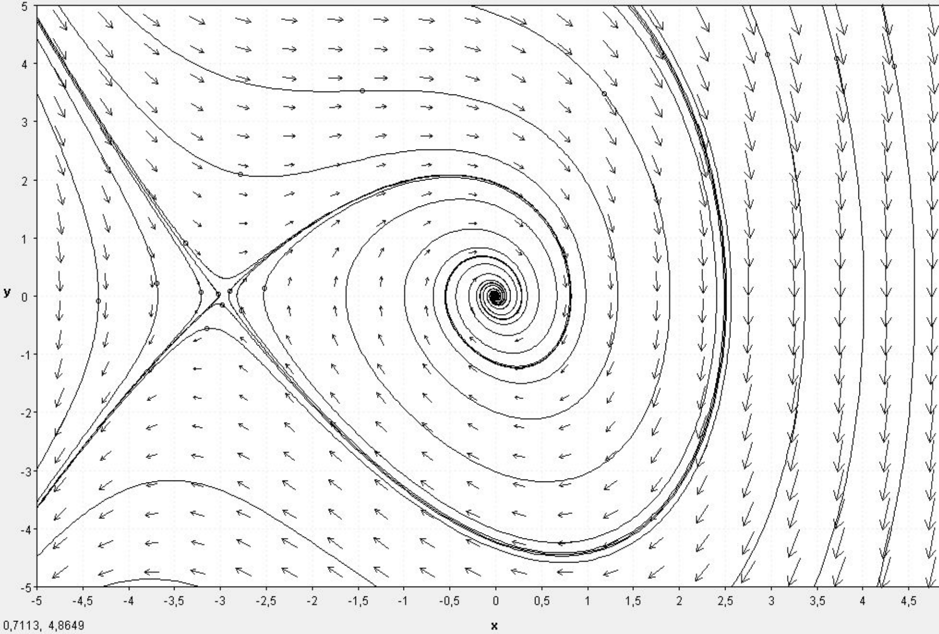


Figure 20: Phase portrait of $\ddot{x} + 0.6\dot{x} + 3x + x^2 = 0$



y
x
0,7113, 4,8649

The system

$$\ddot{x} + 0.6\dot{x} + 3x + x^2 = 0$$

can be written in the state space as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -0.6x_2 - 3x_1 - x_1^2 \end{aligned}$$

Its equilibrium points (in other words, singular points) are found by solving

$$\begin{aligned} 0 &= x_2 \\ 0 &= -0.6x_2 - 3x_1 - x_1^2 \end{aligned}$$

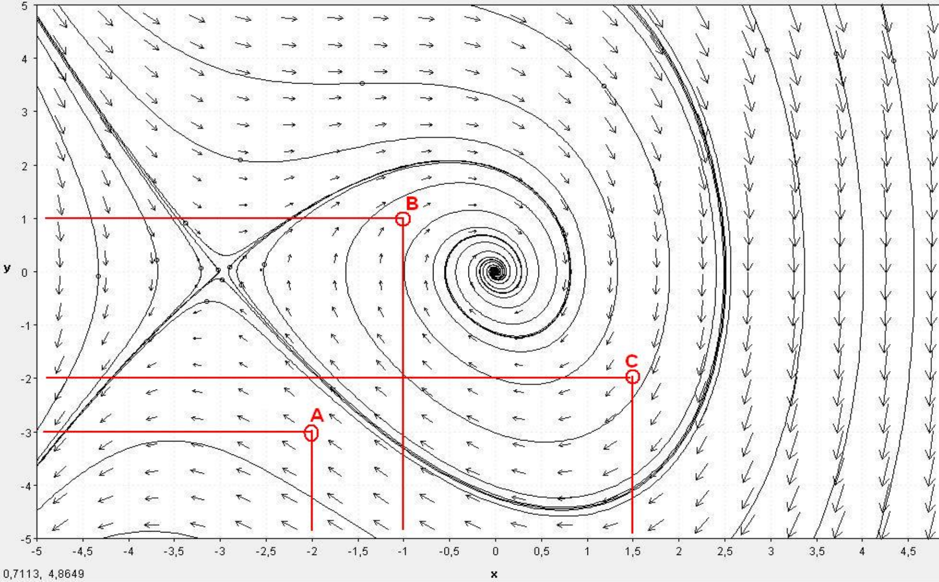
The equation set above has two different solutions: $(0, 0)$ and $(-3, 0)$.
Each one is an equilibrium point.
For a phase portrait plotting in MATLAB, one may use any of pplane5.m-pplane8.m, at

`http://math.rice.edu/~dfield/`

From Eq. (15), the slope of the phase trajectory passing through a point (x_1, x_2) is determined by

$$\frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)} \tag{20}$$

For (20) to have a definite value, f_1 and f_2 must be single value functions, and at the point where we calculate the slope $f_1 \neq 0$ must hold.
Stability of NL systems is characterized in terms of their singular points (EP's). For instance, for a given nonlinear system, some equilibrium points may be stable while some others are unstable.
Phase plane analysis is also useful for first order systems. For the 1st order systems, the phase portrait is composed of a single trajectory.
Now, using (20) we calculate some slopes in the phase portrait of Van der Pol equation:


0,7113, 4,8649
x
y
A
B
C

The Van der Pol equation in the state space form is

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -0.6x_2 - 3x_1 - x_1^2\end{aligned}$$

Recall the slope formula:

$$\frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)} \qquad \text{(cf. 20)}$$

At the point A the slope is

$$\left.\frac{-0.6x_2 - 3x_1 - x_1^2}{x_2}\right|_{(-2,-3)} = \frac{-0.6(-3) - 3(-2) - (-2)^2}{-3} \approx -1.27$$

Likewise, at the points B and C we find 1.4 and 2.78 respectively.

Example 3.3

$$\dot{x} = -4x + x^3$$

EP's: 0,2,-2

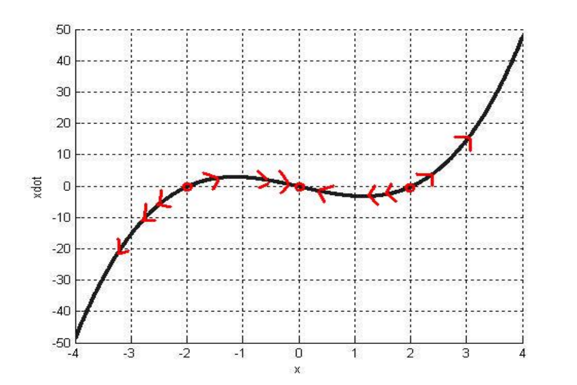


Figure 21: Phase portrait of $\dot{x} = -4x + x^3$

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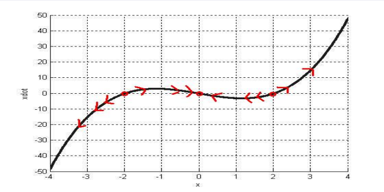
$$\dot{x} = -4x + x^3$$



Examples for determining the arrow direction

At $x = -3$ corresponding $\dot{x} = -15$ implies x is decreasing, therefore the direction should be towards left.

At $x = -1$ corresponding $\dot{x} = 1$ implies x is increasing, therefore the direction should be towards right.

Symmetry about an axis

A phase portrait may have apriori known symmetry properties, which can simplify its generation and study. Recalling the class of second order systems:

$$\ddot{x} + f(x, \dot{x}) = 0 \qquad \text{(cf. 17)}$$

This class can be written in the state space form as

$$\left.\begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & -f(x_1, x_2) \end{array}\right\} \qquad \text{(cf. 18)}$$

Slope at $(x_1^*, x_2^*) = \frac{-f(x_1^*, x_2^*)}{x_2^*}$

Slope at $(x_1^*, -x_2^*) = \frac{-f(x_1^*, -x_2^*)}{-x_2^*}$

We have symmetry about the x_1 axis if

slope at (x_1^*, x_2^*) = -slope at $(x_1^*, -x_2^*)$, $\forall (x_1^*, x_2^*)$

$\rightarrow \frac{-f(x_1^*, x_2^*)}{x_2^*} = -\left(\frac{-f(x_1^*, -x_2^*)}{-x_2^*}\right)$

$$\boxed{\rightarrow f(x_1^*, x_2^*) = f(x_1^*, -x_2^*) \text{ Symmetry condition about } x_1}$$

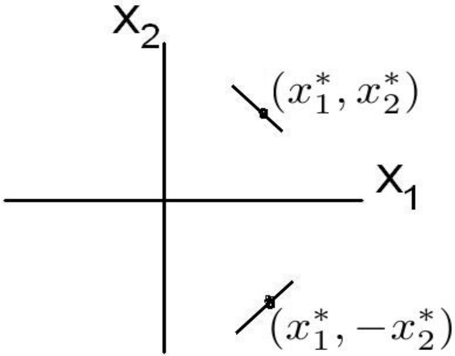


Figure 22: Symmety about the x_1 axis

We have symmetry about the x_2 axis if

slope at (x_1^*, x_2^*) = -slope at $(-x_1^*, x_2^*)$, $\forall (x_1^*, x_2^*)$

$$\rightarrow \frac{-f(x_1^*, x_2^*)}{x_2^*} = -\left(\frac{-f(-x_1^*, x_2^*)}{x_2^*}\right)$$

$$\boxed{f(x_1^*, x_2^*) = -f(-x_1^*, x_2^*) \text{ Symmetry condition about } x_2}$$

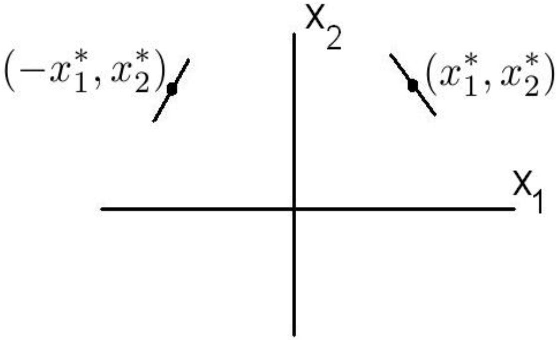


Figure 23: Symmety about the x_2 axis

Symmetry about the origin:

$$f(x_1^*, x_2^*) = -f(-x_1^*, -x_2^*) \text{ Symmetry condition about the origin}$$

Example 3.4

Mass-spring system has the model

$$\left.\begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & -x_1 \end{array}\right\} \tag{21}$$

Considering the class given below

$$\left.\begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & -f(x_1, x_2) \end{array}\right\} \quad \text{(cf. 18)}$$

we have $f(x_1, x_2) = x_1$; it satisfies symmetry conditions about x_1, x_2 and the origin.

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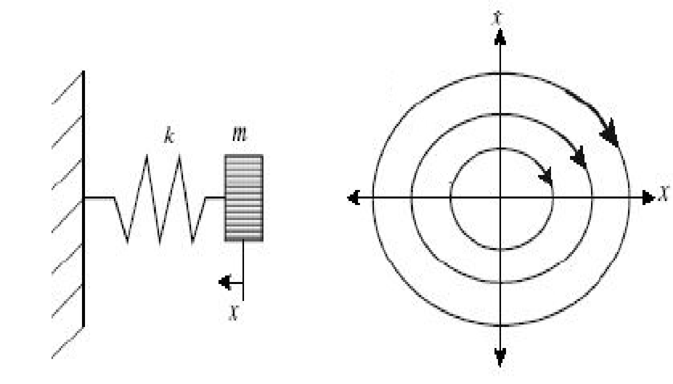


Figure 24: Symmetries in the phase portrait of mas-spring system

Example 3.5

$$\left.\begin{array}{rcl}\dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & -x_1 - x_2^2\end{array}\right\}$$

Considering the class given below

$$\left.\begin{array}{rcl}\dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & -f(x_1, x_2)\end{array}\right\} \qquad \text{(cf. 18)}$$

we have $f(x_1, x_2) = x_1 + x_2^2$, $f(x_1, -x_2) = x_1 + (-x_2)^2 \rightarrow$ symmetry about x_1.
Since $-f(-x_1, x_2) = -(-x_1 + x_2^2) = x_1 - x_2^2 \neq f(x_1, x_2) = x_1 + x_2^2$ there is no symmetry about x_2.
HW. Verify this by constructing the phase portrait.

Example 3.6

By noting that $\ddot{x} = \frac{d\dot{x}}{dx}\frac{dx}{dt}$ the mass spring equation

$$\ddot{x} + x = 0 \qquad \text{(cf. 16)}$$

can be written as

$$\dot{x}\frac{d\dot{x}}{dx} + x = 0$$

$$\dot{x}d\dot{x} + xdx = 0$$

Integration of this equation yields

$$\frac{\dot{x}^2}{2} + \frac{x^2}{2} = c \rightarrow \dot{x}^2 + x^2 = x_0^2$$

where $2c = x_0^2$ is an arbitrary positive constant. Most NL systems cannot be easily solved by any of the techniques we used to solve the mass spring equations. However for piecewise linear systems the method above can be conveniently used.

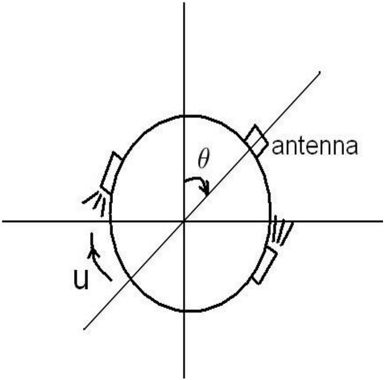
Example 3.7 (Satellite control)

The satellite is simply a rotational inertia unit controlled by a pair of thrusters, which can provide either a positive constant torque A or a negative torque $-A$. The purpose of the control system is to maintain the satellite antenna at zero angle by appropriately firing thrusters. The mathematical model of the satellite is

$$\ddot{\theta} = u$$

where u is the torque provided by the thrusters and θ is the satellite angle. Let us examine on the phase plane the behavior of the control system when the thrusters are fired according to the control law

$$u(t) = \begin{cases} -A & \text{if} \quad \theta > 0 \\ A & \text{if} \quad \theta < 0 \end{cases}$$


antenna
θ
u

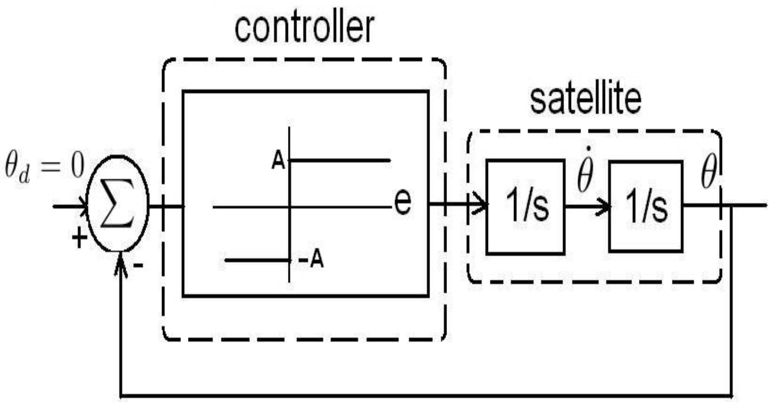


Figure 25: A satellite control system

Continued from the previous page

As the first step of the phase portrait generation, let us consider the phase portrait when the thrusters provide a positive torque A. Positive torque is needed when $\theta < 0$. Under the positive torque A, the dynamics of the system becomes

$$\ddot{\theta} = A$$

which implies that $\dot{\theta} d\dot{\theta} = A d\theta$. (Recall that $\ddot{\theta} = \dot{\theta}\frac{d\dot{\theta}}{d\theta}$.) Integration of $\dot{\theta} d\dot{\theta} = A d\theta$ yields the phase trajectories a family of parabolas

$$\dot{\theta}^2 = 2A\theta + c_1$$

where c_1 is a constant. Corresponding phase portrait of the system is shown in Figure 26.

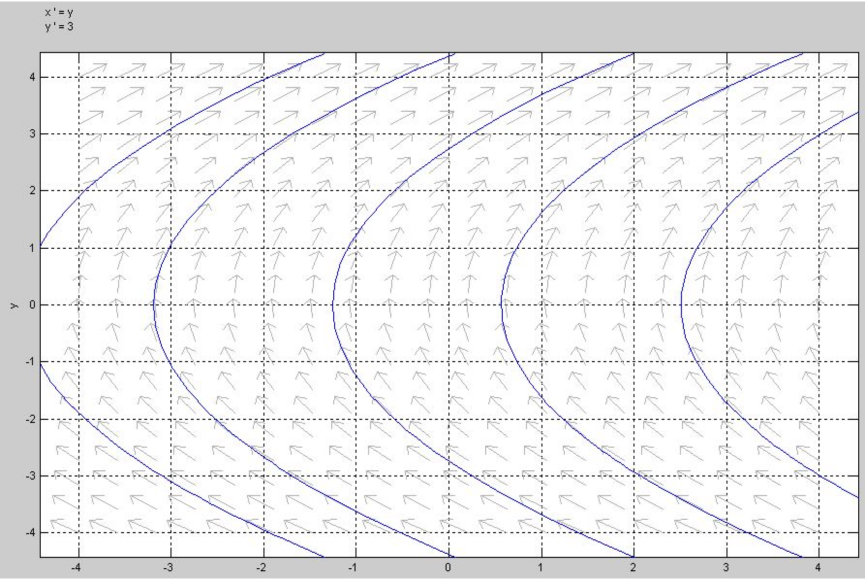


Figure 26: Phase portrait for $\theta < 0$

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When $\theta > 0$ the thrusters provide negative torque $-A$. The phase trajectories are similarly found to be $\dot{\theta}^2 = -2A\theta + c_1$. Corresponding phase portrait of the system is shown in Figure 27.

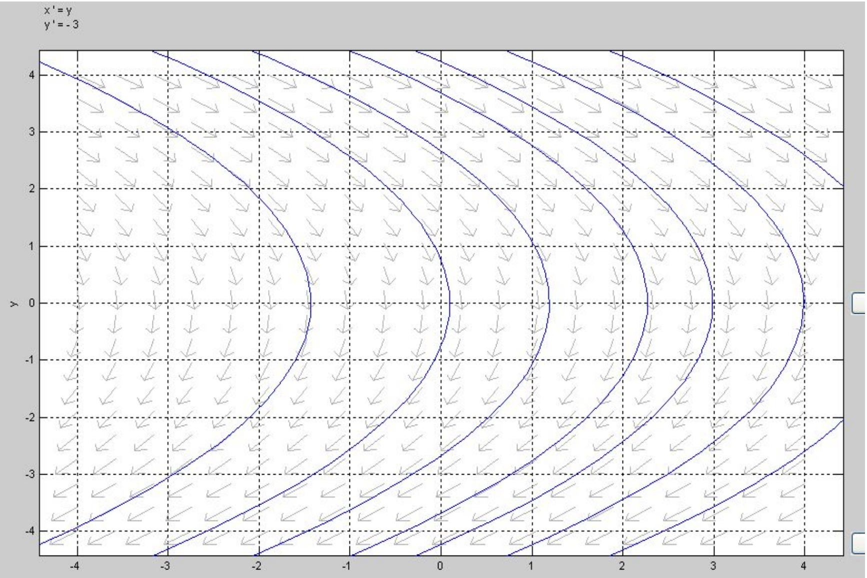

Figure 27: Phase portrait for $\theta > 0$

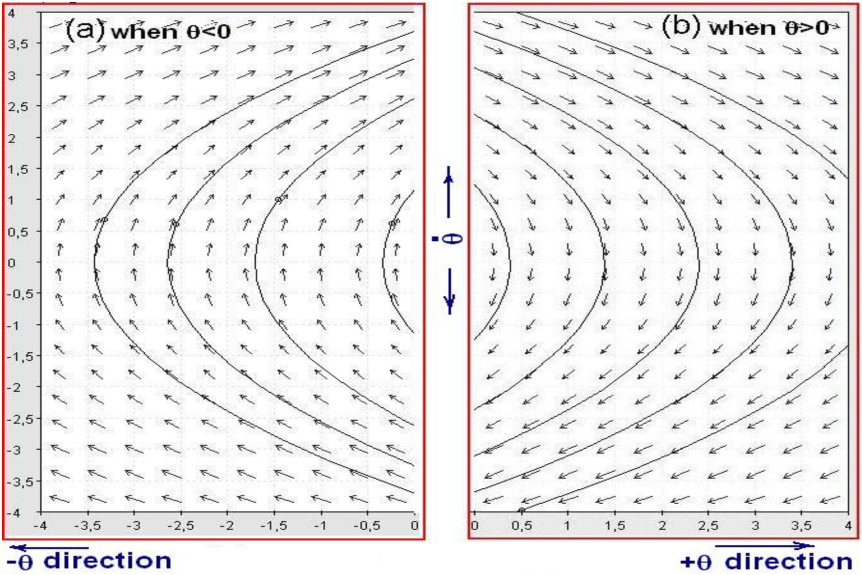

Figure 28: Phase portraits for $\theta < 0$ and $\theta > 0$ together on the same view

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The complete phase portrait of the closed loop system can be obtained simply by connecting the trajectories on the left half of the phase plane in Figure 26 with those on the right half of the phase plane in Figure 27, as shown in Figure 29.
The vertical axis represents a switching line, because the control input and thus the phase trajectories are switched on that line. It is interesting to see that starting from a non zero initial angle, the satellite will oscillate in periodic motions under the action of the thrusters.

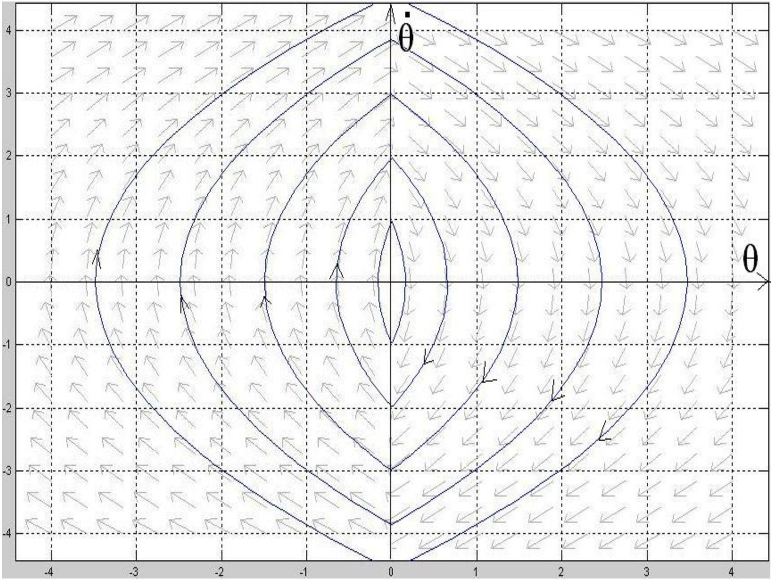

Figure 29: Complete phase portraits when $A = 2$

Constructing the phase portraits

Consider the dynamics

$$\left.\begin{array}{rcl}\dot{x}_1 & = & f_1(x_1, x_2) \\ \dot{x}_2 & = & f_2(x_1, x_2)\end{array}\right\} \qquad \text{(cf. 15)}$$

At a point (x_1, x_2) in the phase plane, the slope of the tangent to the trajectory can be determined by

$$\frac{dx_2/dt}{dx_1/dt} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)}$$

$$\frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)} \qquad \text{(cf. 20)}$$

An **isocline** is defined to be the locus of the points with given tangent slope. An isocline with slope α is thus defined to be

$$\frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)} = \alpha$$

This is to say that points on the curve

$$f_2(x_1, x_2) = \alpha f_1(x_1, x_2)$$

all have the same tangent slope α.
In the method of isoclines, the phase portrait of a system is generated in two steps.
In the first step, **a field of directions of tangents** to the trajectories is obtained.
In the second step, a phase plane trajectories are formed from the field of directions.

Let us explain the isocline method on the mass spring system

$$\left.\begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & -x_1 \end{array}\right\} \quad \text{(cf. 21)}$$

The slope of the trajectories is easily seen to be

$$\frac{dx_2}{dx_1} = -\frac{x_1}{x_2}$$

Therefore, the isocline equation for a slope α is

$$x_1 + \alpha x_2 = 0$$

that is, a straight line. Along the line we can draw a lot of short line segments with slope α. By taking α to be different values, a set of isoclines can be drawn, and a field of directions of tangents to trajectories are generated, as shown in Figure 30. To obtain trajectories from the field of directions, we assume that the tangent slopes are locally constant. Therefore a trajectory starting from any point in the plane can be found by connecting a sequence of line segments.

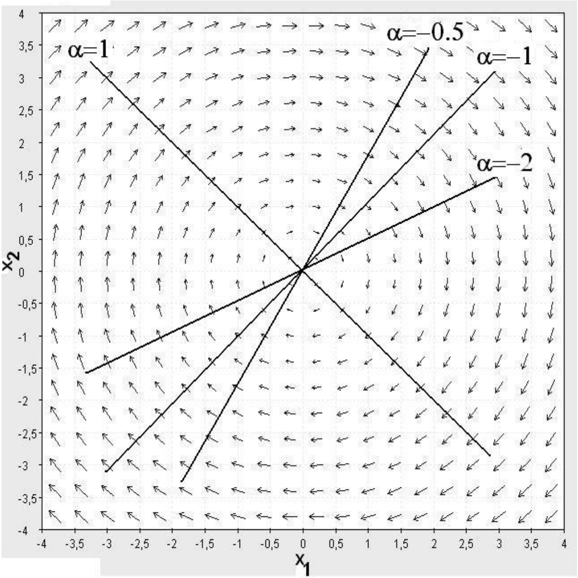

Figure 30: Field of directions for the mass spring system

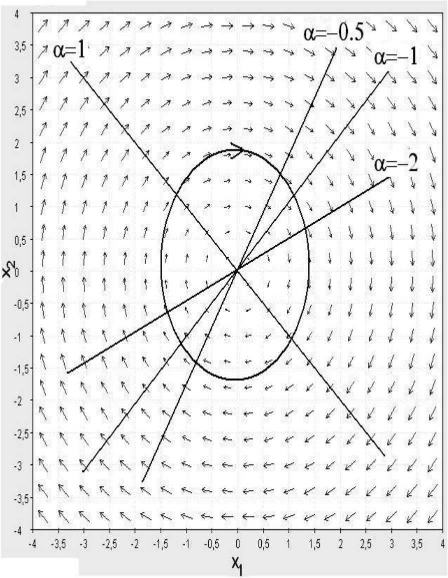

Figure 31: A trajectory on the field of directions for the mass spring system

Let us use the method of isoclines to study the Van der Pol equation, a nonlinear equation.

For the Van der Pol equation

$$\ddot{x} + 0.2(x^2 - 1)\dot{x} + x = 0$$

an isocline of slope α is defined by

$$\frac{d\dot{x}}{dx} = -\frac{0.2(x^2 - 1)\dot{x} + x}{\dot{x}} = \alpha.$$

Therefore, the points on the curve

$$0.2(x^2 - 1)\dot{x} + x + \alpha\dot{x} = 0$$

all have the same slope. By taking α of different values, different isoclines can be obtained, as plotted in Figure 32.

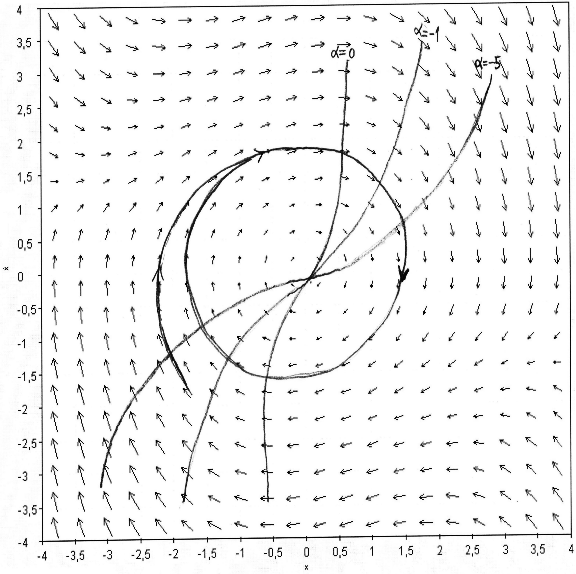

Figure 32: A trajectory on the field of directions for the Van der Pol equation

It is interesting to note that there exists a closed curve in the portrait, and the trajectories starting from both outside and inside both converge to this curve. This closed curve corresponds to a limit cycle.

Determining time from phase portraits

Time t does not explicitly appear in the phase plane having x_1 and x_2 as coordinates. How long it takes for the system to move from a point to another point in a phase plane trajectory can be determined. In short time Δt, the change of x is approximately

$$\Delta x = \dot{x}\Delta t$$

where $\dot{x}$ is the velocity corresponding to the increment Δx. This implies

$$\Delta t = \frac{\Delta x}{\dot{x}}$$

In order to obtain the time corresponding to the motion from one point to another point along a trajectory, one should divide corresponding part of the trajectory into a number of small segments, find the time associated with each segment, and then add up the results.

In the limits $\Delta t \to dt$ and $\Delta x \to dx$, so we can write

$$\Delta t = \frac{\Delta x}{\dot{x}} \to dt = \frac{dx}{\dot{x}}$$

Integrating throughout we obtain

$$t - t_0 = \int_{x_0}^{x} \frac{1}{\dot{x}} dx.$$

We plot the phase portrait with new coordinates x and $\frac{1}{\dot{x}}$, then the area under the resulting curve is the corresponding time interval.

Example 3.8

For the previously examined mass spring system, when $m = k = 1$, the dynamic equations are

$$\ddot{x} + x = 0$$

For the initial condition $(x(0), \dot{x}(0)) = (1, 0)$ its time solution is

$$x(t) = \cos t, \quad \dot{x}(t) = -\sin t$$

and phase plane solution is

$$x^2 + \dot{x}^2 = 1$$

Let us see how much time elapsed from $x(t) = 0.5$ to $x(t) = -0.5$. Then we verify it by the newly learned method.

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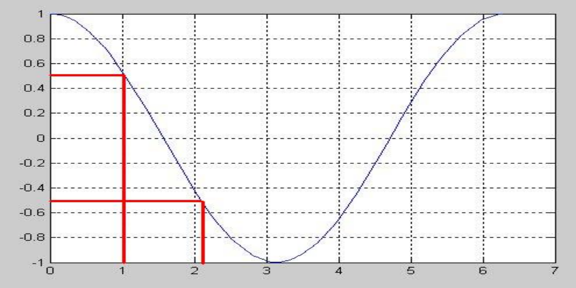

From the graphics, or analytic calculation, $x(t) = 0.5$ at $t = 1.047$ and $x(t) = -0.5$ at $t = 2.094$. So, the time elapsed from $x(t) = 0.5$ to $x(t) = -0.5$ is 1.047 sec.

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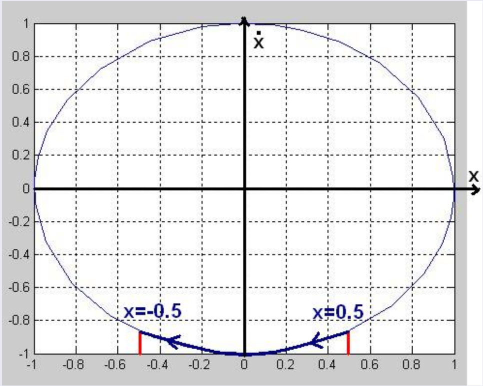

Above, we have the trajectory in the phase plane. In the phase plane $x(t) = 0.5$ and $x(t) = -0.5$ points are shown. How much time does the trajectory need to travel along the trajectory part shown above?

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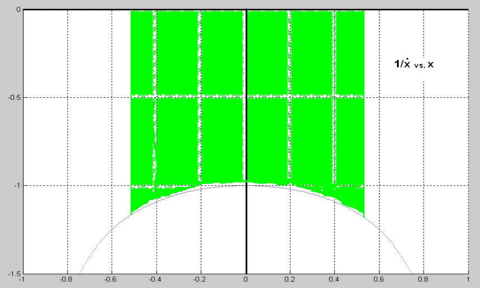


Above, we have the $x \leftrightarrow \frac{1}{\dot{x}}$ graphics for $x = 0.5 \rightarrow -0.5$. To find the elapsed time we need to find shaded area:

$$t_f - t_0 = \int_{0.5}^{-0.5} -\frac{1}{\sqrt{1-x^2}} dx = \int_{-0.5}^{0.5} \frac{1}{\sqrt{1-x^2}} dx$$

$$= \sin^{-1} x = 0.524 - (-0.524) = 1.047 \,\text{sec}$$

This verifies the previous solution!

Phase plane analysis of linear systems

Consider the second order linear differential equation

$$\ddot{x} + a\dot{x} + bx = 0 \tag{22}$$

Its solution is

$$\begin{aligned} x(t) &= k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t}, \text{ for } \lambda_1 \neq \lambda_2 \\ x(t) &= k_1 e^{\lambda_1 t} + k_2 t e^{\lambda_1 t}, \text{ for } \lambda_1 = \lambda_2 \end{aligned}$$

where the constants λ_1 and λ_2 are the solutions of the characteristic equation $s^2 + as + b = (s - \lambda_1)(s - \lambda_2) = 0$. The roots could be written as

$$\lambda_1 = \frac{-a + \sqrt{a^2 - 4b}}{2}, \ \lambda_2 = \frac{-a - \sqrt{a^2 - 4b}}{2}$$

For linear systems described by

$$\ddot{x} + a\dot{x} + bx = 0 \qquad \text{(cf. 22)}$$

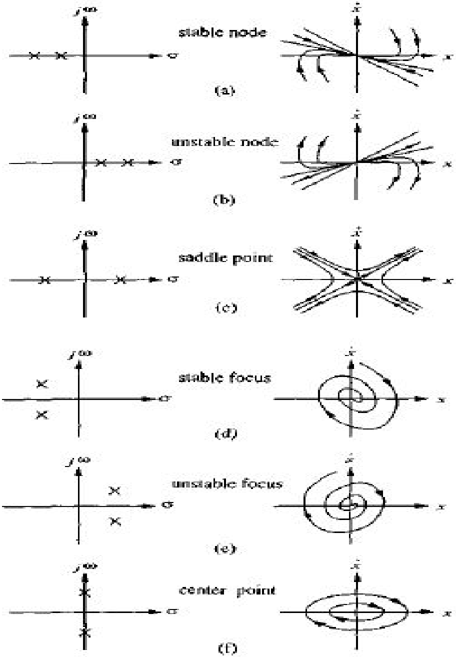
there is only one singular point (assuming $b \neq 0$), namely the origin. Note that (22) has the state space form

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -bx_1 - ax_2 \end{aligned}$$

Its singular points are obtained by solving

$$\begin{aligned} 0 &= x_2 \\ 0 &= -bx_1 - ax_2 \end{aligned}$$

The trajectory in the vicinity of this singularity point can display quite different characteristics, depending on the values of a and b. Root locations and corresponding trajectory behaviors about the origin are given in the next slide.


stable node
(a)
unstable node
(b)
saddle point
(c)
stable focus
(d)
unstable focus
(e)
center point
(f)

Example 3.9

Consider the differential equation set

$$\begin{aligned}\dot{x}_1 &= -2x_1 - x_2 \\ \dot{x}_2 &= -3x_2\end{aligned}$$

Defining

$$\mathbf{x} \triangleq \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The above equations could be written in matrix notation as

$$\dot{\mathbf{x}} = \begin{bmatrix} -2 & -1 \\ 0 & -3 \end{bmatrix} \mathbf{x}$$

Writing the equations in this form allows us to express the solutions in terms of eigenvalues and eigenvectors.

Continued from the previous page

Obtain a general solution of

$$\dot{\mathbf{x}} = \begin{bmatrix} -2 & -1 \\ 0 & -3 \end{bmatrix} \mathbf{x}$$

in terms of eigenstructure. Utilizing this, generate the phase portrait.
A general solution has the form $\mathbf{x}(t) = k_1 \mathbf{v}_1 e^{\lambda_1 t} + k_2 \mathbf{v}_2 e^{\lambda_2 t}$ when $\lambda_1 \neq \lambda_2$, where λ_1, λ_2 are eigenvalues of the system matrix, and $\mathbf{v}_1, \mathbf{v}_2$ are the corresponding eigenvectors.
The eigenvalues satisfy

$$\det(\lambda I - A) = \det\left(\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -2 & -1 \\ 0 & -3 \end{bmatrix}\right) = \det \begin{bmatrix} \lambda + 2 & 1 \\ 0 & \lambda + 3 \end{bmatrix}$$

$$= (\lambda + 2)(\lambda + 3) = 0$$

$$\rightarrow \lambda_1 = -2,\ \lambda_2 = -3$$

Continued from the previous page

The eigenvector $\mathbf{v}_1$ corresponding to $\lambda_1 = -2$ can be found by solving $(\lambda I - A)\mathbf{v} = \mathbf{0}$ at $\lambda = -2$:

$$\left[\begin{array}{cc} \lambda+2 & 1 \\ 0 & \lambda+3 \end{array}\right]_{\lambda=-2} \mathbf{v}_1 = \left[\begin{array}{c} 0 \\ 0 \end{array}\right] \rightarrow \left[\begin{array}{cc} 0 & 1 \\ 0 & 1 \end{array}\right] \mathbf{v}_1 = \left[\begin{array}{c} 0 \\ 0 \end{array}\right]$$

$$\rightarrow \mathbf{v}_1 = c_1 \left[\begin{array}{c} 1 \\ 0 \end{array}\right]$$

The eigenvector corresponding to $\lambda_2 = -3$:

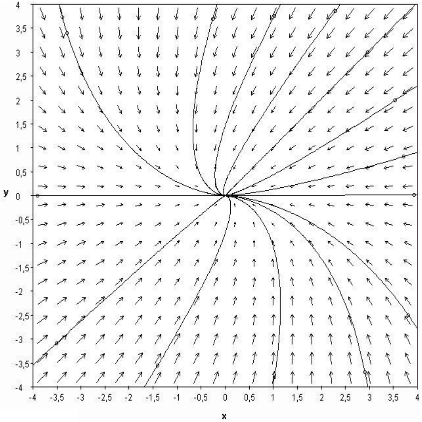
$$\left[\begin{array}{cc} \lambda+2 & 1 \\ 0 & \lambda+3 \end{array}\right]_{\lambda=-3} \mathbf{v}_2 = \left[\begin{array}{c} 0 \\ 0 \end{array}\right] \rightarrow \left[\begin{array}{cc} -1 & 1 \\ 0 & 0 \end{array}\right] \mathbf{v}_2 = \left[\begin{array}{c} 0 \\ 0 \end{array}\right]$$

$$\rightarrow \mathbf{v}_2 = c_2 \left[\begin{array}{c} 1 \\ 1 \end{array}\right]$$

Thus the general solution is:

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-3t}$$

Plotting the general solution for various initial conditions yields the phase portrait shown below. Particularly, notice the progress of trajectories starting on $\mathbf{v}_1$ and $\mathbf{v}_2$.



Example 3.10

Consider

$$\dot{\mathbf{x}} = \begin{bmatrix} -3 & 1 \\ -2 & -2 \end{bmatrix} \mathbf{x}$$

Its eigenvalues are $\lambda_1 = -2.5 + i\sqrt{\frac{7}{4}}$ and $\lambda_2 = -2.5 - i\sqrt{\frac{7}{4}}$ Computing the eigenvector corresponding to λ_1 suffices for the general solution:

$$\mathbf{V}_1 = c_1 \begin{bmatrix} 1 \\ 0.5 + i\sqrt{\frac{7}{4}} \end{bmatrix}$$

A solution to the differential equation is:

$$\mathbf{x}(t) = \begin{bmatrix} 1 \\ 0.5 + i\sqrt{\frac{7}{4}} \end{bmatrix} e^{(-2.5+i\sqrt{\frac{7}{4}})t} = e^{-2.5t} \begin{bmatrix} 1 \\ 0.5 + i\sqrt{\frac{7}{4}} \end{bmatrix} e^{i\sqrt{\frac{7}{4}}t}$$

$$= e^{-2.5t} \begin{bmatrix} 1 \\ 0.5 + i\sqrt{\frac{7}{4}} \end{bmatrix} (\cos\sqrt{\frac{7}{4}}t + i\sin\sqrt{\frac{7}{4}}t)$$

Continued from the previous page

$$\mathbf{x}(t) = e^{-2.5t}\begin{bmatrix} 1 \\ 0.5 + i\sqrt{\frac{7}{4}} \end{bmatrix}(\cos\sqrt{\frac{7}{4}}t + i\sin\sqrt{\frac{7}{4}}t)$$

$$= e^{-2.5t}\left\{\begin{bmatrix} \cos\sqrt{\frac{7}{4}}t \\ 0.5\cos\sqrt{\frac{7}{4}}t - \frac{7}{4}\sin\sqrt{\frac{7}{4}}t \end{bmatrix} + i\begin{bmatrix} \sin\sqrt{\frac{7}{4}}t \\ 0.5\sin\sqrt{\frac{7}{4}}t + \frac{7}{4}\cos\sqrt{\frac{7}{4}}t \end{bmatrix}\right\}$$

Its real and imaginary parts are always linearly independent. Therefore, the general solution can be written as their linear combination:

$$\mathbf{x}(t) = c_1 e^{-2.5t}\begin{bmatrix} \cos\sqrt{\frac{7}{4}}t \\ 0.5\cos\sqrt{\frac{7}{4}}t - \frac{7}{4}\sin\sqrt{\frac{7}{4}}t \end{bmatrix}$$

$$+c_2 e^{-2.5t}\begin{bmatrix} \sin\sqrt{\frac{7}{4}}t \\ 0.5\sin\sqrt{\frac{7}{4}}t + \frac{7}{4}\cos\sqrt{\frac{7}{4}}t \end{bmatrix}$$

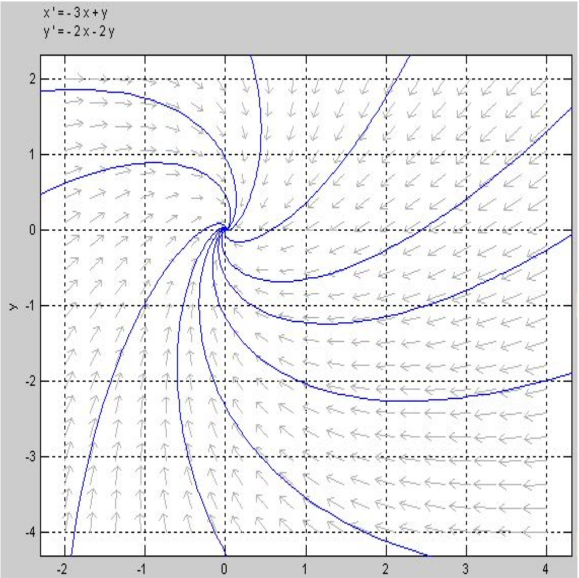


Figure 33: Phase portrait for the example of complex eigenvalues case

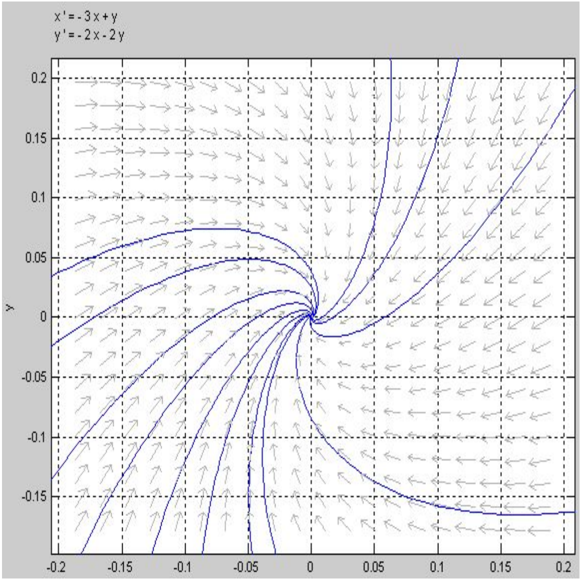


Figure 34: Zoomed phase portrait for the example of complex eigenvalues case

Example 3.11

Stable trajectories may decay faster in certain directions. The one on the right decays faster horizontally. The on in the middle decays faster in vertical direction. The one on the left is heading to the origin directly. We next explain the reason for these behaviors.

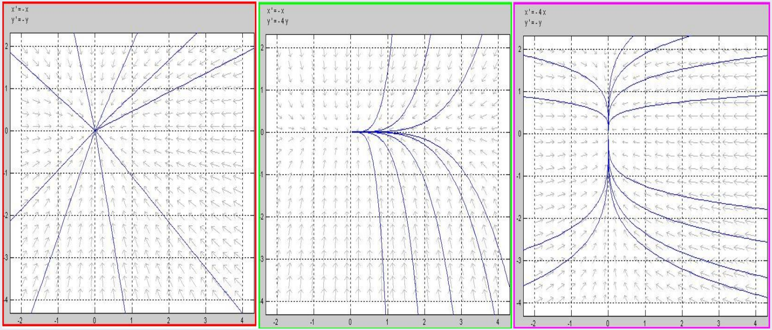
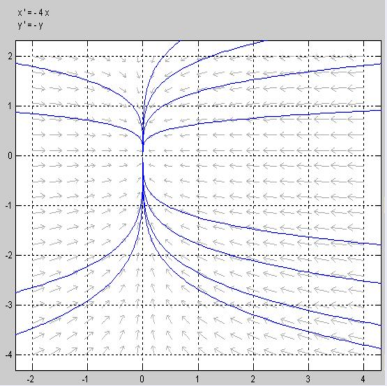


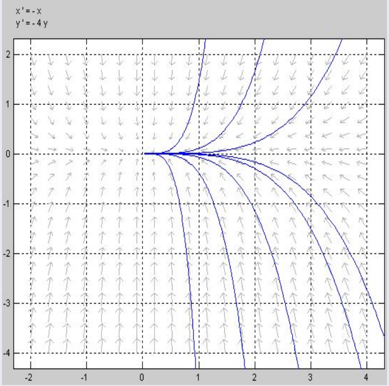
Figure 35: Various trajectory behaviors for negative real eigenvalues



This system has the dynamics $\dot{x} = -4x$, $\dot{y} = -y$ and has a general solution

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-4t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t}$$

$\therefore$ The 1st term dies faster; trajectory decays faster in horizontal direction.



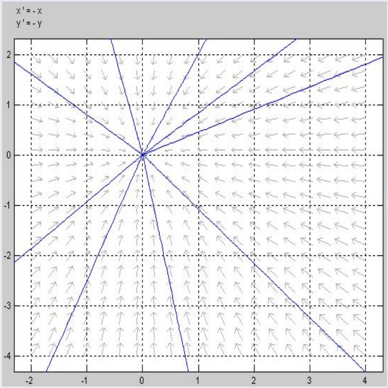
This system has the dynamics $\dot{x} = -x, \ \dot{y} = -4y$ and has a general solution

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-4t}$$

$\therefore$ The 2nd term dies faster; trajectory decays faster in vertical direction.

Continued from the previous page

It is an exercise for a student to find out why the system with dynamics $\dot{x} = -x, \ \dot{y} = -y$ behave in a way different way, compared to the preceding ones.



The solution to a two first order linear differential equations when the real eigenvalues $\lambda_1 \neq \lambda_2$ is

$$\mathbf{x}(t) = c_1 \mathbf{v}_1 e^{\lambda_1 t} + c_2 \mathbf{v}_2 e^{\lambda_2 t}$$

Notice that if, for instance, $\lambda_1 < \lambda_2$ then the solution may be approximated as

$$\mathbf{x}(t) \approx c_2 \mathbf{v}_2 e^{\lambda_2 t}$$

Particularly if both λ_1 and λ_2 are negative,then the motion component in the $\mathbf{v}_1$ direction dies faster, and the motion is represented approximately by the $c_2 \mathbf{v}_2 e^{\lambda_2 t}$ component.

Consider the state space system presented below:

$$\left.\begin{array}{rcl} \dot{x}_1 & = & f_1(x_1, x_2) \\ \dot{x}_2 & = & f_2(x_1, x_2) \end{array}\right\} \qquad \text{(cf. 15)}$$

Let the point $p = (p_1, p_2)$ satisfies

$$\left.\begin{array}{rcl} 0 & = & f_1(x_1, x_2) \\ 0 & = & f_2(x_1, x_2) \end{array}\right\}$$

So, the point $p = (p_1, p_2)$ is an equilibrium point of (15). Also suppose that f_1 and f_2 are continuously differentiable. Expand rhs of (15) into its Taylor series about the point (p_1, p_2).
Next, we will go off-topic and recall Taylor's series expansion.

Digression

Taylor's expansion formula is a way to approximate a function by a polynomial series, centered around a specific point. For a function $f(x)$ and a point a, the Taylor expansion, when exists, is

$$f(x) = f(a) + f'(a)(x-a) + f''(a)(x-a)^2/2! + f'''(a)(x-a)^3/3! + ...$$

where $f'(a)$ is the first derivative of f at a, $f''(a)$ is the second derivative of f at a, and so on.

An example of using Taylor's expansion formula would be to approximate the function e^x near the point $a = 0$. In this case, we have

$$f(x) = e^x,\ f(0) = 1,\ f'(0) = e^0 = 1,\ f''(0) = e^0 = 1,\ f'''(0) = e^0 = 1,\ \ldots$$

So the Taylor expansion of e^x at $a = 0$ is

$$e^x = 1 + x + x^2/2! + x^3/3! + \cdots$$

Taylor's expansion formula for a two-variable function $f(x, y)$ centered around (a, b), when exists, can be expressed as

$$f(x, y) = f(a, b) + (x - a)f_x(a, b) + (y - b)f_y(a, b) + (x - a)^2/2!f_{xx}(a, b)$$

$$+(x - a)(y - b)f_{xy}(a, b) + (y - b)^2/2!f_{yy}(a, b) + ...$$

where $f_x(a, b)$ denotes the partial derivative of f with respect to x evaluated at (a, b), $f_y(a, b)$ is the partial derivative of f with respect to y evaluated at (a, b), $f_{xx}(a, b)$ is the second partial derivative of f with respect to x evaluated at (a, b), and so on.
EOD

$$\left.\begin{array}{rcl}\dot{x}_1 & = & f_1(x_1, x_2) \\ \dot{x}_2 & = & f_2(x_1, x_2)\end{array}\right\} \qquad \text{(cf. 15)}$$

Expand rhs of (15) into its Taylor series about the point (p_1, p_2)

$$\begin{array}{rcl}\dot{x}_1 & = & f_1(p_1, p_2) + a_{11}(x_1 - p_1) + a_{12}(x_2 - p_2) + hot \\ \dot{x}_2 & = & f_2(p_1, p_2) + a_{21}(x_1 - p_1) + a_{22}(x_2 - p_2) + hot\end{array}$$

where

$$a_{11} = \frac{\partial f_1}{\partial x_1}_{x_1=p_1, x_2=p_2}; a_{12} = \frac{\partial f_1}{\partial x_2}_{x_1=p_1, x_2=p_2};$$

$$a_{21} = \frac{\partial f_2}{\partial x_1}_{x_1=p_1, x_2=p_2}; a_{22} = \frac{\partial f_2}{\partial x_2}_{x_1=p_1, x_2=p_2}$$

and *hot* denotes higher order terms.

Since (p_1, p_2) is an EP, we have

$$f_1(p_1, p_2) = 0, \quad f_2(p_1, p_2) = 0$$

Thus

$$\begin{aligned} \dot{x}_1 &= f_1(p_1, p_2) + a_{11}(x_1 - p_1) + a_{12}(x_2 - p_2) + hot \\ \dot{x}_2 &= f_2(p_1, p_2) + a_{21}(x_1 - p_1) + a_{22}(x_2 - p_2) + hot \end{aligned}$$

becomes

$$\begin{aligned} \dot{x}_1 &= a_{11}(x_1 - p_1) + a_{12}(x_2 - p_2) + hot \\ \dot{x}_2 &= a_{21}(x_1 - p_1) + a_{22}(x_2 - p_2) + hot \end{aligned}$$

Using transformations $y_1 = x_1 - p_1$; $y_2 = x_2 - p_2$; the equations become

$$\begin{aligned} \dot{y}_1 &= a_{11}y_1 + a_{12}y_2 + hot \\ \dot{y}_2 &= a_{21}y_1 + a_{22}y_2 + hot \end{aligned}$$

Higher order terms contain $y_1^2, y_2^2, y_1 y_2, y_1^2 y_2, y_1^3, \ldots$, and so on. When (x_1, x_2) is in the neighborhood of (p_1, p_2), the variables y_1, y_2 are in the neighborhood of the origin. In the neighborhood of the origin *hot* can be neglected. The equations become

$$\begin{aligned} \dot{y}_1 &= a_{11} y_1 + a_{12} y_2 \\ \dot{y}_2 &= a_{21} y_1 + a_{22} y_2 \end{aligned}$$

In matrix notation

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{\text{Jacobian matrix of f(x)}} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

In the neigborhood of the equilibrium points nonlinear system behaves as a linear system. It exhibits one of the linear system behaviors we have seen.

Example 3.12

Consider the pendulum equation with friction

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -10\sin x_1 - x_2\end{aligned}$$

Noting that

$$f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} = \begin{bmatrix} x_2 \\ -10\sin x_1 - x_2 \end{bmatrix}$$

we linearize them about the EP $(0,0)$:

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -10\cos x_1 & -1 \end{bmatrix} \rightarrow \frac{\partial f}{\partial x}_{x_1=0, x_2=0} = \begin{bmatrix} 0 & 1 \\ -10 & -1 \end{bmatrix}$$

This matrix has eigenvalues at $-0.5 \pm i3.12$. This leads to the linearized equation

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -10 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Continued from the previous page

Also linearize them about the EP $(\pi, 0)$:

$$\frac{\partial f}{\partial x}_{x_1=\pi, x_2=0} = \underbrace{\begin{bmatrix} 0 & 1 \\ 10 & -1 \end{bmatrix}}$$

This matrix has eigenvalues at -3.7 and 2.7. This leads to the linearized equation

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 10 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

The phase portrait for the nonlinear system is shown in Figure 36. Trajectory behaviors in the vicinity of the equilibrium points are similar to those obtained for the linearized models at the corresponding equilibrium points. Explicitly stating, at the EP $(0, 0)$ linearized model has the eigenvalues $-0.5 \pm i3.12$ which correspond to stable focus behavior. On the other hand, at the EP $(\pi, 0)$ linearized model has the eigenvalues -3.7 and 2.7 which correspond to unstable node behavior.

When the Jacobian has all eigenvalues in the closed LHP, or it has at least one eigenvalue in the closed RHP, then the linearization leads to conclusion on whether NL system's EP attracts all neighboring trajectories or not. Lyapunov's linearization theorem, in the next page, states this.

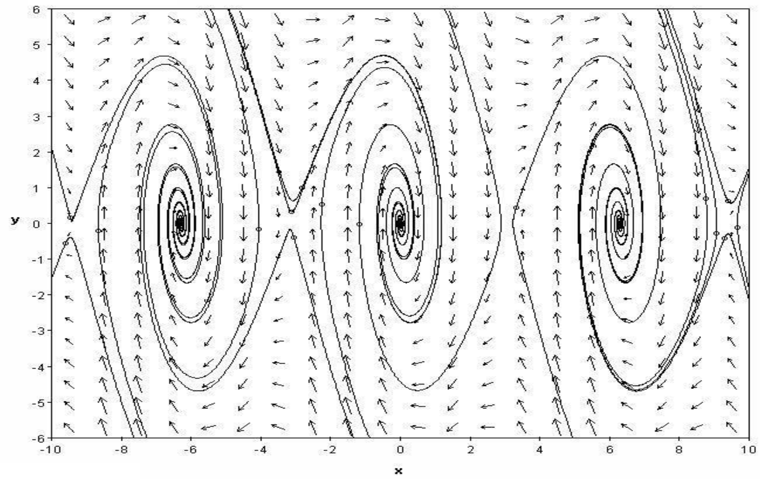


Figure 36: Phase portrait for the pendulum with friction

Basic Sketch of the Lyapunov's Linearization Theorem

If the linearized system's coefficient matrix A has all eigenvalues strictly in the left half complex plane, then in the nonlinear system, the EP attracts the neighboring trajectories.

If the linearized system's coefficient matrix A has at least one eigenvalue strictly in the right half complex plane, then in the nonlinear system, the EP expels some neighboring trajectories.

If the linearized system is marginally stable (i.e., if no eigenvalues of A are in the right half complex plane, and at least one of them is on the imaginary axis), then one cannot conclude anything from the linear approximation (i.e., the EP maybe attracting neighboring trajectories, or maybe expelling some of them).

Example 3.13

Consider the circuit (Figure 37) whose diode characteristics is given by Figure 38. (This corresponds a 10th degree polynomial with coefficients [0.0019 −0.0403 0.3280 −1.2512 1.7495 2.3649 − 10.5278 10.8415 − 3.7931 2.5163 − 0.0070] in MATLAB notation)

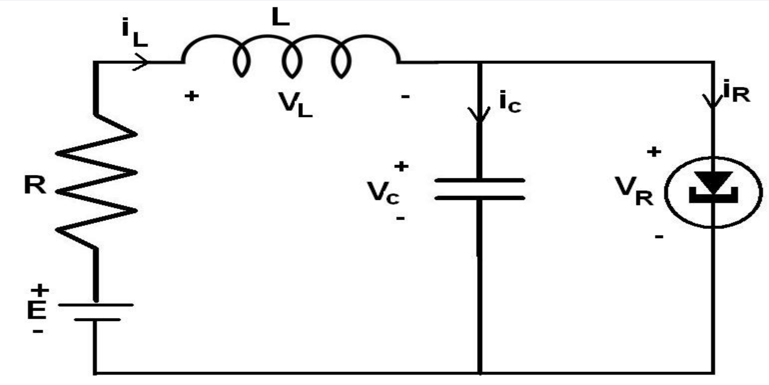


Figure 37: A nonlinear circuit

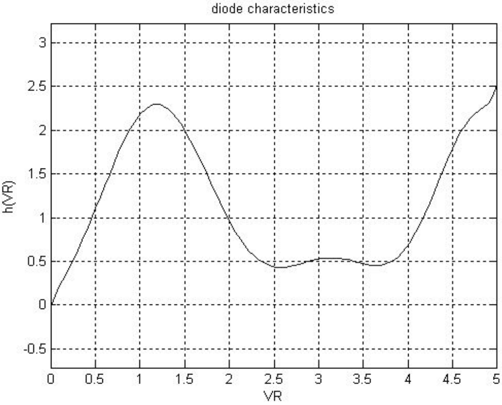


Figure 38: Diode characteristics

The component equations:

$$i_c = C\frac{dV}{dt};\ v_L = L\frac{di}{dt};\ i_R = h(v_R)$$

The component equations:

$$i_c = C\frac{dV}{dt};\ v_L = L\frac{di}{dt};\ i_R = h(v_R)$$

The circuit equations:

$$\begin{array}{rcl} i_c + i_R - i_L & = & 0 \\ v_C - E + Ri_L + v_L & = & 0 \end{array}$$

Definitions: $x_1 := v_C,\ x_2 := i_L$

Circuit equations together with component equations in new variables:

$$\left.\begin{array}{rcl} i_c & = & -h(x_1) + x_2 \\ v_L & = & -x_1 + E - Rx_2 \end{array}\right\} \begin{array}{rcl} \dot{x}_1 & = & \frac{1}{C}\{-h(x_1) + x_2\} \\ \dot{x}_2 & = & \frac{1}{L}\{-x_1 - Rx_2 + E\} \end{array}\Bigg\}$$

Let $C = 1F, L = 1H, R = 2\Omega, E = 5V$.

Let us find the equilibrium points:

$$\begin{array}{rcl} 0 & = & -h(x_1) + x_2 \\ 0 & = & -x_1 - 2x_2 + 5 \end{array} \tag{23}$$

We want to find (x_1, x_2) pairs satisfying both equations in (23). The first one implies $x_2 = h(x_1)$, i.e., x_2 equals the diode characteristics, and the second one implies the line $x_2 = 2.5 - \frac{1}{2}x_1$. The diode characteristics graphics and the line together yield the equilibrium points (Figure 40).

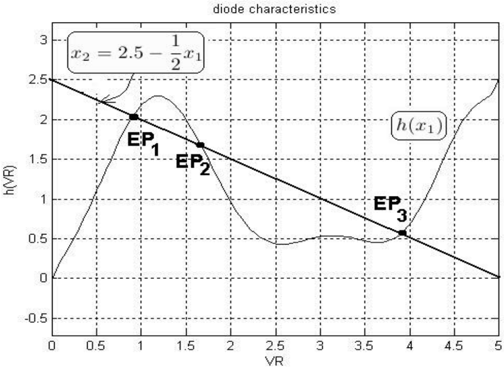


Figure 39: EP's of the circuit

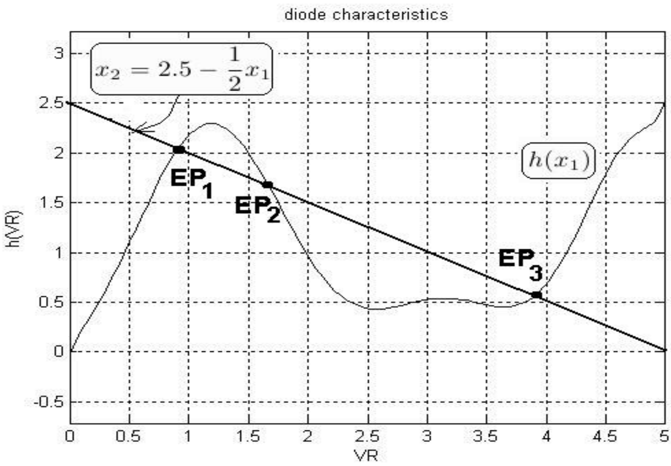


Figure 40: EP's of the circuit

Plotting the phase portrait verifies existence of these equilibrium points.

In the phase plane, a limit cycle is defined as an isolated closed curve. Depending on the motion patterns of the trajectories in the vicinity of the limit cycle, we categorize limit cycles as

1. Stable limit cycle: All trajectories in the vicinity of the limit cycle converge to it as $t \to \infty$
2. Unstable limit cycle: All trajectories in the vicinity of the limit cycle diverge from it as $t \to \infty$
3. Semistable limit cycle: Some of the trajectories in the vicinity converge to it, while the others diverge from it as $t \to \infty$.

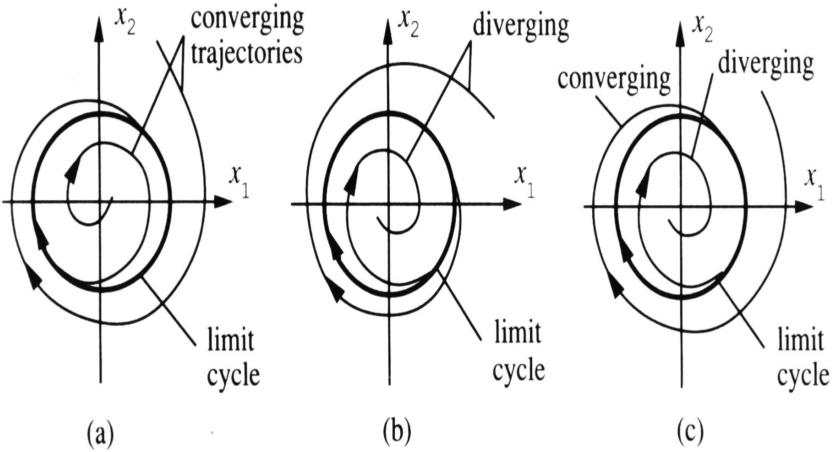


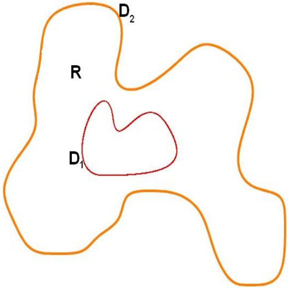
Figure 41: Stable, unstable and semistable limit cycle

Theorem 3.1 (Existence of a limit cycle)

Suppose R is finite region of the plane lying between two simple closed curves D_1 and D_2, and F is the velocity vector field for the system

$$\left.\begin{array}{rcl} \dot{x}_1 & = & f_1(x_1, x_2) \\ \dot{x}_2 & = & f_2(x_1, x_2) \end{array}\right\} \qquad \text{(cf. 15)}$$

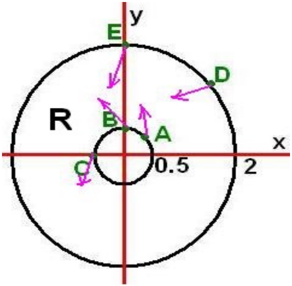
If
(i) at each point of D_1 and D_2, the field F points toward the interior of R, and
(ii) R contains no critical (equilibrium) points
then the system (15) has a closed trajectory lying inside R.



Example 3.14

$$\left.\begin{array}{rcl} \dot{x}_1 & = & -x_2 + x_1(1 - x_1^2 - x_2^2) \\ \dot{x}_2 & = & x_1 + x_2(1 - x_1^2 - x_2^2) \end{array}\right\}$$

Consider the circles with radii $\frac{1}{2}$ and 2 with centers at the origin. Note the directions of the field lines. Also note that the only EP is at $(0,0)$. One can verify that the unit circle is the limit cycle.



Point	Velocity vector
$A : (0.35, 0.35)$	$(\dot{x}_1, \dot{x}_2) = (-0.09, 0.62)$
$B : (0, 0.5)$	$(\dot{x}_1, \dot{x}_2) = (-0.5, 0.38)$
$C : (-0.5, 0)$	$(\dot{x}_1, \dot{x}_2) = (-0.38, -0.5)$
$D : (1.41, 1.41)$	$(\dot{x}_1, \dot{x}_2) = (-5.61, -2.79)$
$E : (0, 2)$	$(\dot{x}_1, \dot{x}_2) = (-2, -6)$

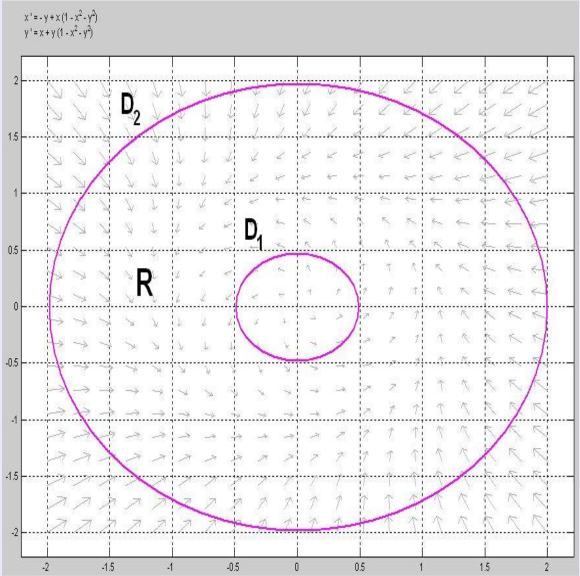


Figure 42: Existence of a limit cycle for Example 3.14

Theorem 3.2 (Nonexistence of limit cycle)

Consider the two dimensional autonomous system

$$\left.\begin{array}{rcl} \dot{x}_1 & = & f_1(x_1, x_2) \\ \dot{x}_2 & = & f_2(x_1, x_2) \end{array}\right\} \qquad \text{(cf 15)}$$

Suppose D is a simply connected open set of R^2. If the expression $\nabla(f_1, f_2) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$ is not identically zero and does not change sign in D, then there are no periodic orbits (limit cycles) of the autonomous system (15) in D.

Example 3.15

$$\left.\begin{array}{rcl} \dot{x}_1 & = & x_1^3 + x_2^3 \\ \dot{x}_2 & = & 3x_1 + x_2^3 + 2x_2 \end{array}\right\}$$

Since $\nabla F = 3x^2 + 3y^2 + 2$ can't be zero and does not change sign anywhere in the xy-plane, there is no closed trajectory in the xy-plane.

Theorem 3.3

A closed trajectory has a critical point in its interior.

Example 3.16

$$\left.\begin{array}{rcl} \dot{x} & = & x^2 + y^2 + 1 \\ \dot{y} & = & x^2 - y^2 \end{array}\right\}$$

$\nabla(f_1, f_2) = 2x - 2y$, which is zero along $x = y$ line.By the nonexistence theorem we are sure that there is no closed trajectory in the region $x > y$ or $x < y$, however we have no conclusion about the regions containing $x = y$. We can invoke the nonexistence theorem 3.3:
Paraphrase of the last theorem "No critical point $\rightarrow$ No surrounding closed trajectory".
This system does not have any critical point in the xy-plane, thus it does not have any closed trajectory in the xy-plane.

Home Exercise Obtain the phase portrait for $\ddot{x} + x - x^3 = 0$. Does it have a limit cycle?

Some definitions The nonlinear system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \tag{24}$$

is said to be autonomous if f does not depend explicitly on time, that is, if the system's state equation can be written as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}). \tag{25}$$

In particular, LTI systems are autonomous, and the LTV systems are nonautonomous.

Example 4.1

$$\left.\begin{array}{rcl} \dot{x}_1 & = & 2x_1 + x_2 \\ \dot{x}_2 & = & t^2 x_1 \end{array}\right\} \text{nonautonomous}$$

Example 4.2

$$\left.\begin{array}{rcl} \dot{x}_1 & = & x_1 + 3x_2 \\ \dot{x}_2 & = & x_1 \sin x_2 \end{array}\right\} \text{autonomous}$$

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \qquad \text{(cf. 25)}$$

Definition 4.1

A state $\mathbf{x}^*$ is an equilibrium state (or EP) of the system (25) if once $\mathbf{x}(t)$ is equal to $\mathbf{x}^*$ it remains equal to $\mathbf{x}^*$ for all future times. This means, $\mathbf{x}^*$ satisfies

$$\mathbf{0} = \mathbf{f}(\mathbf{x}) \tag{26}$$

EP's can be found by solving the nonlinear algebraic equations (26). An LTI system

$$\dot{\mathbf{x}} = A\mathbf{x}$$

has a single EP (the origin 0) if A is nonsingular. If A is singular it has an infinity of EP's. When A is singular

$$\mathbf{0} = A\mathbf{x}$$

has infinitely many solutions. That is, there are infinitely many EP's.

Error Dynamics

Consider the differential equation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \qquad \text{(cf. 25)}$$

Let the solutions be $\mathbf{x}^*(t)$ and $\mathbf{x}(t)$ corresponding to the initial conditions $\mathbf{x_0}$ and $\mathbf{x_0} + \delta$ respectively:

Let $\mathbf{x}^*(t)$ be the solution of (25) corresponding to initial condition $\mathbf{x}^*(0) = \mathbf{x}_0$. $$\dot{\mathbf{x}}^* = \mathbf{f}(\mathbf{x}^*), \quad \mathbf{x}^*(0) = \mathbf{x}_0$$	Let $\mathbf{x}(t)$ be the solution of (25) corresponding to initial condition $\mathbf{x}(0) = \mathbf{x}_0 + \delta$. $$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x}(0) = \mathbf{x}_0 + \delta$$

Let the motion error $\mathbf{e}$ be defined as $\mathbf{e}(t) := \mathbf{x}(t) - \mathbf{x}^*(t)$. In other words, it is the difference between the solutions corresponding to the initial conditions $\mathbf{x_0}$ and $\mathbf{x_0} + \delta$. We show that the difference of the solutions satisfy a differential equation in $\mathbf{e}$. However, this d.e. is not necessariliy autonomous.

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \qquad \text{(cf. 25)}$$

Motion error: $\mathbf{e}(t) := \mathbf{x}(t) - \mathbf{x}^*(t)$.

The error definition leads to

$$\dot{\mathbf{e}} = \mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}^*), \quad \mathbf{e}(0) = \delta$$

This may be written as

$$\dot{\mathbf{e}} \stackrel{\triangle}{=} \mathbf{g}(\mathbf{e}, t), \quad \mathbf{e}(0) = \delta$$

Note that, the error dynamics is nonautonomous. It is illustrated by an example in the next slide.
Since $\mathbf{g}(\mathbf{0}, t) = \mathbf{0}$, the new dynamic system with $\mathbf{e}$ as state and $\mathbf{g}$ in place of $\mathbf{f}$, has an equilibrium point at the origin of the state space.

Example 4.3

Consider the autonomous differential equation $\dot{x} = -x^2$.

$\mathbf{x}^*(t) = \frac{-1}{-1-t}$ is the solution of the equation corresponding to initial condition $x^*(0) = 1$.	$x(t) = \frac{-1}{-0.8-t}$ is the solution of the equation corresponding to initial cond. $x(0) = 1.25$.

Define $e(t) \overset{\triangle}{=} x(t) - x^*(t)$. Then

$$\underbrace{\dot{x}(t) - \dot{x}^*(t)}_{\dot{e}(t)} = -[x(t)]^2 - (-[x^*(t)]^2) = \underbrace{[x^*(t)]^2 - [x(t)]^2}_{(x^*(t)-x(t))(x^*(t)+x(t))}$$

$$\dot{e}(t) = -e(t)\left(\frac{-1}{-0.8-t} + (\frac{-1}{-1-t})\right), \; e(0) = 0.25$$

$$\dot{e}(t) = \underbrace{-e(t)\left(\frac{2t+1.8}{t^2+1.8t+0.8}\right)}_{g(e,t)}, \; e(0) = 0.25$$

The error equation is not autonomous!

Example 4.4

Consider the autonomous mass spring system

$$m\ddot{x} + k_1 x + k_2 x^3 = 0$$

Let $x^*(t)$ with initial position $x^*(0) = x_0$ satisfies the d. e. Also let $x^{**}(t)$ with initial position $x^{**}(0) = x_0 + \delta x_0$ satisfies the d. e.

Define the error by $e(t) \overset{\triangle}{=} x^{**}(t) - x^*(t)$, then the error dynamics is

$$\underbrace{(m\ddot{x}^{**} + k_1 x^{**} + k_2 (x^{**})^3)}_{0} - \underbrace{(m\ddot{x}^{*} + k_1 x^{*} + k_2 x^{*3})}_{0}$$

$$= m\ddot{e} + k_1 e + \underbrace{k_2[e^3 + 3e^2 x^*(t) + 3e x^{*2}(t)]}_{k_2((x^{**})^3 - (x^*)^3)} = 0.$$

Clearly this is a nonautonomous system.

A vector norm on $\mathbb{R}^n$ is a function that assigns to each vector $\mathbf{v} \in \mathbb{R}^n$ a nonnegative real number, called the norm of $\mathbf{v}$ and denoted by $||\mathbf{v}||$, satisfying

(a) $||\mathbf{v}|| \geq 0$, and $||\mathbf{v}|| = 0$ if and only if $\mathbf{v} = \mathbf{0}$

(b) $||c\mathbf{v}|| = |c| \cdot ||\mathbf{v}||$ for any real number c and vector $\mathbf{v}$.

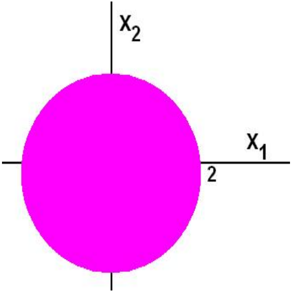
(c) $||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$ for all vectors $\mathbf{u}$ and $\mathbf{v}$.

Let $\mathbf{v} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$. A widely used norm, the Euclidean norm, is defined by

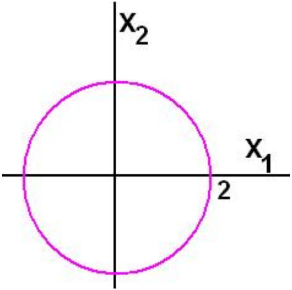
$$||\mathbf{v}|| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

The Euclidean norm may be viewed as a "distance" from the origin.

Let the norm below represent the Euclidean norm.



$$||\mathbf{x}|| < 2 \text{ or } \sqrt{x_1^2 + x_2^2} < 2$$



$$||\mathbf{x}|| = 2 \text{ or } \sqrt{x_1^2 + x_2^2} = 2$$

Interior of a sphere with radius R:

$$||\mathbf{x}|| < R \text{ or } \sqrt{x_1^2 + x_2^2 + x_3^2} < R$$

Surface of a sphere with radius R:

$$||\mathbf{x}|| = R \text{ or } \sqrt{x_1^2 + x_2^2 + x_3^2} = R$$

The idea above can be generalized algebraically to more variables. EOD

Notation Let $\mathbf{B}_R$ denote the spherical region defined by $||\mathbf{x}|| < R$ in the state space, and $\mathbf{S}_R$ the sphere itself, $||\mathbf{x}|| = R$.

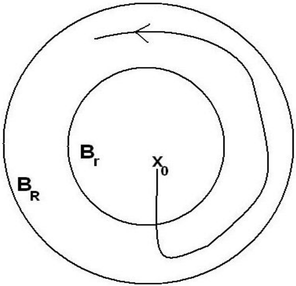
Notice that $\mathbf{B}_R$ is an open set (i.e., not containing the boundaries). This will be regarded in the proof of Lyapunov's 2nd theorem.

Stability

The following stability definitions are for the equilibrium point at the origin. This definition generalizes to the stability of nonzero equilibrium points easily.

Definition 4.2

The equilibrium state $\mathbf{x} = \mathbf{0}$ is said to be stable if, for any $R > 0$, there exists $r > 0$ such that if $||\mathbf{x}(0)|| < r$, then $||\mathbf{x}(t)|| < R$ for all $t \geq 0$. Otherwise the equilibrium point is unstable.



Example 4.5

The Van der Pol oscillator

$$\left.\begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & -x_1 + (1 - x_1^2)x_2 \end{array}\right\}$$

has an equilibrium point at the origin. Looking at the phase portrait, if, for instance, $R = 0.1$, one cannot find any circle centered at the origin with radius r such that all trajectories starting in $||\mathbf{x}(0)|| < r$ remain in $||\mathbf{x}(t)|| < R$ for all $t \geq 0$.

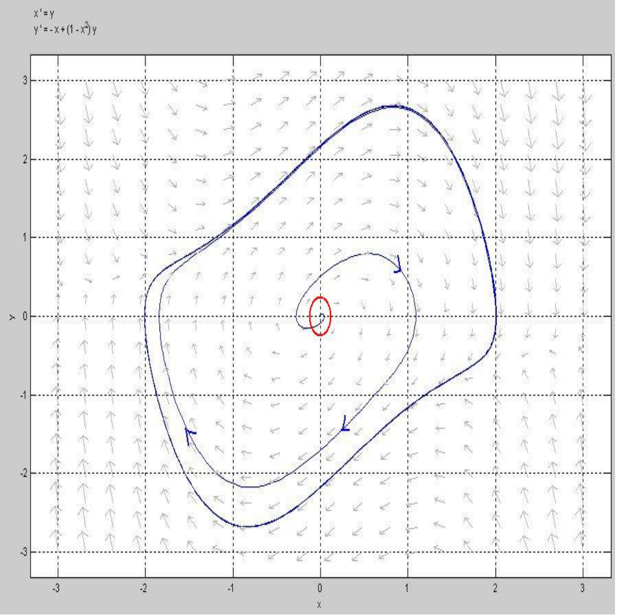


Figure 43: In Example 4.5 we can't keep the trajectory within the specified circle.

Definition 4.3

An equilibrium point $\mathbf{0}$ is asymptotically stable if it is stable, and if in addition there exists some $r > 0$ such that $||\mathbf{x}(0)|| < r$ implies that $\mathbf{x}(t) \to \mathbf{0}$ as $t \to \infty$.

Definition 4.4

An equilibrium point $\mathbf{0}$ is exponentially stable if it is stable and if there exist two strictly positive numbers α and λ such that

$$\forall t > 0, \;\; ||\mathbf{x}(t)|| \leq \alpha ||\mathbf{x}(0)|| e^{-\lambda t}$$

in some ball $\mathbf{B}_r$ around the origin.

Definition 4.5

If asymptotic (or exponential) stability holds for any initial states, the EP is said to be asymptotically (or exponentially) stable in the large. It is also called globally asymptotically (or exponentially) stable.

Stability of nonzero EP's

The definitions on stability can be generalized to nonzero equilibrium points. Consider

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

such that

$$\mathbf{0} = \mathbf{f}(\mathbf{x}_e)$$

That is, $\mathbf{x}_e$ is an equilibrium point of $\mathbf{f}$. Let us do a change of variables: $\mathbf{y} = \mathbf{x} - \mathbf{x}_e$. Then we can write the nonlinear equation as

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y} + \mathbf{x}_e)$$

This system has equilibrium point at $\mathbf{y} = \mathbf{0}$. One can conclude that the EP $\mathbf{x}_e$ of the original system and the EP $\mathbf{0}$ of the transformed system have identical stability properties.

Consider the autonomous system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \tag{cf.25}$$

where $\mathbf{f}(\mathbf{x})$ is continuously differentiable. Using the Taylor's formula, the system dynamics can be written as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{0}) + \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)_{\mathbf{x}=\mathbf{0}} \mathbf{x} + \mathbf{f}_{hot}$$

Noting that $\mathbf{0}$ is an EP we have $\mathbf{f}(\mathbf{0}) = \mathbf{0}$, and $\mathbf{f}_{hot}$ stands for the higher order terms, the above equation becomes

$$\dot{\mathbf{x}} = A\mathbf{x} \tag{27}$$

where $A := \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)_{\mathbf{x}=\mathbf{0}}$ is the Jacobian of $\mathbf{f}$ with respect to $\mathbf{x}$ at $\mathbf{x} = \mathbf{0}$. The Eq. (27) is called the linearization of original nonlinear system (25) at $\mathbf{x} = \mathbf{0}$.

When

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \qquad \text{(cf.25)}$$

linearized at $\mathbf{x} = \mathbf{0}$ we obtain

$$\dot{\mathbf{x}} = A\mathbf{x} \qquad \text{(cf. 27)}$$

where $A := \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)_{\mathbf{x}=\mathbf{0}}$ is the Jacobian of $\mathbf{f}$ with respect to $\mathbf{x}$ at $\mathbf{x} = \mathbf{0}$.

$$\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)_{\mathbf{x}=\mathbf{0}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{\mathbf{x}=\mathbf{0}}$$

The Eq. (27) is called the linearization of original nonlinear system (25) at $\mathbf{x} = \mathbf{0}$.

In the case of nonhomogeneous nonlinear differential equation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}) \tag{28}$$

the linearized system at $\mathbf{x} = \mathbf{0}, \mathbf{u} = \mathbf{0}$ is

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \tag{29}$$

with $A := \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)_{\mathbf{x}=\mathbf{0},\mathbf{u}=\mathbf{0}}$ and $B := \left(\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right)_{\mathbf{x}=\mathbf{0},\mathbf{u}=\mathbf{0}}$ The more explicit expressions for $\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)_{\mathbf{x}=\mathbf{0},\mathbf{u}=\mathbf{0}}$ and $\left(\frac{\partial \ \mathbf{f}}{\partial \mathbf{u}}\right)_{\mathbf{x}=\mathbf{0},\mathbf{u}=\mathbf{0}}$ are

$$\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)_{\mathbf{x}=\mathbf{0},\mathbf{u}=\mathbf{0}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{\mathbf{x}=\mathbf{0},\mathbf{u}=\mathbf{0}},$$

$$\left(\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right)_{\mathbf{x}=\mathbf{0},\mathbf{u}=\mathbf{0}} = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \cdots & \frac{\partial f_2}{\partial u_m} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial f_n}{\partial u_1} & \frac{\partial f_n}{\partial u_2} & \cdots & \frac{\partial f_n}{\partial u_m} \end{bmatrix}_{\mathbf{x}=\mathbf{0},\mathbf{u}=\mathbf{0}}$$

The following theorem (Lyapunov's linearization theorem) states some uses of the linearized systems:

Theorem 4.1

If the linearized system is strictly stable (i.e., if all eigenvalues of A are strictly in the left half complex plane), then the EP is asymptotically stable (for the actual nonlinear system).
If the linearized system is unstable (i.e., if at least one eigenvalue of A is strictly in the right half complex plane), then the EP is unstable (for the actual nonlinear system).
If the linearized system is marginally stable (i.e., if all eigenvalues of A are in the left half complex plane, but at least one of them is on the $j\omega$ axis), then one cannot conclude anything from the linear approximation (the EP may be stable, asymptotically stable, or unstable for the nonlinear system).

One should observe that this theorem formalizes the ideas we studied in the section "Behavior of nonlinear systems near equilibrium points"

Recap

Linearized system $\dot{\mathbf{x}} = A\mathbf{x}$ with all eigenvalues of A are strictly in the left half complex plane $\rightarrow$ The EP of the actual nonlinear system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is asymptotically stable.

Linearized system $\dot{\mathbf{x}} = A\mathbf{x}$ with at least one eigenvalue of A is strictly in the right half complex plane $\rightarrow$ The EP of the actual nonlinear system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is unstable.

Linearized system $\dot{\mathbf{x}} = A\mathbf{x}$ with all eigenvalues of A are in the left half complex plane, but at least one of them is on the imaginary axis $\rightarrow$ One cannot conclude anything from the linear approximation (the EP of the actual nonlinear system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ may be stable, asymptotically stable, or unstable.)

Example 4.6

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -4x_1 - 2x_2 + 4 \\ x_1 x_2 \end{bmatrix}$$

Its equilibrium points satisfy

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -4x_1 - 2x_2 + 4 \\ x_1 x_2 \end{bmatrix}$$

which results in the EP's $(0,2)$ and $(1,0)$. Jacobians at these equilibrium points are

$$\begin{bmatrix} -4 & -2 \\ x_2 & x_1 \end{bmatrix}_{(0,2)} = \begin{bmatrix} -4 & -2 \\ 2 & 0 \end{bmatrix} \rightarrow \lambda_{1,2} = -2, -2 \therefore \text{ stable}$$

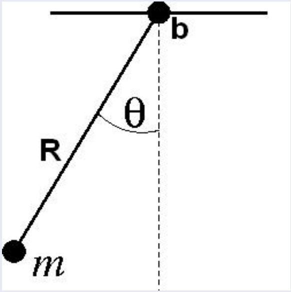
$$\begin{bmatrix} -4 & -2 \\ x_2 & x_1 \end{bmatrix}_{(1,0)} = \begin{bmatrix} -4 & -2 \\ 0 & 1 \end{bmatrix} \rightarrow \lambda_{1,2} = -4, 1 \therefore \text{ unstable}$$

We note that Jacobian can be constructed at EP's not necessarily at the origin, and it leads to the same conclusion.

Example 4.7

Recall that the simple plane pendulum has the dynamics

$$MR^2\ddot{\theta} + b\dot{\theta} + MgR\sin\theta = 0$$



Its equilibrium point $(\theta, \dot{\theta}) = (\pi, 0)$ is unstable. Let us show it by using linearization theorem. The above system could be linearized by linearizing the nonlinear term $\sin\theta$ at the equilibrium point.
At $(\theta, \dot{\theta}) = (\pi, 0)$ we can write the Taylor expansion of $\sin\theta$ as:

$$\sin\theta = \sin\pi + [\cos\theta]_{\theta=\pi}(\theta - \pi) + \text{ h.o.t.} = 0 + (-1)(\theta - \pi) + \text{ h.o.t.}$$

That is, the linearization gives $\sin\theta = -(\theta - \pi)$ at $(\theta, \dot{\theta}) = (\pi, 0)$. The linearized equation of the pendulum becomes

$$MR^2\ddot{\theta} + b\dot{\theta} - MgR(\theta - \pi) = 0$$

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The linearized equation of the pendulum becomes

$$MR^2\ddot{\theta} + b\dot{\theta} - MgR(\theta - \pi) = 0$$

$$\ddot{\theta} + \frac{b}{MR^2}\dot{\theta} - \frac{g}{R}(\theta - \pi) = 0$$

Letting $\tilde{\theta} := \theta - \pi$, the system's linearization about the the EP $(\theta, \dot{\theta}) = (\pi, 0)$ becomes

$$\ddot{\tilde{\theta}} + \frac{b}{MR^2}\dot{\tilde{\theta}} - \frac{g}{R}\tilde{\theta} = 0$$

$\therefore$ The linear approximation is unstable, and therefore so is the nonlinear system at this EP. We next study the same example using the state space notation.

Example 4.8

$$MR^2\ddot{\theta} + b\dot{\theta} + MgR\sin\theta = 0$$

Define $x_1 \triangleq \theta, x_2 \triangleq \dot{\theta}$, and write the d.e. in the state space form

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\tfrac{g}{R}\sin x_1 - \tfrac{b}{MR^2}x_2 \end{aligned}$$

Its equilibrium points are obtained by solving

$$\begin{aligned} 0 &= x_2 \\ 0 &= -\tfrac{g}{R}\sin x_1 - \tfrac{b}{MR^2}x_2 \end{aligned}$$

The equilibrium points are $\{(0,0),(\pi,0),(2\pi,0),\dots\}$. Let us linearize the d.e. at the EP $(\pi,0)$:

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}_{(x_1,x_2)=(\pi,0)} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{R}\cos x_1 & -\frac{b}{MR^2} \end{bmatrix}_{(x_1,x_2)=(\pi,0)} = \begin{bmatrix} 0 & 1 \\ \frac{g}{R} & -\frac{b}{MR^2} \end{bmatrix}$$

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Using the Jacobian, we write the linearized equation as

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ \frac{g}{R} & -\frac{b}{MR^2} \end{bmatrix} \mathbf{x}$$

Eigenvalues of A satisfies

$$\det\left(\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ \frac{g}{R} & -\frac{b}{MR^2} \end{bmatrix}\right) = 0$$

This results in the characteristic equation:

$$\lambda(\lambda + \frac{b}{MR^2}) - \frac{g}{R} = \lambda^2 + \frac{b}{MR^2}\lambda - \frac{g}{R} = 0$$

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$$\lambda(\lambda + \frac{b}{MR^2}) - \frac{g}{R} = \lambda^2 + \frac{b}{MR^2}\lambda - \frac{g}{R} = 0$$

Because the coefficients of the characteristic equation are not of the same sign, at least one of its solutions is in the right half complex plane. An eigenvalue in the RHP means instability of the linear system. Unstable linear system implies that the EP $(\pi, 0)$ of the nonlinear system is not stable.

We did the linearization directly at $(x_1, x_2) = (\pi, 0)$. Transforming $(x_1, x_2) = (\pi, 0)$ to $(y_1, y_2) = (0, 0)$ gives the same result (see the following exercise).

Home Exercise Redo the example above after transforming the EP $(\pi, 0)$ to the origin by the transform $(y_1, y_2) = (x_1, x_2) - (\pi, 0)$.

Example 4.9

Consider the 1st order system

$$\dot{x} = ax + bx^5 \tag{30}$$

Its EP satisfies $0 = ax + bx^5$. The origin 0 is one of the solutions, therefore,0 is an equilibrium point of the system. Let $f(x) \triangleq ax + bx^5$. Then (30) can be linearized by

$$\dot{x} = f(0) + \left.\frac{df}{dx}\right|_{x=0} x + h.o.t.$$

This yields

$$\dot{x} = ax$$

Stability of the system depends on a. The system has the following stability properties:
$a < 0$: asymptotically stable;
$a > 0$: unstable;
$a = 0$: cannot tell from linearization.

Lyapunov's Direct Method

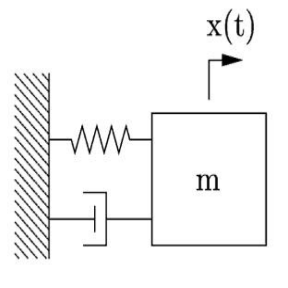
Lyapunov's method is useful in investigating nonlinear systems' stability and designing controllers for them. It is based on the observation: if the total energy of a system is continuously dissipated, then the system, whether linear or nonlinear, must eventually settle down to an equilibrium point.

Example 4.10

Consider the mass-damper-spring system

$$m\ddot{x} + b\dot{x}|\dot{x}| + k_0 x + k_1 x^3 = 0$$

with $b\dot{x}|\dot{x}|$ representing nonlinear dissipation, and $k_0 x + k_1 x^3$ representing nonlinear spring term. Assume that the mass is pulled away from the natural length of the spring by a large distance and then released. Will the resulting motion be stable?



For the analysis, note that the total mechanical energy of the system is the sum of its kinetic energy and its potential energy

$$V(\mathbf{x}) = \frac{1}{2}m\dot{x}^2 + \int_0^x (k_0 x + k_1 x^3)dx = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}k_0 x^2 + \frac{1}{4}k_1 x^4.$$

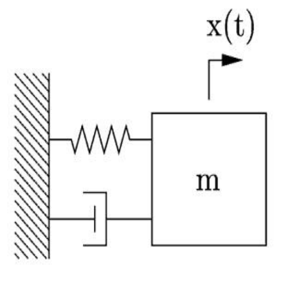
What happens to this energy as time progresses?

Continued from the previous page

The total mechanical energy of the system is the sum of its kinetic energy and its potential energy

$$V(\mathbf{x}) = \frac{1}{2}m\dot{x}^2 + \int_0^x (k_0 x + k_1 x^3)dx = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}k_0 x^2 + \frac{1}{4}k_1 x^4.$$

What happens to this energy as time progresses?
The time derivative may reveal the answer:



$$\dot{V}(\mathbf{x}) = m\dot{x}\ddot{x} + (k_0 x + k_1 x^3)\dot{x} = \dot{x}(-b\dot{x}|\dot{x}|) = -b|\dot{x}|^3$$

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$$\dot{V}(\mathbf{x}) = m\dot{x}\ddot{x} + (k_0 x + k_1 x^3)\dot{x} = \dot{x}(-b\dot{x}|\dot{x}|) = -b|\dot{x}|^3$$

This expression reveals that energy is continuously dissipated by the damper term. Physically, one may predict that the spring finally will settle down to its original length.

Notice that as long as $\dot{x} \neq 0$ (i.e., as long as motion continues), which is the situation until reaching the EP state, the energy decreases.

Knowing the energy of the system helped us in the analysis. In many systems we may not have analytic expressions for the energy; and sometimes energy may have no meaning at all for them. This motivates a systematic approach independent of the systems' physics.

Positive Definite Functions and Lyapunov Functions

The energy function in mass-damper-spring system has two properties.
(1) It is strictly positive unless both state variables x and $\dot{x}$ are zero:

$$V(\mathbf{x}) = \frac{1}{2}m\dot{x}^2 + \int_0^x (k_0 x + k_1 x^3)dx = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}k_0 x^2 + \frac{1}{4}k_1 x^4.$$

(2) It is monotonically decreasing when the variables x and $\dot{x}$ vary according to system dynamics. That is, if the solution of

$$m\ddot{x} + b\dot{x}|\dot{x}| + k_0 x + k_1 x^3 = 0$$

is used in the energy formula we observe that energy decreases monotonically. It is quantified by

$$\dot{V}(\mathbf{x}) = -b|\dot{x}|^3$$

The first property is formalized by the notion of positive definite functions, and the second is formalized by Lyapunov functions.

Definition 4.6

A scalar continuous function $V(\mathbf{x})$ is said to be locally positive definite if $V(\mathbf{0}) = 0$ and in a ball $\mathbf{B}_{R_0}$

$$\mathbf{x} \neq \mathbf{0} \rightarrow V(\mathbf{x}) > 0$$

If $V(\mathbf{0}) = 0$ and the above property holds over the whole state space, then $V(\mathbf{x})$ is said to be globally positive definite.

Example 4.11

Consider

$$V(x_1, x_2) = x_1^2 + x_2^2$$

Clearly, $V(\mathbf{0}) = 0$ and in any ball $\mathbf{B}_{R_0}$ and we have

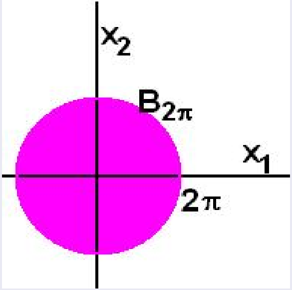
$$\mathbf{x} \neq \mathbf{0} \rightarrow V(\mathbf{x}) > 0$$

Example 4.12

The mechanical energy of of the pendulum:

$$V(\mathbf{x}) = \frac{1}{2}MR^2x_2^2 + MRg(1 - \cos x_1)$$

It is locally positive definite in the ball $\mathbf{B}_{2\pi}$.



Example 4.13

The mechanical energy of mass-damper-spring system

$$V(\mathbf{x}) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}k_0x^2 + \frac{1}{4}k_1x^4.$$

is globally positive definite.

Example 4.14

Note that the kinetic energy

$$V_k = \frac{1}{2}m\dot{x}^2$$

is not positive definite, because it is equal to zero for nonzero values of $\mathbf{x}$. For example, when $(x, \dot{x}) = (a, 0)$ with $a \neq 0$ we have $V_k(\mathbf{x}) = 0$.

The locally pd function V has unique minimum at the origin $\mathbf{0}$. Actually, given any function having a unique minimum in a certain ball, we can construct a locally pd function simply by adding a constant to that function.

Example 4.15

The function $V(\mathbf{x}) = x_1^2 + x_2^2 - 1$ is a lower bounded function with a unique minimum at the origin, and addition of the constant 1 makes it a pd function.

Geometrically thinking, consider a pd function $V(\mathbf{x})$ of two state variables x_1 and x_2 plotted in a 3d space. It looks like an upward cup (Fig. 44).

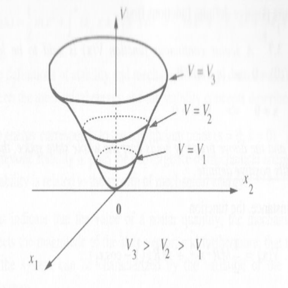


Figure 44: A geometric interpretation of a Lyapunov function

Another geometrical interpretation is as follows. Taking x_1 and x_2 as Cartesian coordinates, the level curves $V(x_1, x_2) = V_\alpha$ typically represents a set of ovals surrounding the origin, with each corresponding to a positive value of V_α (Fig. 45).

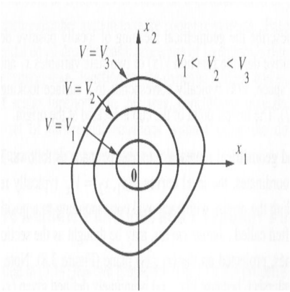


Figure 45: A level curves interpretation of a Lyapunov function

Definition 4.7

A function $V(\mathbf{x})$ is said to be negative definite if $-V(\mathbf{x})$ is positive definite; $V(\mathbf{x})$ is positive semidefinite if $V(\mathbf{0}) = 0$ and $V(\mathbf{x}) \geq 0$ for $\mathbf{x} \neq \mathbf{0}$; $V(\mathbf{x})$ is negative semidefinite if $-V(\mathbf{x})$ is positive semidefinite.

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \qquad \text{(cf.25)}$$

Definition 4.8

If in a ball $\mathbf{B}_{R_0}$, the function $V(\mathbf{x})$ is positive definite and has continuous partial derivatives, and if its time derivative along any state trajectory of system (25) is negative semidefinite, i.e., $\dot{V}(\mathbf{x}) \leq 0$ then $V(\mathbf{x})$ is said to be a Lyapunov function for the system (25).

One should see that as **x** evolves in time the value of $V(x_1, x_2)$ points down the ball (Fig. 46).

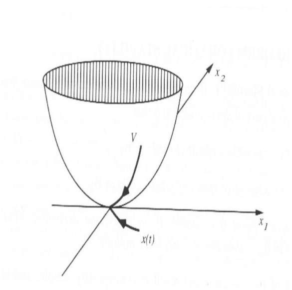


Figure 46: A Lyapunov function as **x** evolves

A geometric interpretation of a Lyapunov function is that in Fig. 46, the point denoting the value of $V(x_1, x_2)$ points down a bowl.

Equilibrium Point Theorems

Lyapunov theorem for local stability

Theorem 4.2

If in a ball $\mathbf{B}_{R_0}$, there exists a scalar function $V(\mathbf{x})$with continuous first partial derivatives such that
- $V(\mathbf{x})$ is positive definite (locally in $\mathbf{B}_{R_0}$)
- $\dot{V}(\mathbf{x})$ is negative semidefinite (locally in $\mathbf{B}_{R_0}$)

then the equilibrium point $\mathbf{0}$ is stable. If, actually, the derivative $\dot{V}(\mathbf{x})$ is locally negative definite in $\mathbf{B}_{R_0}$, then stability is asymptotic.

In applying the above theorem for analysis of a nonlinear system, one goes through the two steps of choosing a positive definite function, and then determining its derivative along the path of the nonlinear system.

Example 4.16

$$\begin{aligned}\dot{x} &= -x + y + xy \\ \dot{y} &= x - y - x^2 - y^3\end{aligned}$$

It has an equilibrium point at $(0, 0)$. To see whether it is stable or not, try the Lyapunov function $V = x^2 + y^2$. Then

$$\begin{aligned}\dot{V} &= 2x\dot{x} + 2y\dot{y} \\ &= 2x(-x + y + xy) + 2y(x - y - x^2 - y^3) \\ &= -2x^2 + 4xy - 2y^2 - 2y^4 \\ &= -2(x - y)^2 - 2y^4 \\ &< 0\end{aligned}$$

$\therefore$ the equilibrium point $(0, 0)$ is asymptotically stable.

Example 4.17

A simple pendulum with viscous damping is described by

$$\ddot{\theta} + \dot{\theta} + \sin\theta = 0 \tag{31}$$

Consider the following scalar function

$$V(\mathbf{x}) = (1 - \cos\theta) + \frac{\dot{\theta}^2}{2}$$

This function is locally positive definite in the ball $\mathbf{B}_{2\pi}$. As a matter of fact, this function represents total energy of the pendulum; sum of its potential and kinetic energies. Its time derivative is

$$\dot{V}(\mathbf{x}) = \dot{\theta}\sin\theta + \dot{\theta}\ddot{\theta} = \dot{\theta}\underbrace{(\sin\theta + \ddot{\theta})}_{-\dot{\theta} \text{ by } (31)} = -\dot{\theta}^2 \le 0$$

By invoking the above theorem, one concludes that the origin is a stable EP. However, one cannot draw a conclusion on the asymptotic stability of the system, because $\dot{V}(\mathbf{x})$ is only negative semidefinite.

Example 4.18

Consider the system defined by

$$\begin{aligned} \dot{x}_1 &= x_1(x_1^2 + x_2^2 - 2) - 4x_1x_2^2 \\ \dot{x}_2 &= 4x_1^2x_2 + x_2(x_1^2 + x_2^2 - 2) \end{aligned}$$

has EP at the origin. Given the positive definite function

$$V(x_1, x_2) = x_1^2 + x_2^2$$

one can find its derivative along the trajectory of the system as

$$\dot{V}(\mathbf{x}) = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 = 2(x_1^2 + x_2^2)(x_1^2 + x_2^2 - 2)$$

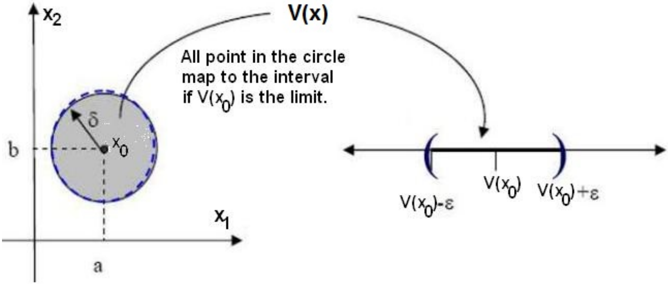
Thus $\dot{V}(\mathbf{x})$ is locally negative definite in the two dimensional ball $\mathbf{B}_2$. Therefore, the above theorem indicates that the origin is asymptotically stable.

Digression: Multivariable function's continuity

A function $V : \mathbb{R}^n \to \mathbb{R}$ is continuous at $\mathbf{x}_0 \in \mathbb{R}^n$ if $V(\mathbf{x}_0)$ exists and

$$\lim_{\mathbf{x} \to \mathbf{x}_0} V(\mathbf{x}) = V(\mathbf{x}_0)$$

Using ϵ, δ style, a consequence of continuity of V at $\mathbf{x}_0$ is that for every given $\epsilon > 0$ we can find $\delta > 0$ such that for any $\mathbf{x}$ in $||\mathbf{x} - \mathbf{x}_0|| < \delta$ we have $|V(\mathbf{x}) - V(\mathbf{x}_0)| < \epsilon$.



EOD

Proof of the Lyapunov Theorem

Recall the hypotheses: In a ball $\mathbf{B}_{R_0}$, there exists a scalar function $V(\mathbf{x})$ with continuous first partial derivatives such that

- $V(\mathbf{x})$ is positive definite (locally in $\mathbf{B}_{R_0}$)
- $\dot{V}(\mathbf{x})$ is negative semidefinite (locally in $\mathbf{B}_{R_0}$)

Let R be a positive number such that $R < R_0$. Then $\mathbf{S}_R$, surface of a sphere with radius R, is inside the ball $\mathbf{B}_{R_0}$. Because V is continuous in the ball $\mathbf{B}_{R_0}$, it is particularly continuous on $\mathbf{S}_R$. The surface $\mathbf{S}_R$ is closed and bounded (i.e., compact), therefore, the continuous function V achieves its minimum on it (Weierstrass theorem). Let the value of this minimum be m. Since V is positive definite, its minimum m is positive. Since V is continuous, in particular at the origin, for any $m > 0$ there exists a $\delta > 0$ such that

$$\mathbf{x}_0 \in \mathbf{B}_\delta \to |V(\mathbf{x}_0) - V(\mathbf{0})| = \underbrace{V(\mathbf{x}_0) < m}_{(*)}$$

Since V is continuous, in particular at the origin, for any $m > 0$ there exists a $\delta > 0$ such that

$$\mathbf{x}_0 \in \mathbf{B}_\delta \to |V(\mathbf{x}_0) - V(\mathbf{0})| = \underbrace{V(\mathbf{x}_0) < m}_{(*)}$$

We claim that for the initial condition $\mathbf{x}_0 \in \mathbf{B}_\delta$ the resulting trajectory never exits the ball $\mathbf{B}_R$, consequently it never exits the ball $\mathbf{B}_{R_0}$. For the sake of contradiction, suppose that the trajectory exits the ball $\mathbf{B}_R$. If trajectory exits the ball $\mathbf{B}_R$ then there exists a time T such that it crosses the surface $\mathbf{S}_R$.

If trajectory exits the ball $\mathbf{B}_R$ then there exists a time T such that it crosses the surface $\mathbf{S}_R$. Then for the exiting trajectory $\mathbf{x}^*$, we have $\underbrace{V(\mathbf{x}^*(T)) \geq m}_{(**)}$. But the derivative of V with respect to time, that is $\dot{V}$, is negative semi-definite, hence V is non-increasing along the corresponding trajectory (that is, $\underbrace{V(\mathbf{x}^*(T)) \leq V(\mathbf{x}^*(t))}_{(***)}$, for all $0 \leq t \leq T$. Particularly $\underbrace{V(\mathbf{x}^*(T)) \leq V(\mathbf{x}_0)}_{(***)}$ holds. Therefore, (*), (**), and (***) together imply

$$m \leq V(\mathbf{x}^*(T)) \leq V(\mathbf{x}_0) < m$$

which is a contradiction, because $V(\mathbf{x}_0)$ can't be both smaller than m, and greater than or equal to m. Hence, the trajectory is contained in the ball $\mathbf{B}_R$ (therefore contained in the ball $\mathbf{B}_{R_0}$). QED

A radially unbounded function is a function $f : \mathbb{R}^n \to \mathbb{R}$ for which $\|\mathbf{x}\| \to \infty \Rightarrow f(\mathbf{x}) \to \infty$.
The function

$$f(\mathbf{x}) = x_1^2 + 2x_2^2 + 5x_3^2$$

is radially unbounded. However, the functions

$$f_1(\mathbf{x}) = (x_1 - x_2)^2$$

$$f_2(\mathbf{x}) = (x_1^2 + x_2^2)/(1 + x_1^2 + x_2^2) + (x_1 - x_2)^2$$

are not radially unbounded since along the line $x_1 = x_2$, the condition is not verified even though the second function is globally positive definite.

EOD

Lyapunov Theorems for Global Stability

Theorem 4.3

Assume that there exists a scalar function V of the state $\mathbf{x}$, with continuous first order partial derivatives such that

- $V(\mathbf{x})$ is positive definite
- $\dot{V}(\mathbf{x})$ is negative definite
- $V(\mathbf{x}) \to \infty$ as $||\mathbf{x}|| \to \infty$

then the equilibrium at the origin is globally asymptotically stable.

The reason for the radial unboundedness condition is to assure that the contour curves (or the contour surfaces in the case of higher order systems) $V(\mathbf{x}) = V_\alpha$ correspond to closed curves. If the curves are not closed, it is possible for the state trajectories to drift away from the EP, even though the state keeps going through contours corresponding to smaller and smaller V_α's. For example the positive definite function $V = [\frac{x_1^2}{1+x_1^2}] + x_2^2$, the curves $V(\mathbf{x}) = V_\alpha$ for $V_\alpha > 1$ are open curves. Figure 47 shows that an initial state can diverge from the equilibrium state at the origin while moving towards lower energy curves.

Figure 47 shows the divergence of the state while moving toward lower energy curves.

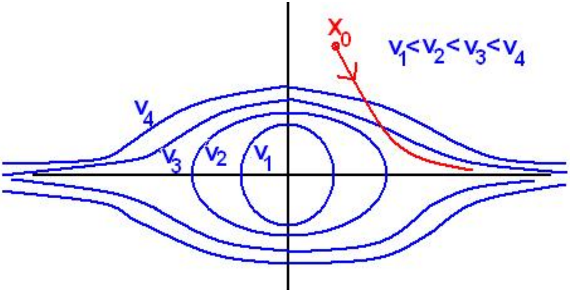


Figure 47: Radially unboundedness condition

Example 4.19

Consider

$$\left.\begin{array}{rcl} \dot{x}_1 &=& (x_2-1)x_1^3 \\ \dot{x}_2 &=& -\frac{x_1^4}{(1+x_1^2)^2}-\frac{x_2}{1+x_2^2} \end{array}\right\} \tag{32}$$

Let us try the pd function

$$V(\mathbf{x})=\frac{x_1^2}{1+x_1^2}+x_2^2$$

Its time derivative is

$$\dot{V}(\mathbf{x})=2\left[\frac{x_1}{1+x_1^2}-\frac{x_1^3}{(1+x_1^2)^2}\right]\dot{x}_1+2x_2\dot{x}_2$$

Along the system trajectories it becomes

$$\dot{V}(\mathbf{x})=-2\frac{x_1^4}{(1+x_1^2)^2}-2\frac{x_2^2}{1+x_2^2}$$

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$$\dot{V}(\mathbf{x}) = -2\frac{x_1^4}{(1+x_1^2)^2} - 2\frac{x_2^2}{1+x_2^2}$$

$\dot{V}$ is negative definite, so, the system is asymptotically stable. However, we cannot conclude that the system is globally asymptotically stable. The pd function $V(\mathbf{x}) = \frac{x_1^2}{1+x_1^2} + x_2^2$ is not radially unbounded therefore, it does not satisfy the third condition of the theorem on being globally asymptotically stable. On the other hand, we cannot conclude that it is not globally asymptotically stable. Such a conclusion requires further analysis. performing a further analysis reveals that the system (32) is not globally asymptotically stable. For instance, for the initial condition $(x_1, x_2) = (3, 1.5)$, the trajectory does not go to the origin (see Figure 48)

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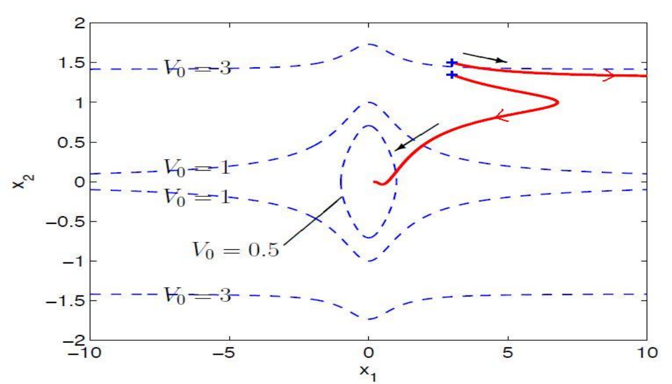

Figure 48: Energy curves and trajectories for Example 4.19

Example 4.20

Consider the NL system

$$\dot{x} + c(x) = 0$$

where c is any continuous function of the same sign as its scalar argument x, i.e.,

$$xc(x) > 0 \text{ for } x \neq 0$$

Since c is continuous, it implies $c(0) = 0$.

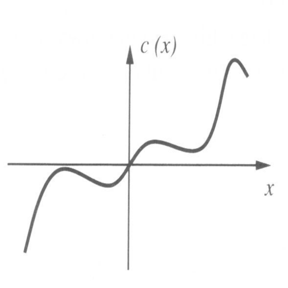

Figure 49: The function $c(x)$

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Consider the Lyapunov function $V(x) = x^2$. This is radially unbounded since it tends to infinity as $|x| \to \infty$. Its derivative is $\dot{V} = 2x\dot{x} = -2xc(x)$. Thus $\dot{V} < 0$ as long as $x \neq 0$, so that $x = 0$ is a globally asymptotically stable EP. For instance, the system

$$\dot{x} = \underbrace{\sin^2 x - x}_{-c(x)}$$

is globally asymptotically convergent to $x = 0$, since for $x \neq 0$, $c(x) \triangleq x - \sin^2 x$ satisfies $xc(x) > 0$. Similarly, the system

$$\dot{x} = \underbrace{-x^3}_{-c(x)}$$

is globally asymptotically convergent to $x = 0$. Notice that this system's linear approximation $\dot{x} = 0 \cdot x$ is inconclusive, even about local stability, however, the actual nonlinear system is globally asymptotically stable.

Example 4.21

Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2 - x_1(x_1^2 + x_2^2)\\ \dot{x}_2 &= -x_1 - x_2(x_1^2 + x_2^2)\end{aligned}$$

The origin of the system is an EP for this system. Let

$$V(x) = x_1^2 + x_2^2$$

The derivative of V along any system trajectory is

$$\dot{V}(x) = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 = -2(x_1^2 + x_2^2)^2$$

which is negative definite. Therefore, the origin is a globally asymptotically stable EP

Invariant set theorems

Using our previous tools, when $\dot{V}$ is only negative semidefinite it is not sufficient for the asymptotic stability. In the case of negative semidefiniteness we can still draw conclusions on the asymptotic stability by using the invariant set theorems.

Definition 4.9

A set **G** is invariant set for a dynamic system if every system trajectory which starts from a point in **G** remains in **G** for all future times.

Example 4.22

Some invariant sets

Any equilibrium point

Whole state space

For an autonomous system, any of the trajectories In the state space

Limit cycles

In some cases, we can prove asymptotic stability even if $\dot{V} \leq 0$. The following La Salle's theorem makes reference to the autonomous system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \qquad \text{(cf. 25)}$$

Theorem 4.4

Consider an autonomous system of the form (25), with $\mathbf{f}$ continuous , and let $V(\mathbf{x})$ be scalar function with continuous partial derivatives. Assume that

- for some $L > 0$, the region Ω_L defined by $V(\mathbf{x}) < L$ is bounded
- $\dot{V}(\mathbf{x}) \leq 0$ for all $\mathbf{x} \in \Omega_L$.

Let $\mathbf{E}$ be the set of all points within Ω_L where $\dot{V}(\mathbf{x}) = 0$, and $\mathbf{N}$ be the largest invariant set in $\mathbf{E}$. Then every solution $\mathbf{x}(t)$ originating in Ω_L tends to $\mathbf{N}$ as $t \to \infty$.

Highlights of La Salle's Theorem

- $V : D \to \mathbb{R}$ is continuously differentable (where $D \subset \mathbb{R}^n$)
- The bounded set Ω_L is defined by $V(\mathbf{x}) < L$
- $\dot{V}(\mathbf{x}) \leq 0$ for all $\mathbf{x} \in \Omega_L$.
- $\mathbf{E} \stackrel{\Delta}{=} \{\mathbf{x} \in \Omega_L : \ \dot{V}(\mathbf{x}) = 0\}$
- $\mathbf{N}$: largest invariant set in $\mathbf{E}$.

Then every solution starting in Ω_L approaches $\mathbf{N}$ as $t \to \infty$.

Remark Unlike Lyapunov theory, La Salle's theorem requires V to be continuously differentiable but not necessarily positive definite.

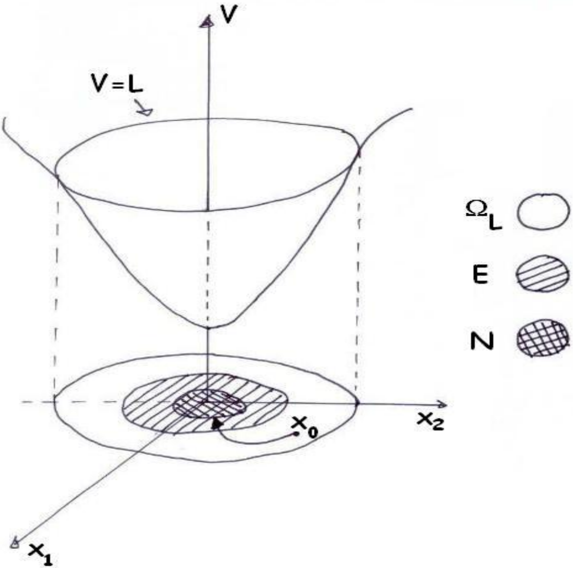


Figure 50: La Salle's theorem definitions

Example 4.23

For the mass-damper-spring system

$$m\ddot{x} + b\dot{x}|\dot{x}| + k_0 x + k_1 x^3 = 0$$

using the energy function

$$V(\mathbf{x}) = \frac{1}{2}m\dot{x}^2 + \int_0^x (k_0 x + k_1 x^3)dx = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}k_0 x^2 + \frac{1}{4}k_1 x^4.$$

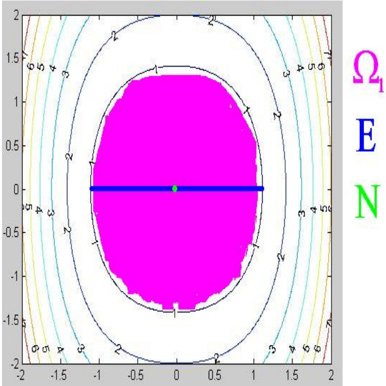
we can conclude only the marginal stability because $\dot{V}$ is only negative semidefinite, i.e.,

$$\dot{V}(\mathbf{x}) = m\dot{x}\ddot{x} + (k_0 x + k_1 x^3)\dot{x} = \dot{x}(-b\dot{x}|\dot{x}|) = -b|\dot{x}|^3$$

Now let us use the invariant set theorem. The set $\mathbf{E}$ is defined by $\dot{x} = 0$, the collection of states with zero velocity, or the whole horizontal axis in the phase plane $(x, \dot{x})$. The largest invariant set $\mathbf{N}$ in it contains only the origin. Suppose not. ...

$$V(\mathbf{x}) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}k_0x^2 + \frac{1}{4}k_1x^4$$
$$\dot{V}(\mathbf{x}) = -b|\dot{x}|^3$$

Let $L = 1$, then $V(\mathbf{x}) < 1$, i.e., Ω_1 is the set of (x, y) pairs satisfying $\frac{1}{2}m\dot{x}^2 + \frac{1}{2}k_0x^2 + \frac{1}{4}k_1x^4 < 1$ (pink); $\dot{V}(\mathbf{x}) = 0$, i.e., $\mathbf{E}$, is the set of points in Ω_1 satisfying $\dot{x} = 0$ (blue), and the largest invariant set $\mathbf{N}$ (green), equals origin, shown below:



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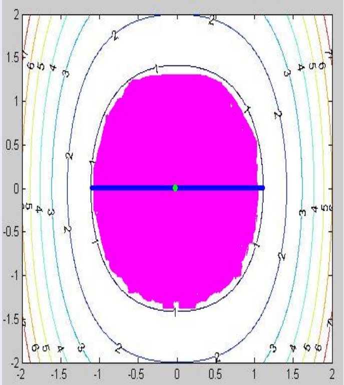
Now let us use the invariant set theorem. The set **E** is defined by $\dot{x} = 0$, the collection of states with zero velocity, or the whole horizontal axis in the phase plane $(x, \dot{x})$. The largest invariant set **N** in it contains only the origin. Suppose not. Let **N** contains a nonzero position x_1, then the acceleration at that point is $\ddot{x} = -(\frac{k_0}{m})x - (\frac{k_1}{m})x^3$.

In the state space form, with $x_1 \triangleq x$ and $x_2 \triangleq \dot{x}$, we have

$$\dot{x}_1 = \underbrace{x_2}_{0}, \quad \dot{x}_2 = \frac{1}{m}[-k_0 x_1 - k_1 x_1^3 - \underbrace{b x_2 |x_2|}_{0}]$$

This implies that the trajectory immediately moves out of the set **E**, and thus also out of the set **N**. This causes a contradiction.



Ω_1

E

N

Example 4.24

Consider the system

$$\begin{aligned}\dot{x}_1 &= -x_1 - x_2 \\ \dot{x}_2 &= x_1\end{aligned}$$

This system is linear and its EP at the origin is asymptotically stable. Let us utilize La Salle's method for showing the asymptotic stability of the origin. Let the Lyapunov function candidate be $V(\mathbf{x}) = x_1^2 + x_2^2$. Then $\dot{V}(\mathbf{x}) = 2x_1(-x_1 - x_2) + 2x_2(x_1) = -2x_1^2$ is negative semidefinite. We can conclude only the stability of the EP. Let us form a bounded set, for instance, $\Omega_5 = \{(x_1, x_2) : \ x_1^2 + x_2^2 < 5\}$. Then $\mathbf{E} = \{(x_1, x_2) : \ x_1 = 0\} \cap \Omega_5$. On $(0, x_2)$, $x_2 \neq 0$, vector field is perpendicular to the x_2 axis. Clearly, only the trajectory starting at $\mathbf{0}$ stays in $\mathbf{E}$. Thus $\mathbf{N} = \mathbf{0}$. La Salle's theorem implies that $\mathbf{0}$ is asymptotically stable.

We could have, of course, used a better Lyapunov function, for instance, $V(\mathbf{x}) = 3x_1^2 + x_1x_2 + 2x_2^2$ so that we could have reached the asymptotic stability conclusion directly.

Example 4.25

Consider the system

$$\begin{aligned}\dot{x}_1 &= x_1(x_1^2 + x_2^2 - 2) - 4x_1x_2^2 \\ \dot{x}_2 &= 4x_1^2x_2 + x_2(x_1^2 + x_2^2 - 2)\end{aligned}$$

For $L = 2$ the region Ω_2 is defined by $V(\mathbf{x}) = x_1^2 + x_2^2 < 2$, is bounded. The set $\mathbf{E}$ is simply the origin $\mathbf{0}$, which is an invariant set. . All the conditions of the local invariant set theorem are satisfied and, therefore, any trajectory starting within the circle converges to the origin.

Example 4.26

Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 - x_1^7[x_1^4 + 2x_2^2 - 10] \\ \dot{x}_2 &= -x_1^3 - 3x_2^5[x_1^4 + 2x_2^2 - 10] \end{aligned}$$

Notice that the set $\{(x_1, x_2) : \ x_1^4 + 2x_2^2 - 10 = 0\}$ is invariant since if (x_1, x_2) satisfies this once, it satisfies it for all future times. The (x_1, x_2) coordinates satisfying $x_1^4 + 2x_2^2 - 10 = 0$ form a closed curve (see the next slide). For the system dynamics given, the $x_1^4 + 2x_2^2 - 10$ value never changes if (x_1, x_2) is on the closed curve $x_1^4 + 2x_2^2 - 10 = 0$. Let us look at the change in it:

$$\frac{d}{dt}(x_1^4 + 2x_2^2 - 10) = -(4x_1^{10} + 12x_2^6)(x_1^4 + 2x_2^2 - 10)$$

which is zero on the set $\{(x_1, x_2) : \ x_1^4 + 2x_2^2 - 10 = 0\}$. The motion on this invariant set is described by

$$\dot{x}_1 = x_2, \ \ \dot{x}_2 = -x_1^3$$

One can verify that the invariant set $x_1^4 + 2x_2^2 - 10 = 0$ is a closed curve containing the origin. The contour curves are shown on the right. Its MATLAB codes are below:

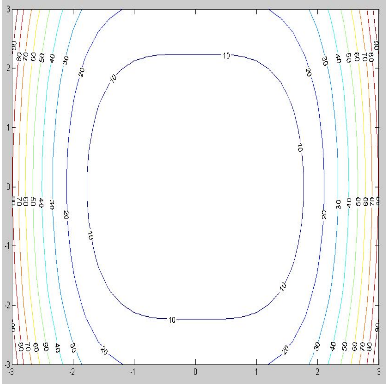


Figure 51: Contours for $x_1^4 + 2x_2^2$

```
x = -3:0.2:3;
y = -3:0.2:3;
[X,Y] = meshgrid(x,y);
Z = X.^4+2*Y.^2;
figure
contour(X,Y,Z,'ShowText','on')
```

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The invariant set $x_1^4 + 2x_2^2 - 10 = 0$ represents a limit cycle, along which the state vector moves clockwise. The dynamics on this set in state space form

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1^3 \end{aligned}$$

reveals the direction of the motion.

Is this limit cycle attractive? Define a function

$$V(\mathbf{x}) = (x_1^4 + 2x_2^2 - 10)^2$$

which represents a measure of the distance to the limit cycle. It is 0 on the limit cycle, and gets larger as (x_1, x_2) moves away from the LC. For an arbitrary positive number L, the region Ω_L, which surrounds the limit cycle is bounded. Using our earlier calculation, we obtain

$$\dot{V} = -8(x_1^{10} + 3x_2^6)(x_1^4 + 2x_2^2 - 10)^2$$

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$$\dot{V} = -8(x_1^{10} + 3x_2^6)(x_1^4 + 2x_2^2 - 10)^2$$

Thus $\dot{V}$ is strictly negative, except if

$$x_1^4 + 2x_2^2 = 10 \text{ or } x_1^{10} + 3x_2^6 = 0$$

in which case $\dot{V} = 0$. The first equation is defining the limit cycle while the second equation is verified only at the origin. Union of these two sets is **E**. Since both the limit cycle and the origin are invariant sets the set **N** simply consists of their union. Thus all the system trajectories starting in Ω_L converge either to the limit cycle or the origin.

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It can be shown that the origin is unstable. When this is shown, any trajectory in Ω_L can tend to the limit cycle only. The instability of the origin cannot be shown by linearization, because, the linearized system ($\dot{x}_1 = 0,\ \dot{x}_2 = 0$) is only marginally stable. This does not let us conclude anything about the stability or instability. Now consider the region Ω_{100}: This region is $V(\mathbf{x}) < 100$ or, more explicitly, $(x_1^4 + 2x_2^2 - 10)^2 < 100$. At the origin: $\left[(x_1^4 + 2x_2^2 - 10)^2\right]_{(0,0)} < 100 \rightarrow 100 < 100$ does not hold, therefore the origin does not belong to Ω_{100} (in other words, origin corresponds to the local maximum of V. Indeed, its value is 100 at the origin and lower than that in its close neighborhood). The expression $\dot{V} = 0$ is the same as before, However, the points satisfying it don't have the point $(0, 0)$ anymore. Therefore, the set $\mathbf{E}$ is just the limit cycle, so is the largest invariant set $\mathbf{N}$ in it. Thus, reapplication of the invariant set theorem shows that any trajectory starting from the region within the limit cycle, excluding the origin, actually converges to the limit cycle. In particular this implies that the EP at the origin is unstable.

There is no general way of finding Lyapunov functions for nonlinear systems. This is a fundamental drawback. To find Lyapunov function one has to use experience and intuition. However, Lyapunov functions can systematically be found for stable linear systems.

Definition 4.10

A square matrix $\mathbf{M}$ is symmetric if $\mathbf{M} = \mathbf{M}^T$ (In other words, if $\forall i, j\ M_{ij} = M_{ji}$). A square matrix $\mathbf{M}$ is skew symmetric if $\mathbf{M} = -\mathbf{M}^T$ (i.e., if $\forall i, j\ M_{ij} = -M_{ji}$).

Fact Any square $n \times n$ matrix $\mathbf{M}$ can be represented as the sum of a symmetric matrix and a skew symmetric matrix:

$$\mathbf{M} = \frac{\mathbf{M} + \mathbf{M}^T}{2} + \frac{\mathbf{M} - \mathbf{M}^T}{2}$$

The first term on the left is symmetric, and the second term is skew-symmetric.

Fact The quadratic function associated with a skew symmetric matrix is always zero. Definition of a skew symmetric matrix implies

$$\mathbf{x}^T\mathbf{M}\mathbf{x} = -\mathbf{x}^T\mathbf{M}^T\mathbf{x}$$

Since $\mathbf{x}^T\mathbf{M}^T\mathbf{x}$ is a scalar, the right-hand side of this equation can be replaced by its transpose. The we have

$$\mathbf{x}^T\mathbf{M}\mathbf{x} = -\mathbf{x}^T\mathbf{M}\mathbf{x}$$

This shows that

$$\forall \mathbf{x},\ \mathbf{x}^T\mathbf{M}\mathbf{x} = 0 \tag{33}$$

Actually the property (33) is a necessary and sufficient condition for a matrix $\mathbf{M}$ to be skew symmetric. To demonstrate this apply (33) to the basis vectors:

$$\mathbf{e}_i^T \mathbf{M}_s \mathbf{e}_i = 0,\ i = 1, 2, \ldots, n \rightarrow M_{ii} = 0,\ i = 1, 2, \ldots, n$$

$\therefore$ Diagonal elements of a skew symmetric matrix are zero.

Example 4.27

Observe that $\mathbf{e}_i^T \mathbf{M} \mathbf{e}_i$ results in the i-th diagonal element of $\mathbf{M}$.

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}^T \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 5, \ldots$$

Also

$$[\forall(i,j),\ (\mathbf{e}_i+\mathbf{e}_j)^T\mathbf{M}_s(\mathbf{e}_i+\mathbf{e}_j)=0] \rightarrow [\forall(i,j),\ \cancel{M_{ii}}+M_{ij}+M_{ji}+\cancel{M_{jj}}=0]$$

Considering that the diagonal elements of a skew symmetric matrix are zero, we must have

$$M_{ij}=-M_{ji},\ \ \forall(i,j)$$

Example 4.28

$$\begin{aligned}(\mathbf{e}_1+\mathbf{e}_2)^T\mathbf{M}_s(\mathbf{e}_1+\mathbf{e}_2) &= \left(\begin{bmatrix}1\\0\\0\end{bmatrix}+\begin{bmatrix}0\\1\\0\end{bmatrix}\right)^T\begin{bmatrix}0 & 2 & 3\\-2 & 0 & 6\\-3 & -6 & 0\end{bmatrix}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}+\begin{bmatrix}0\\1\\0\end{bmatrix}\right)\\ &= M_{11}+M_{12}+M_{21}+M_{22}\\ &= 0+2+(-2)+0\\ &= 0\end{aligned}$$

In our later analysis of linear systems we use $\mathbf{x}^T\mathbf{M}\mathbf{x}$ as Lyapunov function candidates; we assume WLOG M is symmetric. Note that

$$\mathbf{x}^T\mathbf{M}\mathbf{x} = \mathbf{x}^T(\frac{\mathbf{M}+\mathbf{M}^T}{2} + \frac{\mathbf{M}-\mathbf{M}^T}{2})\mathbf{x} = \mathbf{x}^T\frac{\mathbf{M}+\mathbf{M}^T}{2}\mathbf{x}$$

Thus we can replace $\mathbf{M}$ with $\frac{\mathbf{M}+\mathbf{M}^T}{2}$, and the values of $\mathbf{x}^T\mathbf{M}\mathbf{x}$ are not affected by this.

Exercise

For the matrix $\mathbf{M} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ compare $\mathbf{x}^T\mathbf{M}\mathbf{x}$ with $\mathbf{x}^T(\frac{\mathbf{M}+\mathbf{M}^T}{2})\mathbf{x}$

Definition 4.11

A square $n \times n$ matrix $\mathbf{M}$ is positive definite (p.d.) if

$$\mathbf{x} \neq \mathbf{0} \rightarrow \mathbf{x}^T \mathbf{M} \mathbf{x} > 0$$

In other words, a matrix $\mathbf{M}$ is positive definite if the quadratic function $\mathbf{x}^T \mathbf{M} \mathbf{x}$ is a positive definite function.
Geometrically, $\mathbf{x}^T \mathbf{M} \mathbf{x}$ can be viewed as dot product of $\mathbf{x}$ and $\mathbf{Mx}$. Thus, positive definiteness implies

$$< \mathbf{x}, \mathbf{Mx} > \; = \; |\mathbf{x}| \times |\mathbf{Mx}| \times \cos\alpha \; > \; 0$$

where α i the angle between $\mathbf{x}$ and $\mathbf{Mx}$. If this angle is less than 90^0 then the result $< \mathbf{x}, \mathbf{Mx} >> 0$ holds.

A necessary condition for a square matrix $\mathbf{M}$ to be p.d. is that its diagonal elements be strictly positive. Also, Sylvester's theorem says that, assuming $\mathbf{M}$ is symmetric, a necessary and sufficient condition for $\mathbf{M}$ to be p.d. is that its principal minors (i.e., $M_{11}, M_{11}M_{22} - M_{21}M_{12}, \ldots, \det \mathbf{M}$) all be strictly positive. In particular, a symmetric p.d. matrix is always invertible, because its determinant is nonzero.

Example 4.29

For a 4×4 matrix, matrix elements used for the principal minors are as follows:

$$\begin{bmatrix} \bullet & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \end{bmatrix}, \begin{bmatrix} \bullet & \bullet & \square & \square \\ \bullet & \bullet & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \end{bmatrix}, \begin{bmatrix} \bullet & \bullet & \bullet & \square \\ \bullet & \bullet & \bullet & \square \\ \bullet & \bullet & \bullet & \square \\ \square & \square & \square & \square \end{bmatrix} \begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{bmatrix}$$

A symmetric positive definite matrix $\mathbf{M}$ can always be decomposed as

$$\mathbf{M} = \mathbf{U}^T \Lambda \mathbf{U} \tag{34}$$

where $\mathbf{U}$ is a matrix of eigenvectors and satisfies $\mathbf{U}^T\mathbf{U} = \mathbf{I}$, and Λ is a diagonal matrix containing the eigenvalues of the matrix $\mathbf{M}$. Let $\lambda_{min}(\mathbf{M})$ denote the smallest eigenvalue of $\mathbf{M}$ and $\lambda_{max}(\mathbf{M})$ the largest. Then it follows from (34) that

$$\lambda_{min}(\mathbf{M})||\mathbf{x}||^2 \leq \mathbf{x}^T\mathbf{M}\mathbf{x} \leq \lambda_{max}(\mathbf{M})||\mathbf{x}||^2$$

A square $n \times n$ matrix $\mathbf{M}$ is said to be positive semidefinite (p.s.d.) if

$$\forall x, \ \mathbf{x}^T \mathbf{M} \mathbf{x} \geq 0$$

A matrix inequality of the form $\mathbf{M}_1 > \mathbf{M}_2$ means that $\mathbf{M}_1 - \mathbf{M}_2 > \mathbf{0}$.

Definition 4.12

A matrix $\mathbf{M}$ is negative definite iff $-\mathbf{M}$ is positive definite.

Definition 4.13

A matrix $\mathbf{M}$ is negative semidefinite iff $-\mathbf{M}$ is positive semidefinite.

Given a linear system of the form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, consider a Lyapunov function candidate

$$V(\mathbf{x}) = \mathbf{x}^T\mathbf{P}\mathbf{x}$$

where $\mathbf{P}$ is a given p.d. matrix. Differentiating the p.d. function V along the system trajectory yields :

$$\begin{aligned}
\dot{V}(\mathbf{x}) &= \dot{\mathbf{x}}^T\mathbf{P}\mathbf{x} + \mathbf{x}^T\mathbf{P}\dot{\mathbf{x}} \\
&= (\mathbf{A}\mathbf{x})^T\mathbf{P}\mathbf{x} + \mathbf{x}^T\mathbf{P}(\mathbf{A}\mathbf{x}) \\
&= \mathbf{x}^T\mathbf{A}^T\mathbf{P}\mathbf{x} + \mathbf{x}^T\mathbf{P}\mathbf{A}\mathbf{x} \\
&= \mathbf{x}^T \underbrace{(\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A})}_{-\mathbf{Q}} \mathbf{x} \\
&= -\mathbf{x}^T\mathbf{Q}\mathbf{x}
\end{aligned}$$

Now we need to determine whether the symmetric matrix $\mathbf{Q}$ defined above is p.d. If it is the case then V satisfies the conditions of the basic Lyapunov theorem (4.2). However, this approach may sometimes lead to inconclusive results, that is, $\mathbf{Q}$ may not be p.d. even for a stable system.

Example 4.30

Consider

$$\dot{\mathbf{x}} = \underbrace{\begin{bmatrix} 0 & 4 \\ -8 & -12 \end{bmatrix}}_{\mathbf{A}} \mathbf{x}$$

Take $\mathbf{P} = \mathbf{I}$, then $V(\mathbf{x}) = \mathbf{x}^T \mathbf{I} \mathbf{x} = \mathbf{x}^T \mathbf{x}$ is the Lyapunov function. This leads to

$$-\mathbf{Q} = \mathbf{P}\mathbf{A} + \mathbf{A}^T \mathbf{P} = \begin{bmatrix} 0 & -4 \\ -4 & -24 \end{bmatrix}$$

The matrix $\mathbf{Q}$ is not p.d. Therefore, no conclusion can be drawn from the Lyapunov function on whether the system is stable or not.

A conclusive approach

Choose a positive definite matrix $\mathbf{Q}$

Solve for $\mathbf{P}$ from the Lyapunov equation $\mathbf{A}^T\mathbf{P} + \mathbf{PA} = -\mathbf{Q}$

Check whether $\mathbf{P}$ is p.d.

If $\mathbf{P}$ is p.d. $\mathbf{x}^T\mathbf{P}\mathbf{x}$ is a Lyapunov function for the linear system and global asymptotic stability is guaranteed. This approach always leads to conclusive results.

Theorem 4.5

A necessary and sufficient condition for a LTI system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ to be strictly stable is that, for any symmetric p.d. matrix $\mathbf{Q}$, the unique matrix $\mathbf{P}$ solution of the Lyapunov equation $\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{Q}$ be symmetric positive definite.

The above theorem shows that any p.d. matrix $\mathbf{Q}$ can be used to determine the stability of a linear system. A simple choice of $\mathbf{Q}$ is the identity matrix.

Example 4.31

Consider

$$\dot{\mathbf{x}} = \underbrace{\begin{bmatrix} 0 & 4 \\ -8 & -12 \end{bmatrix}}_{A} x$$

and take $\mathbf{Q} = \mathbf{I}$. Let $\mathbf{P}$ be

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

Noting the symmetry of $\mathbf{P}$, we have $p_{12} = p_{21}$, then the expression $\mathbf{A}^T\mathbf{P} + \mathbf{PA} = -\mathbf{Q}$ becomes

$$\begin{bmatrix} 0 & -8 \\ 4 & -12 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 4 \\ -8 & -12 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

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$$\begin{bmatrix} 0 & -8 \\ 4 & -12 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 4 \\ -8 & -12 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

which takes the form

$$\begin{bmatrix} f_1(p_{11}, p_{12}, p_{22}) & f_2(p_{11}, p_{12}, p_{22}) \\ f_3(p_{11}, p_{12}, p_{22}) & f_4(p_{11}, p_{12}, p_{22}) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

This has the solution

$$p_{11} = \frac{5}{16}, \; p_{12} = p_{22} = \frac{1}{16}$$

The corresponding matrix

$$\mathbf{P} = \frac{1}{16} \begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix}$$

is p.d., and therefore the linear system is globally asymptotically stable.

Krasovskii's Method

Krasovskii's method suggests a simple form of Lyapunov function for the autonomous nonlinear systems of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}). \qquad \text{(cf. 25)}$$

when the Jacobian **A** of the system satisfies a certain definiteness condition.

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}). \qquad \text{(cf. 25)}$$

Theorem 4.6

Consider the autonomous system defined by (25), with the equilibrium point of interest being the origin. Let $\mathbf{A}(\mathbf{x})$ denote the Jacobian matrix of the system, i.e.,

$$\mathbf{A}(\mathbf{x}) = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}$$

If the matrix $\mathbf{F} = \mathbf{A} + \mathbf{A}^T$ is negative definite in a neighborhood Ω, then the equilibrium point at the origin is asymptotically stable. A Lyapunov function for this system is

$$V(\mathbf{x}) = \mathbf{f}^T(\mathbf{x})\mathbf{f}(\mathbf{x})$$

If Ω is the entire state space and, in addition, $V(\mathbf{x}) \to \infty$ as $||\mathbf{x}|| \to \infty$, then the equilibrium point is globally asymptotically stable.

Example 4.32

Consider the nonlinear system

$$\begin{aligned}\dot{x}_1 &= 6x_1 + 2x_2 \\ \dot{x}_2 &= 2x_1 - 6x_2 - 2x_2^3\end{aligned}$$

We have

$$\mathbf{A} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \begin{bmatrix} -6 & 2 \\ 2 & -6 - 6x_2^2 \end{bmatrix}, \ \mathbf{F} = \mathbf{A} + \mathbf{A}^T = \begin{bmatrix} -12 & 4 \\ 4 & -12 - 12x_2^2 \end{bmatrix}$$

The matrix $\mathbf{F}$ is negative definite over the whole state space. Therefore, the origin is asymptotically stable, and the Lyapunov function candidate is

$$V(\mathbf{x}) = \mathbf{f}^T(\mathbf{x})\mathbf{f}(\mathbf{x}) = (-6x_1 + 2x_2)^2 + (2x_1 - 6x_2 - 2x_2^3)^2$$

Since $V(\mathbf{x}) \to \infty$ as $||\mathbf{x}|| \to \infty$, the equilibrium state at the origin is globally asymptotically stable.

Theorem 4.7

Consider the autonomous system defined by (25), with the equilibrium point of interest being the origin. Let $\mathbf{A}(\mathbf{x})$ denote the Jacobian matrix of the system. Then a sufficient condition for the origin to be asymptotically stable is that there exist two symmetric positive definite matrices $\mathbf{P}$ and $\mathbf{Q}$ such that $\forall \mathbf{x} \neq \mathbf{0}$, the matrix

$$\mathbf{F}(\mathbf{x}) = \mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{Q}$$

is negative semidefinite in some neighborhood Ω of the origin. The function $V(\mathbf{x}) = \mathbf{f}^T\mathbf{P}\mathbf{f}$ is then a Lyapunov function for the system. If the region Ω is the whole state space, and if in addition, $V(\mathbf{x}) \to \infty$ as $||\mathbf{x}|| \to \infty$, then the system is globally asymptotically stable.

The objective of the control system design can be stated as follows: Given a physical system to be controlled and the specifications of its desired behavior, construct a feedback control law to make the closed loop system display the desired behavior.

Stabilization Problems Asymptotic Stabilization Problem:
Given a nonlinear dynamic system described by

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t)$$

find a control law $\mathbf{u}$ such that, starting from anywhere in a region Ω, the state $\mathbf{x}$ tends to $\mathbf{0}$ as $t \to \infty$.

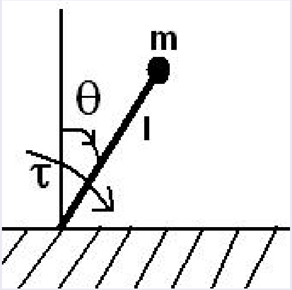
If the control law depends on measurement signals directly, it is said to be **a static control law**. If it depends on the measurements through a differential equation, the control law is said to be **a dynamic control law**. For example, in linear control a proportional controller is a static controller, while a lead lag controller is a dynamic controller.

Example 5.1

Consider the pendulum on the right-hand side. Its dynamics is

$$J\ddot{\theta} - mgl\sin\theta = \tau$$

where $J = ml^2$. Assume that our task is to bring the pendulum from a large initial angle, say $\theta(0) = 60^0$, to the vertical up position.



$$J\ddot{\theta} - mgl\sin\theta = \tau$$

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One choice of stabilizer is

$$\tau = -k_d\dot{\theta} - k_p\theta - mgl\sin\theta$$

with k_d and k_p denoting positive constants. This leads to the following globally stable closed loop dynamics

$$J\ddot{\theta} + k_d\dot{\theta} + k_p\theta = 0$$

Another interesting controller is

$$\tau = -k_d\dot{\theta} - 2mgl\sin\theta$$

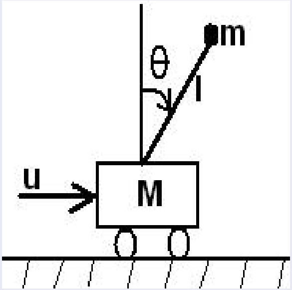
which leads to the stable closed loop dynamics

$$J\ddot{\theta} + k_d\dot{\theta} + mgl\sin\theta = 0$$

Example 5.2

However, many nonlinear stabilization problems are not easy to solve. Consider the inverted pendulum on the right. It has the following dynamics:

$$\begin{aligned}(M+m)\ddot{x} + ml\cos\theta\ddot{\theta} - ml\sin\theta\dot{\theta}^2 &= 0\\ m\ddot{x}\cos\theta + ml\ddot{x} - mg\sin\theta &= 0\end{aligned}$$



A particularly interesting task is to design a controller to bring the inverted pendulum from a vertical down position at the middle of the lateral track to a vertical up position at the same lateral point. This seemingly simple NL control problem is difficult to solve; because, it has two degrees of freedom and only one input.

If the task is to drive the state to some nonzero set point $\mathbf{x}_d$, then we can simply transform the problem into a zero point regulation problem by defining $\mathbf{y} \overset{\triangle}{=} \mathbf{x} - \mathbf{x}_d$ as the state. When the d.e. in $\mathbf{y}$ tends to zero, this means $\mathbf{x} - \mathbf{x}_d$ tends to zero, or $\mathbf{x} \to \mathbf{x}_d$

Asymptotic tracking problem Given a nonlinear dynamics system described by

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{u}, t) \\ \mathbf{y} &= \mathbf{h}(\mathbf{x}) \end{aligned}$$

and a desired output trajectory $\mathbf{y}_d$, find a control law for the input $\mathbf{u}$ such that starting from any initial state in a region Ω, the tracking errors $\mathbf{y}(t) - \mathbf{y}_d(t)$ go to zero, while the whole state $\mathbf{x}$ is bounded.

When the closed loop system is such that proper initial states imply zero tracking error for all time,

$$\mathbf{y}(t) = \mathbf{y}_d(t), \ \forall t \geq 0$$

the control system is said to be capable of perfect tracking. Asymptotic tracking implies that perfect tracking is asymptotically achieved. Exponential tracking convergence is defined similarly.
For nonminimum phase systems perfect tracking and asymptotic tracking cannot be achieved, as seen in the following example.

Definition 5.1

Minimum Phase Systems have neither poles nor zeros in the RHP. Otherwise a system is called a nonminimum phase system.

Example 5.3

Consider the linear system

$$\ddot{y} + 2\dot{y} + 2y = -\dot{u} + u \tag{35}$$

The system is nonminimum phase because it has a zero at 1. Note that

$$\frac{Y}{U} = -\frac{s-1}{s^2+2s+2}.$$

Assume that perfect tracking is achieved. Such u satisfies

$$\dot{u} - u = -(\ddot{y}_d + 2\dot{y}_d + 2y_d)$$

Under this u, the dynamics (35) becomes

$$\ddot{y} + 2\dot{y} + 2y = -\dot{u} + u = \ddot{y}_d + 2\dot{y}_d + 2y_d \to \ddot{e} + 2\dot{e} + 2e = 0$$

Seemingly error tends to zero, that is, $y \to y_d$.

Assume that perfect tracking is achieved. Such u satisfies

$$\dot{u} - u = -(\ddot{y}_d + 2\dot{y}_d + 2y_d)$$

Under this u, the dynamics (35) becomes

$$\ddot{y} + 2\dot{y} + 2y = -\dot{u} + u = \ddot{y}_d + 2\dot{y}_d + 2y_d \rightarrow \ddot{e} + 2\dot{e} + 2e = 0$$

Seemingly error tends to zero, that is, $y \rightarrow y_d$.
Note that, we defined u by

$$\dot{u} - u = \underbrace{-(\ddot{y}_d + 2\dot{y}_d + 2y_d)}_{f(t)}$$

The d.e. $\dot{u} - u = f(t)$ has a solution component e^t, so, this u is obviously evergrowing. This is caused by unstable dynamics of u. Above tracking problem requires infinite control input, which is not practical.

Specifying the desired behavior

In linear control, the desired behavior of a control system can be systematically specified either in the time domain (in terms of rise time, overshoot and settling time corresponding to a step command) or in the frequency domain (in terms of bandwidths, cut off frequencies, etc.). In linear control design, one first lays down the quantitative specifications of the closed loop control system, and then synthesizes a controller which meets these specifications. However, systematic specification for nonlinear systems is much less obvious because the response of NL system to one command does not reflect its response to another command, and furthermore frequency domain description is not possible. As a result, for nonlinear systems, one often looks instead for some qualitative specifications of the desired behavior in the operating region of interest. These may be

Stability

Accuracy and speed of response

Robustness

Cost.

A procedure for control design

Given a physical system to be controlled, one typically goes through the following standard procedure, possibly few iterations:

1. specify the desired behavior, and select actuators and sensors
2. model the physical plant by a set of differential equations
3. design a control law for the system
4. analyze and simulate the resulting control system
5. implement the control system in hardware.

Modeling nonlinear system

Modeling is basically the process of constructing a mathematical description (usually a set of differential equations) for the physical system to be controlled. Note that more accurate models are not always better because they may require unnecessarily complex control design and analysis and more demanding computation. The key here is to keep the essential effects and discard the insignificant effects in the system dynamics in the operating range of interest. Second, modeling is more than obtaining a nominal model for the physical system: it should also provide some characterization of the model uncertainties, which may be used for robust design, adaptive design, or merely simulation.

Model uncertainties are differences between the model and the real physical system. Uncertainties in parameters are called parametric uncertainties while the others are called nonparametric uncertainties. For example, for the model of a controlled mass

$$m\ddot{x} = u$$

the uncertainty in m is parametric uncertainty, while the neglected motor dynamics, mesurement noise, sensor dynamics are nonparametric uncertainties.

In NL control design, feedforward is used to cancel the effects of known disturbances and provide anticipative actions in tracking tasks.
Consider a linear (controllable and observable) minimum phase system in the form

$$A(p)y = B(p)u \tag{36}$$

where

$$A(p) = a_0 + a_1 p + \cdots + a_{n-1}p^{n-1} + p^n$$

$$B(p) = b_0 + b_1 p + \cdots + b_m p^m$$

The control objective is to make the output $y(t)$ follow a time-varying desired trajectory $y_d(t)$. We assume that only the output $y(t)$ is measured, and that $y_d, \dot{y}_d, \ldots, y_d^{(r)}$ are known with r being the relative degree (excess of poles over zeros) of the transfer function (thus $r = n - m$).

Digression

State controllability

The state of a system, which is a collection of the system's variables values, completely describes the system at any given time. In particular, no information on the past of a system will help in predicting the future, if the states at the present time are known.

Complete state controllability describes the ability of an external input to move the internal state of a system from any initial state to any other final state in a finite time interval.

Observability

A system is said to be observable if, for any possible sequence of state and control vectors, the current state can be determined in finite time using only the outputs. EOD

$$A(p)y = B(p)u \qquad \text{(cf. 36)}$$

$$A(p) = a_0 + a_1 p + \cdots + a_{n-1} p^{n-1} + p^n$$

$$B(p) = b_0 + b_1 p + \cdots + b_m p^m$$

Example 5.4

$$2\frac{d^5 y}{dt^5} + \frac{d^4 y}{dt^4} + 3\frac{d^3 y}{dt^3} + 6\frac{d^2 y}{dt^2} + 8\frac{dy}{dt} + 5y = 3\frac{d^2 u}{dt^2} + 4\frac{du}{dt} + u$$

$$A(p) = 2p^5 + p^4 + 3p^3 + 6p^2 + 8p + 5$$

$$B(p) = 3p^2 + 4p + 1$$

Relative degree: $\mathbf{r} = 5 - 2 = \mathbf{3}$

$$A(p)y = B(p)u \qquad \text{(cf. 36)}$$

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The control design can be achieved in two steps . First let us take the control law in the form of

$$u = v + \frac{A(p)}{B(p)} y_d \qquad (37)$$

where v is a new input, called "equivalent" or "synthetic" input, to be determined. Substitution of (37) in (36) leads to

$$\begin{aligned} A(p)y &= B(p)\left[v + \frac{A(p)}{B(p)} y_d\right] \\ A(p)y &= B(p)v + A(p)y_d \\ A(p)(y - y_d) &= B(p)v \\ A(p)e &= B(p)v \end{aligned}$$

Continued from the previous page

$$A(p)y = B(p)\left[v + \frac{A(p)}{B(p)}y_d\right]$$

$$A(p)e = B(p)v$$

where $e(t) \stackrel{\triangle}{=} y(t) - y_d(t)$ is the tracking error.
Now we have dynamics of the error. We want error to be zero. We design the equivalent input v accordingly.
The feedforward signal $\frac{A}{B}y_d$ can be computed as

$$\frac{A}{B}y_d = \alpha_1 y_d^{(r)} + \cdots + \alpha_r y_d + w$$

where the α_i $(i = 1, \ldots, r)$ are constants obtained from dividing A by B, and w is the filtered version of y_d.

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where the α_i $(i = 1, \ldots, r)$ are constants obtained from dividing A by B, and w is the filtered version of y_d.

Example 5.5

$$A(p) = p^5 + 3p^4 + 5p + 1$$

$$B(p) = p^2 - 2p + 1$$

$$\begin{aligned}
\frac{A(p)}{B(p)}y_d &= (p^3 + 5p^2 + 9p + 13)y_d + \underbrace{\frac{22p - 12}{p^2 - 2p + 1}y_d}_{w} \\
&= \dddot{y}_d + 5\ddot{y}_d + 9\dot{y}_d + 13y_d + w
\end{aligned}$$

Continued from the previous page

The second step is to construct input u so that the error dynamics is asymptotically stable. Since e is known (by subtracting the known y_d from the measured y), while its derivatives are not. One can stabilize e by using standard linear techniques, that is, pole placement together with a Luenberger observer. A simpler way of deriving the control law is to let

$$v = \frac{C(p)}{D(p)} e$$

with C and D being polynomials of order $(n - m)$. With this control law, the closed loop dynamics is

$$A(p)e = B(p) \left[\frac{C(p)}{D(p)} e \right]$$

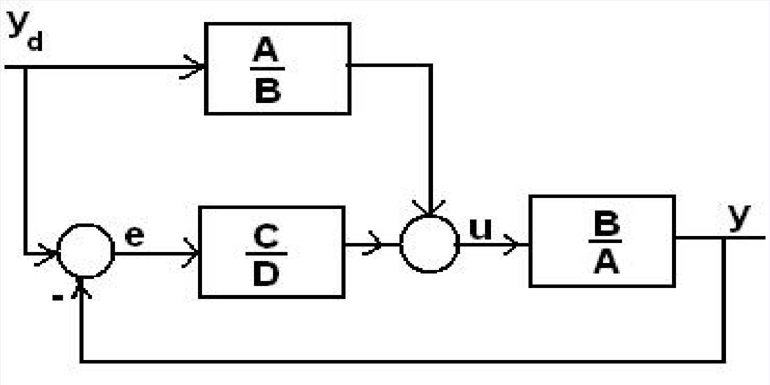
$$(AD - BC)e = 0$$

If the coefficients of C and D are chosen properly, the poles of the closed loop polynomial can be placed anywhere in the complex plane.

Thus the control law

$$u = \frac{A}{B} y_d + \frac{C}{D} e$$

guarantees that the tracking error $e(t)$ remains at zero if initial conditions satisfy $y^{(i)}(0) = y_d^{(i)}(0),\ i = 1, \ldots, r$, and exponentially converges to zero if the initial conditions do not satisfy these conditions.



Digression

For a continuous-time linear system

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\ \mathbf{x} + \mathbf{B}\ \mathbf{u} \\ \mathbf{y} &= \mathbf{C}\ \mathbf{x} \end{aligned}$$

where $\mathbf{x} \in \mathbb{R}^n, \mathbf{u} \in \mathbb{R}^m, \mathbf{y} \in \mathbb{R}^r$, the Luenberger observer has the dynamics

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\ \mathbf{u} + \mathbf{L}\left(\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}\right)$$

The observer error $\mathbf{e} \stackrel{\triangle}{=} \hat{\mathbf{x}} - \mathbf{x}$ satisfies the equation

$$\dot{\mathbf{e}} = \left(\mathbf{A} - \mathbf{L}\mathbf{C}\right)\ \mathbf{e}.$$

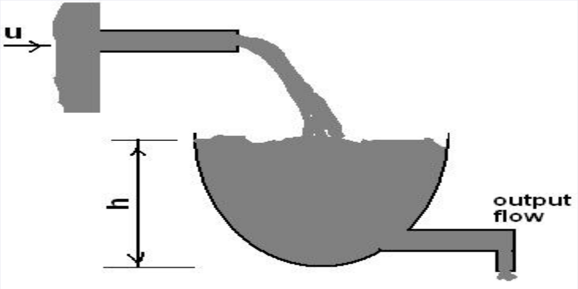
The eigenvalues of the matrix $\mathbf{A} - \mathbf{L}\mathbf{C}$ can be made arbitrarily by appropriate choice of the observer gain $\mathbf{L}$ when the pair $[\mathbf{A}, \mathbf{C}]$ is observable. In particular, $\mathbf{A} - \mathbf{L}\mathbf{C}$ can be made Hurwitz, so the observer error $\mathbf{e}(t) \to \mathbf{0}$ when $t \to \infty$. EOD

FEEDBACK LINEARIZATION

The main idea of the approach is to algebraically transform a nonlinear system dynamics into a (fully or partly) linear one, so that linear control techniques can be applied.

Example 6.1

Consider the control of the level h of the fluid tank to a specified level h_d. The control input is the flow u into the tank, and the initial level is h_0.



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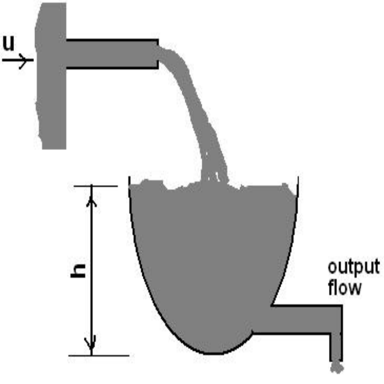
The dynamic model of the tank is

$$\frac{d}{dt}\left[\int_0^h A(h)dh\right] = u(t) - a\sqrt{2gh} \tag{38}$$



where $A(h)$ is the cross section of the tank and a is the cross section of the outlet pipe. If the initial level h_0 is quite different from the desired level h_d the control of h involves a nonlinear regulation problem.

The dynamics (38) can be written as

$$A(h)\dot{h} = u - a\sqrt{2gh}$$

A Simplified Version (Applicable to our case)
If f is continuous on $[a, b]$ and if $u(x)$ and $v(x)$ are differentiable functions of x whose values lie in $[a, b]$ then

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)dt = f(v(x))\frac{dv}{dx} - f(u(x))\frac{du}{dx}$$

A more general version
Let $f(x, \theta)$ be a function such that $f_\theta(x, \theta)$ exists, and is continuous. Then,
$\frac{d}{d\theta}\left(\int_{a(\theta)}^{b(\theta)} f(x, \theta)\, dx\right) = \int_{a(\theta)}^{b(\theta)} f_\theta(x, \theta)\, dx + f(b(\theta), \theta)b'(\theta) - f(a(\theta), \theta)a'(\theta)$
where the partial derivative of f indicates that inside the integral only the variation of $f(x, \theta)$ with θ is considered in taking the derivative. EOD

Continued from the previous page

$$A(h)\dot{h} = u - a\sqrt{2gh}$$

If $u(t)$ is chosen as

$$u(t) = a\sqrt{2gh} + A(h)v$$

with v being the "equivalent input" to be specified, the resulting dynamics is linear

$$A(h)\dot{h} = a\sqrt{2gh} + A(h)v - a\sqrt{2gh} \rightarrow \dot{h} = v$$

Choosing v as

$$v = \alpha\tilde{h}$$

with $\tilde{h} = h(t) - h_d$ being the level error, and α being a strictly positive constant, the resulting closed loop dynamics is

$$\dot{h} + \alpha\tilde{h} = 0$$

Since $\dot{h} = (h - h_d)' = \dot{\tilde{h}}$, we have $\dot{\tilde{h}} + \alpha\tilde{h} = 0$

Choosing v as

$$v = \alpha\tilde{h}$$

with $\tilde{h} = h(t) - h_d$ being the level error, and α being a strictly positive constant, the resulting closed loop dynamics is

$$\dot{\tilde{h}} + \alpha\tilde{h} = 0$$

This implies that $\tilde{h}(t) = Ce^{-\alpha t}$, so, $\tilde{h} \to 0$ as $t \to \infty$. So the actual input flow is determined by the nonlinear control law

$$u(t) = a\sqrt{2gh} - A(h)\alpha\tilde{h}$$

Similarly, if the desired level is a known time varying function $h_d(t)$, the equivalent input v can be chosen as

$$v = \dot{h}_d(t) - \alpha\tilde{h}$$

so as to still yield $\tilde{h}(t) \to 0$ as $t \to \infty$.

The idea of feedback linearization, i.e., cancelling the nonlinearities and imposing a desired linear dynamics, can simply be applied to a class of nonlinear systems described by the so called "companion form" or "controllability canonical form". A system is said to be in companion form if its dynamics is represented by

$$x^{(n)} = f(\mathbf{x}) + b(\mathbf{x})u$$

where u is the scalar control input, x is the scalar output of interest, $\left[x, \dot{x}, \ddot{x}, \ldots, x^{(n-1)}\right]^T$ is the state vector, and $f(\mathbf{x})$ and $b(\mathbf{x})$ are nonlinear functions of the states. In state space representation this form can be written as

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \\ f(\mathbf{x}) + b(\mathbf{x})u \end{bmatrix}$$

Continued from the previous page

For the systems which can be expressed in the controllability canonical form, using the control input (assuming b to be nonzero)

$$u = \frac{1}{b}[v - f]$$

we can cancel the nonlinearities and obtain a simple input output relation

$$x^{(n)} = v$$

Thus the control law

$$v = -k_0 x - k_1 \dot{x} - \cdots - k_{n-1} x^{(n-1)}$$

with the k_i chosen so that the polynomial $p^n + k_{n-1}p^{n-1} + \cdots + k_0$ has all its roots strictly in the left-half plane leads to the exponentially stable dynamics ...

$$x^{(n)} = f(\mathbf{x}) + b(\mathbf{x})u$$

$$x^{(n)} = v$$

Continued from the previous page

Thus the control law

$$v = -k_0 x - k_1 \dot{x} - \cdots - k_{n-1} x^{(n-1)}$$

with the k_i chosen so that the polynomial $p^n + k_{n-1}p^{n-1} + \cdots + k_0$ has all its roots strictly in the left-half plane, leads to the exponentially stable dynamics

$$x^{(n)} = -k_{n-1}x^{(n-1)} - \cdots - k_0 x$$

or

$$x^{(n)} + k_{n-1}x^{(n-1)} + \cdots + k_0 x = 0$$

which implies that $x(t) \to 0$.

$$x^{(n)} = v$$

Continued from the previous page

For tasks involving the tracking of a desired output $x_d(t)$ the control law

$$v = x_d^{(n)} - k_0 e - k_1 \dot{e} - \cdots - k_{n-1} e^{(n-1)}$$

$$x^{(n)} = x_d^{(n)} - k_0 e - k_1 \dot{e} - \cdots - k_{n-1} e^{(n-1)}$$

$$\underbrace{x^{(n)} - x_d^{(n)}}_{e^{(n)}} + k_0 e + k_1 \dot{e} + \cdots + k_{n-1} e^{(n-1)} = 0$$

$$e^{(n)} + k_{n-1} e^{(n-1)} + \cdots + k_1 \dot{e} + k_0 e = 0$$

where $e(t)$ is the tracking error, leads to exponentially convergent tracking.

Example 6.2

Consider the two link robot model with each joint equipped with a motor for providing input torque, an encoder for measuring joint position, and a tachometer for measuring joint velocity. The objective of the control design is to make the joint positions q_1 and q_2 follow designed position histories $q_{d1}(t)$ and $q_{d2}(t)$.

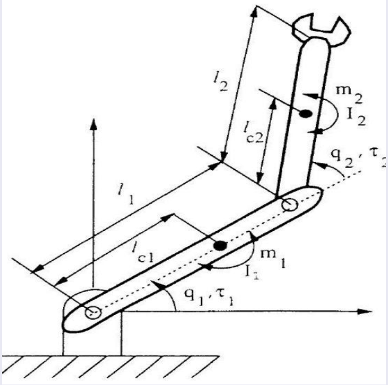


Figure 52: 2-link robot manipulator

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The dynamic equations of the robot is

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} -h\dot{q}_2 & -h\dot{q}_1 - h\dot{q}_2 \\ h\dot{q}_1 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

with $\mathbf{q} = [q_1 \; q_2]^T$ being the two joint angles, $\boldsymbol{\tau} = [\tau_1 \; \tau_2]^T$ being the joint inputs, and

$$\begin{aligned}
H_{11} &= m_1 l_{c_1}^2 + I_1 + m_2[l_1^2 + l_{c_2}^2 + 2 l_1 l_{c_2} \cos q_2] + I_2 \\
H_{22} &= m_2 l_{c_2}^2 + I_2 \\
H_{12} &= H_{21} = m_2 l_1 l_{c_2} \cos q_2 + m_2 l_{c_2}^2 + I_2 \\
h &= m_2 l_1 l_{c_2} \sin q_2 \\
g_1 &= m_1 l_{c_1} g \cos q_1 + m_2 g [l_{c_2} \cos(q_1 + q_2) + l_1 \cos q_1] \\
g_2 &= m_2 l_{c_2} g \cos(q_1 + q_2)
\end{aligned}$$

This equation can be compactly expressed as

$$\mathbf{H}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \boldsymbol{\tau}$$

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} -h\dot{q}_2 & -h\dot{q}_1 - h\dot{q}_2 \\ h\dot{q}_1 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

Continued from the previous page

To achieve the tracking task we use the control

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} -h\dot{q}_2 & -h\dot{q}_1 - h\dot{q}_2 \\ h\dot{q}_1 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \tag{39}$$

The robot dynamics now reduces to

$$\ddot{\mathbf{q}} = \mathbf{v}$$

where $\mathbf{v} = [v_1 \;\; v_2]^T$. Now choose

$$\mathbf{v} = \ddot{\mathbf{q}}_d - 2\lambda\dot{\tilde{\mathbf{q}}} - \lambda^2\tilde{\mathbf{q}}$$

with $\tilde{\mathbf{q}} = \mathbf{q} - \mathbf{q}_d$ being the position tracking error, and λ a positive number.

The robot dynamics now reduces to

$$\ddot{\mathbf{q}} = \mathbf{v}$$

where $\mathbf{v} = [v_1 \;\; v_2]^T$. Now choose

$$\mathbf{v} = \ddot{\mathbf{q}}_d - 2\lambda\dot{\tilde{\mathbf{q}}} - \lambda^2\tilde{\mathbf{q}}$$

with $\tilde{\mathbf{q}} = \mathbf{q} - \mathbf{q}_d$ being the position tracking error, and λ a positive number.

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Then the tracking error $\tilde{\mathbf{q}}$ satisfies

$$\ddot{\tilde{\mathbf{q}}} + 2\lambda\dot{\tilde{\mathbf{q}}} + \lambda^2\tilde{\mathbf{q}} = \mathbf{0}$$

and converges to zero exponentially. The control law (39) is commonly called "computed torque" in robotics. It can be applied to robots with arbitrary number of joints.

When the nonlinear dynamics is not in a controllability canonical form, one may have to use algebraic transformations to first put the dynamics into the controllability form before using the feedback linearization design.

Input-State Linearization

Consider the problem of designing the control input u for a single input nonlinear system of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, u)$$

The technique of input-state linearization solves this problem in two steps.

(1) One finds a state transformation $\mathbf{z} = \mathbf{z}(\mathbf{x})$ and an input transformation $u = u(\mathbf{x}, v)$ so that the nonlinear system dynamics is transformed into an equivalent linear time invariant dynamics $\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{b}v$.

(2) One uses standard linear techniques (such as pole placement) to design v.

Example 6.3

$$\begin{aligned}\dot{x}_1 &= -2x_1 + ax_2 + \sin x_1 \\ \dot{x}_2 &= -x_2 \cos x_1 + u\cos(2x_1)\end{aligned}$$

We want to move system states to $(0,0)$ from any given initial condition. For instance, using an input of the form

$$u = \frac{1}{\cos(2x_1)}\left[x_2 \cos x_1 + v\right]$$

results in

$$\begin{aligned}\dot{x}_1 &= -2x_1 + ax_2 + \sin x_1 \\ \dot{x}_2 &= v\end{aligned}$$

When, for instance, $v = -x_2$ we have

$$\begin{aligned}\dot{x}_1 &= -2x_1 + ax_2 + \sin x_1 \\ \dot{x}_2 &= -x_2\end{aligned}$$

This linearized only the second equation. Obviously, nonlinearities cannot be cancelled directly by the control input u.

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$$
\begin{aligned}
\dot{x}_1 &= -2x_1 + ax_2 + \sin x_1 \\
\dot{x}_2 &= -x_2 \cos x_1 + u \cos(2x_1)
\end{aligned}
$$

Obviously, nonlinearities cannot be cancelled directly by the control input u. However, we may use the transformation

$$
\begin{aligned}
z_1 &= x_1 \\
z_2 &= ax_2 + \sin x_1
\end{aligned}
$$

then the new state equations are

$$
\begin{aligned}
\dot{z}_1 &= -2z_1 + z_2 \\
\dot{z}_2 &= -2z_1 \cos z_1 + \cos z_1 \sin z_1 + au \cos(2z_1)
\end{aligned}
$$

We can cancel the nonlinearities by the control law

$$
u = \frac{1}{a\cos(2z_1)}(v - \cos z_1 \sin z_1 + 2z_1 \cos z_1)
$$

which leads to

Continued from the previous page

$$\begin{aligned} \dot{z}_1 &= -2z_1 + z_2 \\ \dot{z}_2 &= -2z_1 \cos z_1 + \cos z_1 \sin z_1 + au \cos(2z_1) \end{aligned}$$

We can cancel the nonlinearities by the control law

$$u = \frac{1}{a\cos(2z_1)}(v - \cos z_1 \sin z_1 + 2z_1 \cos z_1)$$

which leads to

$$\begin{aligned} \dot{z}_1 &= -2z_1 + z_2 \\ \dot{z}_2 &= v \end{aligned}$$

Now let us select v as $v = -k_1 z_1 - k_2 z_2$ with proper choices of feedback gains. In case $v = -2z_2$ we will have

$$\begin{aligned} \dot{z}_1 &= -2z_1 + z_2 \\ \dot{z}_2 &= -2z_2 \end{aligned}$$

whose poles are both placed at-2.

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The control input, when $= -2z_2$, is

$$u = \frac{1}{a\cos(2z_1)}(-2z_2 - \cos z_1 \sin z_1 + 2z_1 \cos z_1)$$

In terms of the original variables x_1, x_2 the control input is

$$u = \frac{1}{a\cos(2x_1)}(-2ax_2 - 2\sin x_1 - cosx_1 \sin x_1 + 2x_1 \cos x_1)$$

The original state $\mathbf{x}$ is given from $\mathbf{z}$ by

$$\begin{aligned} x_1 &= z_1 \\ x_2 &= (z_2 - \sin z_1)/a \end{aligned}$$

Since both z_1 and z_2 converge to zero, the original state $\mathbf{x}$ converges to zero.

Input-output linearization

Consider the system

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, u) \\ y &= h(\mathbf{x}) \end{aligned} \tag{40}$$

and assume that our objective is to make the output $y(t)$ track a desired trajectory $y_d(t)$ while keeping the whole state bounded, where $y_d(t)$ and its time derivatives up to a sufficiently high order are assumed to be known and bounded. Apparent difficulty with this model is that output y is only indirectly related to the input u, through the state variable x and the nonlinear state equations (40). Therefore it is not easy to design the control input u to control the tracking behavior of y.

Example 6.4

Consider the third order system

$$\begin{aligned} \dot{x}_1 &= \sin x_2 + (x_2+1)x_3 \\ \dot{x}_2 &= x_1^5 + x_3 \\ \dot{x}_3 &= x_1^2 + u \\ y &= x_1 \end{aligned} \tag{41}$$

To generate a direct relationship between the output y and the input u, let us differentiate the output y

$$\dot{y} = \dot{x}_1 = \sin x_2 + (x_2+1)x_3$$

Since $\dot{y}$ is not directly related to u let us differentiate again:

$$\ddot{y} = (x_2+1)u + f_1(x) \tag{42}$$

where $f_1(x)$ is a function of the state defined by

$$f_1(x) = (x_1^5 + x_3)(x_3 + \cos x_2) + (x_2+1)x_1^2$$

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$$\dot{y} = \dot{x}_1 = \sin x_2 + (x_2 + 1)x_3$$

Let us show that $\ddot{y}$ is as calculated in the preceding slide:

$$\ddot{y} = \frac{d^2 y}{dt^2} = \frac{dy}{dx_2}\frac{dx_2}{dt} + \frac{dy}{dx_3}\frac{dx_3}{dt} = (\cos x_2 + x_3)\dot{x}_2 + (x_2 + 1)\dot{x}_3$$

$$\ddot{y} = (\cos x_2 + x_3)(x_1^5 + x_3) + (x_2 + 1)(x_1^2 + u)$$

$$\ddot{y} = \underbrace{(\cos x_2 + x_3)(x_1^5 + x_3) + (x_2 + 1)x_1^2}_{f_1(\mathbf{x})} + (x_2 + 1)u$$

Since $\dot{y}$ is not directly related to u let us differentiate again:

$$\ddot{y} = (x_2 + 1)u + f_1(x) \qquad \text{(cf. 42)}$$

$$\ddot{y} = (x_2 + 1)u + f_1(x) \qquad \text{(cf. 42)}$$

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Clearly, (42) represents an explicit relationship between y and u. If we choose the control input to be in the form

$$u = \frac{1}{x_2 + 1}(v - f_1) \tag{43}$$

where v is a new input to be determined, the nonlinearity in (42) is cancelled, and we obtained

$$\ddot{y} = v$$

Let $e(t) := y(t) - y_d(t)$ and choose the new input v as

$$v = \ddot{y}_d - k_1 e - k_2 \dot{e}$$

with $k_1,\ k_2$ being positive constants, the tracking error of the closed loop system is given by

$$\ddot{e} + k_2 \dot{e} + k_1 e = 0 \tag{44}$$

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$$\ddot{y} = v$$

Let $e(t) := y(t) - y_d(t)$ and choose the new input v as

$$v = \ddot{y}_d - k_1 e - k_2 \dot{e}$$

with $k_1,\ k_2$ being positive constants, the tracking error of the closed loop system is given by

$$\ddot{e} + k_2 \dot{e} + k_1 e = 0 \tag{44}$$

which represents an exponentially stable dynamics. Therefore, if $e(0) = \dot{e}(0) = 0$, then $e(t) = 0,\ \forall t > 0$, i.e., perfect tracking is achieved; otherwise, $e(t)$ converges to zero exponentially.

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Recall the control law:

$$u = \frac{1}{x_2 + 1}(v - f_1) \qquad \text{(cf. 43)}$$

Note that the control law is defined everywhere, except at the singularity points such that $x_2 = -1$.
Full state measurement is necessary for the implementation of the control law because we use $\mathbf{x}$ in computing the u.

If we need to differentiate the output of a system r times to generate an explicit relationship between the output y and the input u, the system is said to have "**relative degree of r**". Thus the system in the above example has relative degree 2. Relative degree in linear systems is defined as excess of poles over zeros. It can be shown that for a controllable system of order n, it will take at most n differentiations of any output for the control input to appear, i.e., $r \leq n$.

$$\begin{aligned} \dot{x}_1 &= \sin x_2 + (x_2+1)x_3 \\ \dot{x}_2 &= x_1^5 + x_3 \\ \dot{x}_3 &= x_1^2 + u \end{aligned} \qquad \text{(System dynamics 41)}$$

$$\ddot{e} + k_2\dot{e} + k_1 e = 0 \qquad \text{(Error dynamics 44)}$$

Notice that the error dynamics, which says that we can track $y_d(t)$, has order 2, while the whole system dynamics has order 3 (the same as that of the plant, because the controller (43) introduces no extra dynamics). Thus one state is unobservable in input-output linearized dynamics. This part of the dynamics will be called "**internal dynamics**". For the above example the internal state is x_3.

$$\dot{x}_3 = x_1^2 + \frac{1}{x_2+1}(\ddot{y}_d - k_1 e - k_2\dot{e} - f_1)$$

If this internal dynamics is stable, then our tracking control design problem has indeed been solved. Otherwise, the above tracking controller is practically meaningless, ...

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For the above example the internal state is x_3.

$$\dot{x}_3 = x_1^2 + \frac{1}{x_2 + 1}(\ddot{y}_d - k_1 e - k_2 \dot{e} - f_1)$$

If this internal dynamics is stable, then our tracking control design problem has indeed been solved. Otherwise, the above tracking controller is practically meaningless, because the instability of the internal dynamics would imply undesirable phenomena such as the burning-up of fuses or the violent vibration of mechanical members.

Example 6.5

Consider

$$\begin{aligned}\dot{x}_1 &= x_2^3 + u \\ \dot{x}_2 &= u \\ y &= x_1\end{aligned}$$

Assume that the control objective is to make y track $y_d(t)$. Differentiation of y simply leads to the first state equation:

$$\dot{y} = \dot{x}_1 = x_2^3 + u$$

Thus, choosing the control law

$$u = -x_2^3 - e(t) + \dot{y}_d(t)$$

yields

$$\dot{y} = x_2^3 - x_2^3 - e(t) + \dot{y}_d(t) = -e(t) + \dot{y}_d(t)$$

with defining $e \stackrel{\triangle}{=} y - y_d$ we get

$$\dot{e} + e = 0 \tag{45}$$

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... defining $e \overset{\triangle}{=} y - y_d$ we get

$$\dot{e} + e = 0 \qquad \text{(cf. 45)}$$

in which e exponentially convergences to zero. This error dynamics is first order. However, original system dynamics is 2nd order:

$$\begin{aligned} \dot{x}_1 &= x_2^3 + u \\ \dot{x}_2 &= u \end{aligned}$$

One state is missing, equivalently not observable, in the error dynamics. If the missing part of the dynamics is stable, then the conclusion of the error dynamics, that is, error approaches zero asymptotically, is valid.
The 1st state equation is used in the error dynamics, therefore, the missing dynamics is related to the 2nd state equation. We need to use the u in the 2nd equation.

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Apply the same input to the second equation:

$$\dot{x}_2 + x_2^3 = \dot{y}_d(t) - e(t) \tag{46}$$

This is nonautonomous and nonlinear. However, in view of e is bounded by (45), and $\dot{y}_d$ is assumed to be bounded, we have

$$|\dot{y}_d - e| \leq D$$

where D is a positive constant. Define $f_1(t) \triangleq \dot{y}_d - e(t)$, then (46) appears neater:

$$\dot{x}_2 + x_2^3 = f_1(t)$$

How does the solution of this equation behave as $t \to \infty$?

$$\dot{x}_2 + x_2^3 = f_1(t)$$

Continued from the previous page

We claim that $|x_2| \leq D^{\frac{1}{3}}$. It can be justified by noting that

if $x_2 > D^{\frac{1}{3}}$ then $x_2^3 > D$ and $\dot{x}_2 = f_1(t) - D < 0$ and x_2 decreases;

and if $x_2 < -D^{\frac{1}{3}}$ then then $x_2^3 < D$ and $\dot{x}_2 = f_1(t) - D > 0$ and x_2 increases.
$\therefore |x_2| \leq D^{\frac{1}{3}}$

Therefore, the chosen u represents a satisfactory tracking control law for any given trajectory y_d such that $\dot{y}_d$ is bounded.

In general, it is difficult to determine whether the internal dynamics is stable or not. For a simpler case we consider the stability of internal dynamics in the linear systems context.

Example 6.6

Consider the simple controllable and observable linear system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 + u \\ u \end{bmatrix}, \; y = x_1 \tag{47}$$

where $y(t)$ is required to track a desired output $y_d(t)$. Differentiation of the output yields

$$\dot{y} = x_2 + u$$

which explicitly contains u.

Continued from the previous page

Differentiation of the output yields

$$\dot{y} = x_2 + u$$

which explicitly contains u. Thus the control law

$$u = -x_2 + \dot{y}_d - (y - y_d) \tag{48}$$

yields the tracking error equation

$$\dot{e} + e = 0$$

(where $e = y - y_d$) and the internal dynamics

$$\dot{x}_2 + x_2 = \dot{y}_d - e(t)$$

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We see from these equations that while $y(t)$ tends to $y_d(t)$ (and $\dot{y}$ tends to $\dot{y}_d$), x_2 remains bounded. Therefore by the control input (48) tracking objective is achieved.

Now let us consider a slightly different system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 + u \\ -u \end{bmatrix}, \; y = x_1 \tag{49}$$

The same control law as above yields the same tracking error dynamics, but now leads to the internal dynamics

$$\dot{x}_2 - x_2 = \dot{y}_d - e(t)$$

This implies that x_2, accordingly u, both go to infinity as $t \to \infty$. Therefore, the control input (48) does not achieve the tracking task.

In the above example the transfer function for the first case is $W_1 = \frac{s+1}{s^2}$, and for the second case is $W_2 = \frac{s-1}{s^2}$. Note that W_1 is the tf of a minimum phase system (i.e., all the zeros are in the left half plane), and W_2 is tf of a nonminimum phase system i.e., at least one zero is in the right half plane. For the above example, we observe that the internal dynamics is stable for the minimum phase one. It can be shown that this observation is true for all linear systems.

Digression: Companion form

Given the transfer function

$$H(s) = \frac{Y(s)}{U(s)} = \frac{b_0 s^2 + b_1 s + b_2}{s^3 + a_1 s^2 + a_2 s + a_3}$$

we want to express it in controllable (companion) state space form.
Let us write

$$H(s) = \frac{Y(s)}{U(s)} = \frac{(b_0 s^2 + b_1 s + b_2) Z(s)}{(s^3 + a_1 s^2 + a_2 s + a_3) Z(s)}$$

This leads to

$$Y(s) = (b_0 s^2 + b_1 s + b_2) Z(s), \quad y = b_0 \ddot{z} + b_1 \dot{z} + b_2 z$$

$$U(s) = (s^3 + a_1 s^2 + a_2 s + a_3) Z(s), \quad u = \dddot{z} + a_1 \ddot{z} + a_2 \dot{z} + a_3 z$$

Now let $x_1 = z,\ x_2 = \dot{z},\ x_3 = \ddot{z}$, then

$$\begin{aligned} \dot{x}_1 &= \dot{z} = x_2 \\ \dot{x}_2 &= \ddot{z} = x_3 \\ \dot{x}_3 &= \dddot{z} = u - a_1 \ddot{z} - a_2 \dot{z} - a_3 z \\ &= u - a_1 x_3 - a_2 x_2 - a_3 x_1 \end{aligned}$$

The output equations are

$$y = b_0 \ddot{z} + b_1 \dot{z} + b_2 z = b_0 x_3 + b_1 x_2 + b_2 x_1$$

Now we can use the matrix notation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

Now we can use the matrix notation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_2 & b_1 & b_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

EOD

Example 6.7

Consider the state space model

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{b}u, \quad y = \mathbf{c}^T\mathbf{z} \tag{50}$$

Let it have one zero (hence two more poles than zeros). Its transfer function is

$$Y = \mathbf{c}^T(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}U = \frac{b_0 + b_1 s}{a_0 + a_1 s + a_2 s^2 + s^3} U \tag{51}$$

We can write it companion form as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_0 & b_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Continued from the previous page

Let us perform input-output linearization based on this form. First differentiation yields:

$$\dot{y} = b_0\dot{x}_1 + b_1\dot{x}_2 = b_0x_2 + b_1x_3$$

Second differentiation yields:

$$\ddot{y} = b_0\dot{x}_2 + b_1\dot{x}_3 = b_0x_3 + b_1(-a_0x_1 - a_1x_2 - a_2x_3 + u)$$

It is seen that the input u appears in the second differentiation, which means that the required number of differentiation s (relative degree) is indeed the same as the excess of poles over zeros (of course, since the input output relation of y to u is independent of the choice of the state variables, it would also take two differentiations for u to appear if we use the original state space representation (50).

Continued from the previous page

Thus the control law

$$u = (a_0 x - 1 + a_1 x_2 + a_2 x_3 - \frac{b_0}{b_1} x_3) + \frac{1}{b_1}(-k_1 e - k_2 \dot{e} + \ddot{y}_d)$$

where $e = y - y_d$, yields an exponentially stable tracking error

$$\ddot{e} + k_2 \dot{e} + k_1 e = 0$$

Since this is a second order dynamics, the internal dynamics of our third order system can be described by only one state equation. For the internal dynamics x_1 can be used, because it can be shown that $x_1, y, \dot{y}$ are related to x_1, x_2, x_3 through a one to one transformation.

Digression

$$\begin{bmatrix} y \\ \dot{y} \\ x_1 \end{bmatrix} = \begin{bmatrix} b_0 & b_1 & 0 \\ 0 & b_0 & b_1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Thus $x_1, y, \dot{y}$ are related to x_1, x_2, x_3 through a 1-1 xform. EOD

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The internal dynamics is

$$\dot{x}_1 = x_2 = \frac{1}{b_1}(y - b_0 x_1) \rightarrow \dot{x}_1 + \frac{b_0}{b_1} x_1 = \frac{1}{b_1} y \tag{52}$$

Since y is bounded ($y = e + y_d$), we see that the internal stability of the internal dynamics depends on the location of zero $-\frac{b_0}{b_1}$ of the tf in (51). If the system is minimum phase, then the zero is in the left half plane, which implies that the internal dynamics in (52) is stable, independently of the initial conditions and of the magnitudes of the desired $y_d, \dot{y}_d, \ldots, y_d^{(r)}$ (where r is the relative degree.

Feedback linearization using Lie derivative notation

Consider the system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u} \quad (53)$$

$$\mathbf{y} = \mathbf{h}(\mathbf{x}) \quad (54)$$

where $\mathbf{x} \in \mathbb{R}^n$ is the state vector, $\mathbf{u} \in \mathbb{R}^m$ is the input vector, and $\mathbf{y} \in \mathbb{R}^p$ is the output vector.The goal is to develop a control input

$$\mathbf{u} = \mathbf{a}(\mathbf{x}) + \mathbf{b}(\mathbf{x})\mathbf{v} \quad (55)$$

that renders a linear input–output map between the new input $\mathbf{v}$ and the output. An outer-loop control strategy for the resulting linear control system can then be applied.

Next, we consider the case of feedback linearization of a single-input single-output (SISO) system. Similar results can be extended to multiple-input multiple-output (MIMO) systems. In this case, $u \in \mathbb{R}$ and $y \in \mathbb{R}$. We wish to find a coordinate transformation $\mathbf{z} = \mathbf{T}(\mathbf{x})$ that transforms our system (53) into the so-called normal form which will reveal a feedback law of the form

$$u = a(\mathbf{x}) + b(\mathbf{x})v \tag{56}$$

that will render a linear input–output map from the new input $v \in \mathbb{R}$ to the output y.

The goal of feedback linearization is to produce a transformed system whose states are the output y and its first (n-1) derivatives. To understand the structure of this target system, we use the Lie derivative. Consider the time derivative of $y = h(\mathbf{x})$, which we can compute using the chain rule,

$$\dot{y} = \frac{\mathrm{d}\, h(\mathbf{x})}{\mathrm{d}\, t} = \frac{\mathrm{d}\, h(\mathbf{x})}{\mathrm{d}\, \mathbf{x}}\dot{\mathbf{x}} = \frac{\mathrm{d}\, h(\mathbf{x})}{\mathrm{d}\, \mathbf{x}} f(\mathbf{x}) + \frac{\mathrm{d}\, h(\mathbf{x})}{\mathrm{d}\, \mathbf{x}} g(\mathbf{x}) u$$

Now we can define the Lie derivative of $h(\mathbf{x})$ along $f(\mathbf{x})$ as,

$$L_f h(\mathbf{x}) = \frac{\mathrm{d}\, h(\mathbf{x})}{\mathrm{d}\, \mathbf{x}} f(\mathbf{x}),$$

and similarly, the Lie derivative of $h(\mathbf{x})$ along $g(\mathbf{x})$ as,

$$L_g h(\mathbf{x}) = \frac{\mathrm{d}\, h(\mathbf{x})}{\mathrm{d}\, \mathbf{x}} g(\mathbf{x}).$$

With this new notation, we may express $\dot{y}$ as,

$$\dot{y} = L_f h(\mathbf{x}) + L_g h(\mathbf{x}) u$$

Note that the notation of Lie derivatives is convenient when we take multiple derivatives with respect to either the same vector field, or a different one. For example,

$$L_f^2 h(\mathbf{x}) = L_f L_f h(\mathbf{x}) = \frac{\mathrm{d}(L_f h(\mathbf{x}))}{\mathrm{d}\,\mathbf{x}} f(\mathbf{x}),$$

and

$$L_g L_f h(\mathbf{x}) = \frac{\mathrm{d}(L_f h(\mathbf{x}))}{\mathrm{d}\,\mathbf{x}} g(\mathbf{x}).$$

Example 6.8

$$h(\mathbf{x}) = \frac{1}{2}(x_1^2 + x_2^2), \; f(\mathbf{x}) = \begin{bmatrix} -x_2 \\ -x_1 - \mu(1 - x_1^2)x_2 \end{bmatrix}, \; g(\mathbf{x}) = \begin{bmatrix} -x_1 - x_1 x_2^2 \\ -x_2 + x_1^2 x_2 \end{bmatrix}$$

$$L_f h(\mathbf{x}) = \frac{\partial h}{\partial \mathbf{x}} f(\mathbf{x}) = \begin{bmatrix} x_1 x_2 \end{bmatrix} \begin{bmatrix} -x_2 \\ -x_1 - \mu(1 - x_1^2)x_2 \end{bmatrix} = -\mu(1 - x_1^2)x_2^2$$

$$L_g h(\mathbf{x}) = \frac{\partial h}{\partial \mathbf{x}} g(\mathbf{x}) = \begin{bmatrix} x_1 x_2 \end{bmatrix} \begin{bmatrix} -x_1 - x_1 x_2^2 \\ -x_2 + x_1^2 x_2 \end{bmatrix} = -(x_1^2 + x_2^2)$$

$$L_f L_g h(\mathbf{x}) = \frac{\partial (L_g h)}{\partial \mathbf{x}} f(\mathbf{x}) = -2 \begin{bmatrix} x_1 x_2 \end{bmatrix} \begin{bmatrix} -x_2 \\ -x_1 - \mu(1 - x_1^2)x_2 \end{bmatrix} = 2\mu(1 - x_1^2)x_2^2$$

In our feedback linearized system made up of a state vector of the output y and its first (n-1) derivatives, we must understand how the input u enters the system. To do this, we introduce the notion of relative degree. Our system given by (53) is said to have relative degree r at a point $\mathbf{x}_0$ if,
$L_g L_f^k h(\mathbf{x}) = 0 \qquad \forall \mathbf{x}$ in a neighbourhood of $\mathbf{x}_0$ and all $k \leq r - 2$
$L_g L_f^{r-1} h(\mathbf{x}_0) \neq 0$
Considering this definition of relative degree in light of the expression of the time derivative of the output y, we can consider the relative degree of our system (53) to be the number of times we have to differentiate the output y before the input u appears explicitly. In an LTI system, the relative degree is the difference between the degree of the transfer function's denominator polynomial (i.e., number of poles) and the degree of its numerator polynomial (i.e., number of zeros).

Linearization by feedback

For the discussion that follows, we will assume that the relative degree of the system is n. In this case, after differentiating the output n times we have,

$$\begin{aligned}
y &= h(\mathbf{x}) \\
\dot{y} &= L_f h(\mathbf{x}) \\
\ddot{y} &= L_f^2 h(\mathbf{x}) \\
&\vdots \\
y^{(n-1)} &= L_f^{n-1} h(\mathbf{x}) \\
y^{(n)} &= L_f^n h(\mathbf{x}) + L_g L_f^{n-1} h(\mathbf{x}) u
\end{aligned}$$

where the notation $y^{(n)}$ indicates the nth derivative of y. Because we assumed the relative degree of the system is n, the Lie derivatives of the form $L_g L_f^i h(\mathbf{x})$ for $i = 1, \ldots, n-2$ are all zero. That is, the input u has no direct contribution to any of the first $(n-1$th derivatives.

The coordinate transformation $\mathbf{T}(\mathbf{x})$ that puts the system into normal form comes from the first $(n-1)$ derivatives.

In particular,

$$\mathbf{z} = \mathbf{T}(\mathbf{x}) = \begin{bmatrix} z_1(\mathbf{x}) \\ z_2(\mathbf{x}) \\ \vdots \\ z_n(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \\ \vdots \\ y^{(n-1)} \end{bmatrix} = \begin{bmatrix} h(\mathbf{x}) \\ L_f h(\mathbf{x}) \\ \vdots \\ L_f^{n-1} h(\mathbf{x}) \end{bmatrix}$$

transforms trajectories from the original $\mathbf{x}$ coordinate system into the new $\mathbf{z}$ coordinate system. So long as this transformation is a diffeomorphism, smooth trajectories in the original coordinate system will have unique counterparts in the $\mathbf{z}$ coordinate system that are also smooth. Those $\mathbf{z}$ trajectories will be described by the new system,

$$\begin{cases} \dot{z}_1 & = L_f h(\mathbf{x}) = z_2(\mathbf{x}) \\ \dot{z}_2 & = L_f^2 h(\mathbf{x}) = z_3(\mathbf{x}) \\ & \vdots \\ \dot{z}_n & = L_f^n h(\mathbf{x}) + L_g L_f^{n-1} h(\mathbf{x}) u \end{cases} .$$

Hence, the feedback control law

$$u = \frac{1}{L_g L_f^{n-1} h(\mathbf{x})} (-L_f^n h(\mathbf{x}) + v)$$

renders a linear input–output map from v to $z_1 = y$. The resulting linearized system

$$\begin{cases} \dot{z}_1 & = z_2 \\ \dot{z}_2 & = z_3 \\ & \vdots \\ \dot{z}_n & = v \end{cases}$$

is a cascade of n integrators, and an outer-loop control v may be chosen using standard linear system methodology. In particular, a state-feedback control law of

$$v = -\mathbf{Kz} \qquad ,$$

where the state vector $\mathbf{z}$ is the output y and its first (n-1) derivatives, results in the LTI system

$$\dot{\mathbf{z}} = \mathbf{Az}$$

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z}$$

with,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -k_1 & -k_2 & -k_3 & \dots & -k_n \end{bmatrix}.$$

So, with the appropriate choice of k, we can arbitrarily place the closed-loop poles of the linearized system.

Unstable zero dynamics

Feedback linearization can be accomplished with systems that have relative degree less than n. However, the normal form of the system will include zero dynamics (i.e., states that are not observable from the output of the system) that may be unstable. In practice, unstable dynamics may have deleterious effects on the system (e.g., it may be dangerous for internal states of the system to grow unbounded). These unobservable states may be stable or at least controllable, and so measures can be taken to ensure these states do not cause problems in practice.

In this section basic principles of the SMC are introduced and several examples are provided for it. The examples in this section involve the second order systems which allow usage of phase portraits in the presentation.

Basically, a sliding mode control system contains two or more feedback subsystems which each of them has a fixed structure. According to a control law (alternatively called a decision rule) only one of the feedback subsystems is employed during the operation of the system. At each region of the state space, the control mechanism employs a proper fixed feedback subsystem. These regions are specified by the designer; and their boundaries are called switching (or discontinuity) surfaces. When a switching surface is crossed by the system trajectory, the feedback subsystem just before the crossing is unemployed and a new feedback subsystem specified for the newly entered region is employed.

The SMC system may have new valuable properties such that they cannot be obtained by using any single fixed feedback subsystem alone. For instance, each of the two fixed feedback subsystems may cause instability alone. However, switching between them appropriately may result in a stable system.

In SMC systems, after some transients, the state is restricted to a switching surface, called the sliding surface. System behavior on this surface is called the sliding mode. Restricting the states of a dynamic system to a surface means a static relationship among the state components. Due to the static relationship, the order of dynamic system on the surface reduces. Therefore, the system behavior on the surface is completely characterized by solving a differential equation whose order is less than that of the unrestricted system. The reduced order differential equation depends on the surface parameters. Therefore, designer must select a surface that gives rise to a desired reduced order differential equation.

The design of an SMC system has two phases:
(1) Designing the surface parameters causing a desired reduced order differential equation.
(2) Designing a control law to reach this surface; and once it is reached to keep the trajectory on it for all subsequent time.
In the following example a simple SMC system is analyzed. Within this example the basic concepts of the SMC systems are presented.

Example 7.1

Consider the system model

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{57}$$

where $\mathbf{x} := [x_1\ x_2] \in \mathbb{R}2$ and $u \in \mathbb{R}$ represent state vector and control input of the system respectively. The control input has the structure

$$u = \begin{bmatrix} k_1 & k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{58}$$

that is, u is a linear function of states. Depending on the system states, the feedback coefficients (k_1, k_2) assume values from the set $\{(2, -1), ((-1, 0)\}$. The system states and their associated feedback coefficients are given by

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given by

$$[k_1\ k_2] = [2 \quad -1] \text{ when } s(\mathbf{x}(t))x_1(t) < 0 \tag{59}$$

and

$$[k_1\ k_2] = [-1 \quad 0] \text{ when } s(\mathbf{x}(t))x_1(t) > 0 \tag{60}$$

where the switching function s is defined by $s(x(t)) := x_2(t) + x_1(t)$[a] The region of the state space satisfying condition (59) is shaded in the figure below. Throughout this example it will be called Region A. The complementary region in the state space satisfies condition (60), and it will be called region B.

[a] For the sake of simple presentation, the arguments of the functions may be suppressed after their initial appearance in the text

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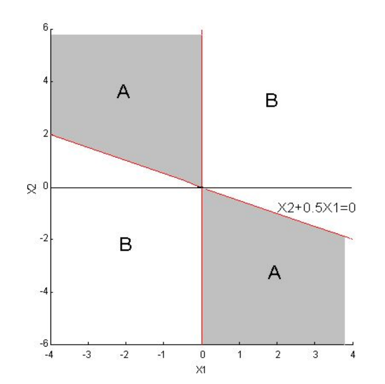


The lines $S := \{(x_1, x_2) \in \mathbb{R} : \;\; x_2 + 0.5x_1 = 0\}$ and $x_1 = 0$ will be referred to as switching lines. It will be shown that, under the control law (58)-(60)only the line S attracts all the neighboring trajectories, and is called the sliding line.

System (57) with feedback (58) can be written as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

or

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ k_1 & k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{61}$$

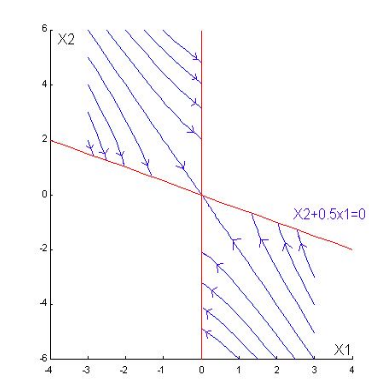
We next present the analysis of the system (57)-(60) in a phase portrait form. This can be done by plotting system trajectories for various initial conditions. Let us start with the initial condition in Region A, that is, the initial state $(x_1(0), x_2(0))$ satisfies $s(\mathbf{x}(0))x_1(0) < 0$. This implies that the feedback (59) is active. Then the corresponding system can be written as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{62}$$

format, is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-2t} \tag{63}$$

where c_1, c_2 are constant scalars depending on the initial conditions. In the figure below the phase portrait of this system (62) overlapping on region A is given.



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The solution of this equation, in eigenstructure Notice that every trajectory starting in region A leaves the region and crosses into region B. The exceptions are the ones starting on the line $x_2 + 2x_1 = 0$ Note that this line represents restriction of the solution (63) to its stable mode. The unstable mode of (63) is the line $x_2 - x_1 = 0$, which lies in Region B. Now consider the remaining case where the initial state $(x_1(0), x_2(0))$ satisfies the condition $s(x(0))x_1 > 0$, that is, the condition of being in Region B. In this region feedback coefficients (k_1, k_2) equal $(-1, 0)$. This gives rise to the system dynamics

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{64}$$

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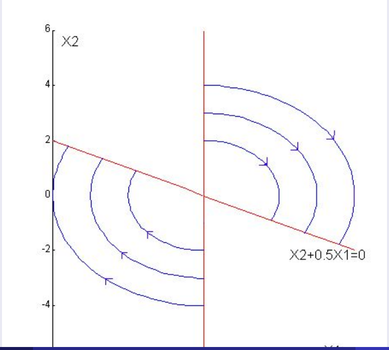
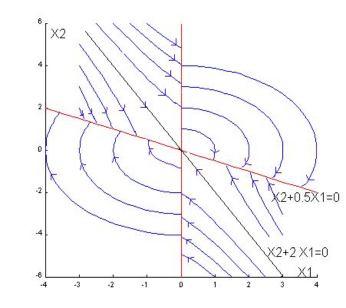
The solution of this equation, in eigenstructure format, is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + c_2 \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} \tag{65}$$

where c_1, c_2 are constant scalars depending on the initial conditions. On the phase plane, solution (65) corresponds clockwise circular motions. Its portion overlapping on Region B is shown below.



The complete phase portrait of (57)-(60) is obtained by combining the individual phase portraits obtained for Regions A and B. It is given below.



Looking at the combined phase portrait of the system, observe that the line S attracts all the trajectories in its neighborhood. Because of this, the line S is called the "sliding line" (or "sliding surface"). Dynamic behavior of the system on this line is called the "sliding mode" (or "sliding regime").

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Analyzing the phase portrait of the current example, the sliding line, in a finite time, attracts all the trajectories no only in its neighborhood but also in the complete state space with the exception of the ones on the line $x_2 + 2x_1 = 0$.

Let us consider the case where a trajectory starts on the sliding line. Theoretically, this is equivalent to the analysis of the trajectories after the moment it intercepts the sliding line. Regarding 1st row of Equation (57)

$$x_2 = \dot{x}_1 \tag{66}$$

and the static relationship $x_2 + 0.5x_1 = 0$ between the state components on the sliding line, one may write the system dynamics on the sliding line as

$$\dot{x}_1 + 0.5x_1 = 0$$

This has the general solution

$$x_1(t) = Ae^{-0.5t}$$

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one may write the system dynamics on the sliding line as

$$\dot{x}_1 + 0.5x_1 = 0$$

This has the general solution

$$x_1(t) = Ae^{-0.5t}$$

where the arbitrary constant A is the projection of the initial point on the x_1 axis. Using the relationship (66), the other component of the state can be obtained:

$$x_2(t) = 0.5Ae^{-0.5t}$$

Noting that both x_1 and x_2 are decreasing, all the trajectories on the sliding line tend to the origin. Therefore the sliding mode associated with the line $x_2 + 0.5x_1 = 0$ is stable.

Example 7.2

For the preceding example, a different switching line may or may not attract the trajectories in its neighborhood. Consider the system given by (57)-(58) with a new switching line

$$S_2 = \{(x_1, x_2) \in \mathbb{R}^2 : \ x_2 + 3x_1 = 0\}$$

Using this switching line and and the control law (59)-(60) with a redefined switching function

$$s(x(t)) := x_2(t) + 3x_1(t)$$

we obtained the phase portrait shown below.

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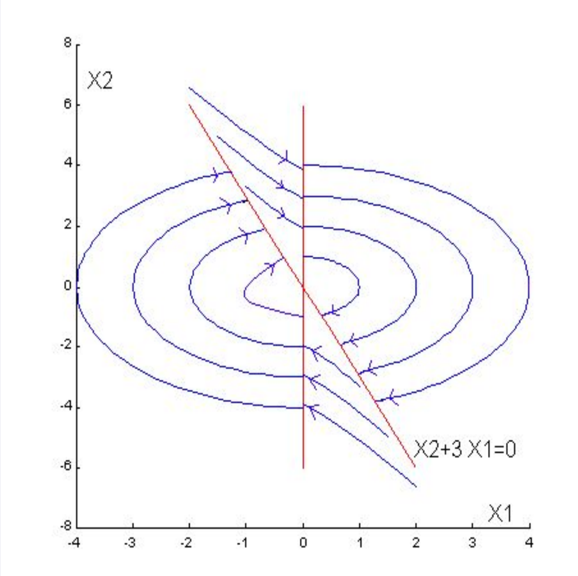

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Looking at the phase portrait above, we see that the switching line $x_2 + 3x_1 = 0$ does not attract all the trajectories in its neighborhood. However, it can be shown that if the control strategy is redesigned appropriately then this line attracts all the neighboring trajectories. So, it can be a stable sliding line.

Not all the lines can be a stable sliding line by designing the control law. Some are pre-qualified for being stable. We should choose the switching line appropriately so that it can work as a stable sliding line.

Designing a Stable Sliding Surface

Consider the LTI system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \tag{67}$$

where $\mathbf{x} \in \mathbb{R}^n$ and $u \in \mathbb{R}$are the state vector and the control input respectively. The sliding surface to be designed is represented by

$$S \stackrel{\triangle}{=} \{\mathbf{x} \in \mathbb{R} : \ \mathbf{G}\mathbf{x} = 0\} \tag{68}$$

where $\mathbf{G} \in \mathbb{R}^{1\times n}$. The set (68) represents a surface on the n-dimensional plane. We want to find an input u_{eq} so that any trajectory that starts on S will stay on it for ever. Define the function s as $s \stackrel{\triangle}{=} Gx$ and note that $s = 0$ means $\mathbf{x}$ is on the surface S.

Let at some time $\mathbf{x}$ satisfies $s = 0$. Now, if $\dot{s} = 0$ then $\mathbf{x}$ keeps staying on S afterwards. We want to find u that satisfies

$$\dot{s} = \mathbf{G}\dot{\mathbf{x}} = \mathbf{G}(\mathbf{A}\mathbf{x} + \mathbf{B}u_{eq}) = 0$$

This results in

$$u_{eq} = -(\mathbf{G}\mathbf{B})^{-1}\mathbf{G}\mathbf{A}\mathbf{x} \stackrel{\triangle}{=} [k_1 \ k_2]\mathbf{x}$$

The "equivalent control input" u_{eq} keeps the trajectory on the surface S for all subsequent times. The coefficient matrix $-(\mathbf{G}\mathbf{B})^{-1}\mathbf{G}\mathbf{A}$ is called the "equivalent feedback matrix". For the inverse to be defined we assume invertibility of $\mathbf{G}\mathbf{B}$.

Le us find out how the trajectories behave on the surface S. To find out the answer we use the equivalent input u_{eq} in (67).

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}(-(\mathbf{G}\mathbf{B})^{-1}\mathbf{G}\mathbf{A}\mathbf{x}) = [\mathbf{A} - \mathbf{B}(\mathbf{G}\mathbf{B})^{-1}\mathbf{G}\mathbf{A}]\mathbf{x} \stackrel{\triangle}{=} [\mathbf{A} + \mathbf{B}\mathbf{F}_{eq}]\mathbf{x} \quad (69)$$

This is the dynamics restricted to the sliding surface. On the sliding surface, the trajectory may or may not go to the origin of the state space. If it approaches the origin then we call it a stable sliding line. We find out the stability by analyzing the system dynamics restricted to the sliding line. It is

$$\dot{\mathbf{x}} = [\mathbf{A} + \mathbf{B}\mathbf{F}_{eq}]\mathbf{x}$$

The above system is stable if all the eigenvalues of $\mathbf{A} + \mathbf{B}\mathbf{F}_{eq}$ are in the negative half of complex plane.

Motivations

Describing function method reveals some frequency response features of nonlinear systems.

It is an analysis based on approximation and it facilitates prediction nonlinear behavior.

Its main use is to predict the limit cycles in nonlinear systems.

"It is only an approximation method, there exists inaccurate predictions:

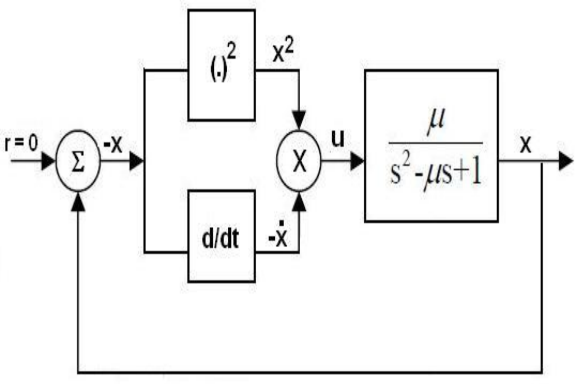
a. There exists difference between the prediction values and actual values in amplitude and frequency of limit cycle.

b. The limit cycle predicted by the describing function method may not exist.

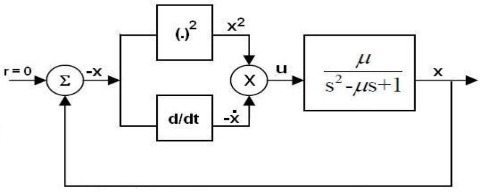
c. The actual exist limit cycle may not be predicted by the describing function."

Van der Pol Equation

Example 8.1



We want analyze this system. We start with the i-o relationship in the time domain.



Consider the linear block with the transfer function $\frac{X}{U} = \frac{\mu}{s^2 - \mu s + 1}$. In the differential equation domain it is

$$\mu u = \ddot{x} - \mu \dot{x} + x$$

Tracking the signal flow from the output of the linear block reveals that $u = -x^2 \dot{x}$. Thus the differential equation above becomes

$$-\mu x^2 \dot{x} = \ddot{x} - \mu \dot{x} + x$$

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$$

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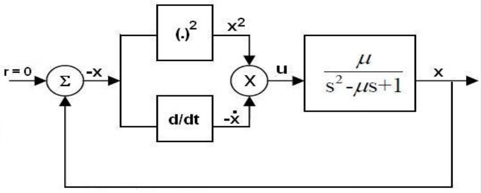

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0, \qquad \text{(Van der Pol Equation)}$$

Now let us assume that the signal at the output of the linear block is sinusoidal: $x(t) = A \sin wt$. Tracking the signal in the direction of the arrows we see that $u = -x^2\dot{x} = -A^2 \sin^2(wt)Aw\cos(wt)$

$$
\begin{aligned}
u &= -\frac{A^3 w}{2}[(1 - \cos(2wt))\cos(wt)] \\
\rightarrow \qquad &= -\frac{A^3 w}{2}[\cos(wt) - \frac{\cos(wt)+\cos(3wt)}{2}] \\
&= -\frac{A^3 w}{2}[\frac{\cos(wt)-\cos(3wt)}{2}]
\end{aligned}
$$

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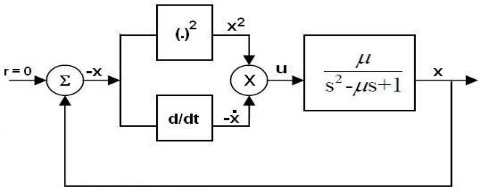

Linear blocks input is $u = -\frac{A^3 w}{2}[\frac{\cos(wt)-\cos(3wt)}{2}]$. This has two different frequency components, one at w and the other at $3w$. However, at the output of the block we assumed that the only frequency component is at w. This may be explained by noting that the linear block has a low pass property, so it blocks the signal at $3w$. Because of this, we may view the signal u as

$$u = -\frac{A^3 w}{4}\cos(wt) = \frac{A^2 w}{4}\frac{d}{dt}(-A\sin(wt)) = \frac{A^2}{4}\frac{d}{dt}(-x)$$

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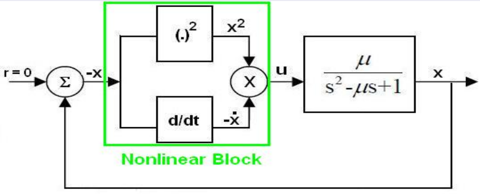

$$u = \frac{A^2}{4}\frac{d}{dt}(-x)$$

Now let us investigate what has been done by the nonlinear block. Clearly it differentiates the input and multiplies by the gain $\frac{A^2 w}{4}$. The combined effect of differentiation and amplification in the frequency domain is

$$N(A, w) \triangleq \frac{A^2(iw)}{4}$$

The term $N(A, w)$ is called describing function of the nonlinear block above.

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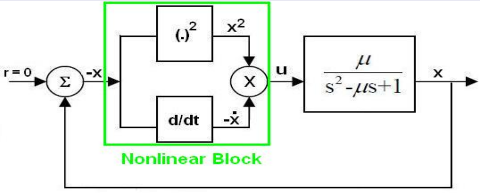

The term $N(A, w)$ is called describing function of the nonlinear block above. In some sense, it may be viewed as nonlinear block's transfer function. Notice that it depends on both the magnitude and frequency of the input signal.

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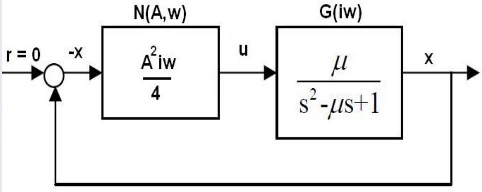


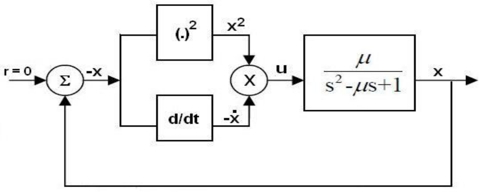
$$
\begin{aligned}
x &= A\sin(wt) \\
&= G(iw)u \\
&= G(iw)N(A,w)(-x)
\end{aligned}
$$

Thus

$$[1 + G(iw)N(A,w)]x = 0$$

$$[1 + \frac{A^2 iw}{4} \frac{\mu}{(iw)^2 - \mu(iw) + 1}]x = 0$$

Equating real and imaginary parts of the above equation to zero yields $A = 2$ and $w = 1$.



Our assumption that $x = A \sin wt$ holds true when $A = 2$ and $w = 1$. For the stability analysis of the system let us write the characteristic equation of the linear approximation:

$$1 + \frac{A^2 s}{4} \frac{\mu}{s^2 - \mu s + 1} = 0$$

Its eigenvalues are:

$$\lambda_{1,2} = -\frac{1}{8}\mu(A^2 - 4) \pm \sqrt{\frac{1}{64}\mu^2(A^2 - 4)^2 - 1}$$

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$$\lambda_{1,2} = -\frac{1}{8}\mu(A^2 - 4) \pm \sqrt{\frac{1}{64}\mu^2(A^2 - 4)^2 - 1}$$

When $A = 2$, the eigenvalues are real, $\pm iw$, as expected.
When $A > 2$, we have eigenvalues with negative real part, therefore, the amplitude decreases to $A = 2$.
When $A < 2$, we have eigenvalues with positive real part, therefore, the amplitude increases to $A = 2$.
Thus the limit cycle corresponding to the amplitude $A = 2$ is stable.

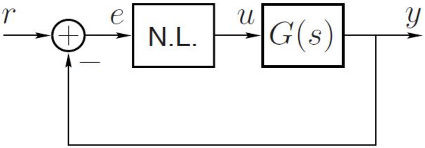
Fourier Series

A periodic function $u(t) = u(t+T)$ has a Fourier series representation

$$\begin{aligned} u(t) &= \tfrac{a_0}{2} + \textstyle\sum_1^\infty (a_n \cos nwt + b_n \sin nwt) \\ &= \tfrac{a_0}{2} + \textstyle\sum_1^\infty \sqrt{a_n^2 + b_n^2} \sin[nwt + \arctan \tfrac{a_n}{b_n}] \end{aligned}$$

where $w = \frac{2\pi}{T}$ and

$$a_n(w) = \frac{2}{T} \int_0^T u(T) \cos(nwt) dt, \quad b_n(w) = \frac{2}{T} \int_0^T u(T) \sin(nwt) dt$$



$e(t) = A\sin(wt)$ results in

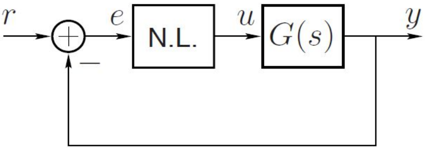
$$u(t) = \frac{a_0}{2} + \sum_1^{\infty} \sqrt{a_n^2 + b_n^2}\,\sin[nwt + \arctan\frac{a_n}{b_n}]$$

$$y(t) \approx |G(iw)|\sqrt{a_n^2 + b_n^2}\,\sin[nwt + \arg G(iw) + \arctan\frac{a_n}{b_n}]$$

Definition of Describing Function

The describing function is

$$n(A,w) = \frac{b_1(w) + ia_(w)}{A}$$



If G is low pass and $a_0 = 0$, then

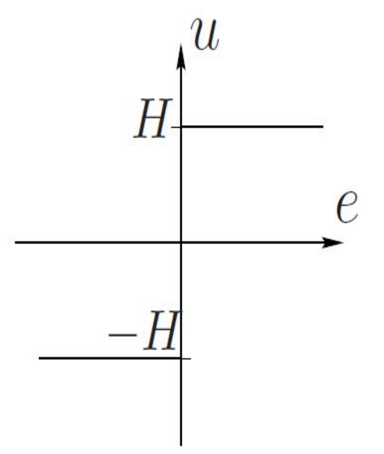
$$\hat{u} = |N(A,w)| A \sin(wt + \arg N(A,w))$$

can be used instead of u(t) to analyze the system.

Example 8.2

Describing function of a relay
Describing function of a relay

Let $\phi \overset{\triangle}{=} \frac{2\pi}{T} t$ then $dt = \frac{T}{2\pi} d\phi$; integration factor becomes $\frac{1}{\pi}$ and integral limits become 0 to 2π.



$$a_1 = \frac{1}{\pi} \int_0^{2\pi} u(\phi) \cos\phi d\phi = 0$$

$$b_1 = \frac{1}{\pi} \int_0^{2\pi} u(\phi) \sin\phi d\phi$$

$$= \frac{2}{\pi} \int_0^{\pi} H \sin\phi d\phi = \frac{4H}{\pi}$$

The describing function is

$$N(A) = \frac{4H}{\pi A}$$

Odd static nonlinearities

If f and g are odd static nonlinearities with describing functions N_f and N_g, then

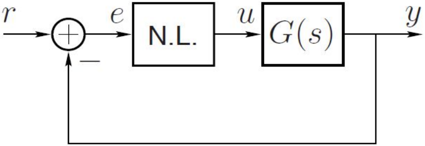
$$\mathrm{Im}\ N_f(A, w) = 0$$

$$N_f(A, w) = N_f(A)$$

$$N_{\alpha f}(A) = \alpha N_f(A)$$

$$N_{f+g}(A) = N_f(A) + N_g(A)$$

Periodic Solutions



$$y = G(iw)u = -G(iw)N(A)y \rightarrow y(1 + G(iw)N(A)) = 0$$

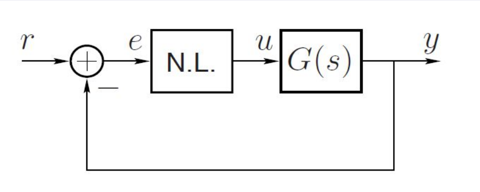
For this system to have nonzero response we must have

$$1 + G(iw)N(A) = 0$$

This is the periodicity condition. In an other arrangement

$$G(iw) = -\frac{1}{N(A)}$$

Example 8.3



Given $G(s) = \frac{3}{(s+1)^3}$ and $u = \operatorname{sgn} e$, find the limit cycles if any exists.
For the relay $N(A) = \frac{4}{\pi A}$ which is real. Therefore,

$$G(iw) = -\frac{1}{N(A)}$$

holds for a real $G(iw)$. Plotting $G(iw)$, one observes that it is real (crossing the real axis) at -0.375 Now we have $-0.375 = -\frac{1}{N(A)}$
$\rightarrow A = 0.478$ The real axis crossing of $G(iw)$ occurs at $w = 1.75$

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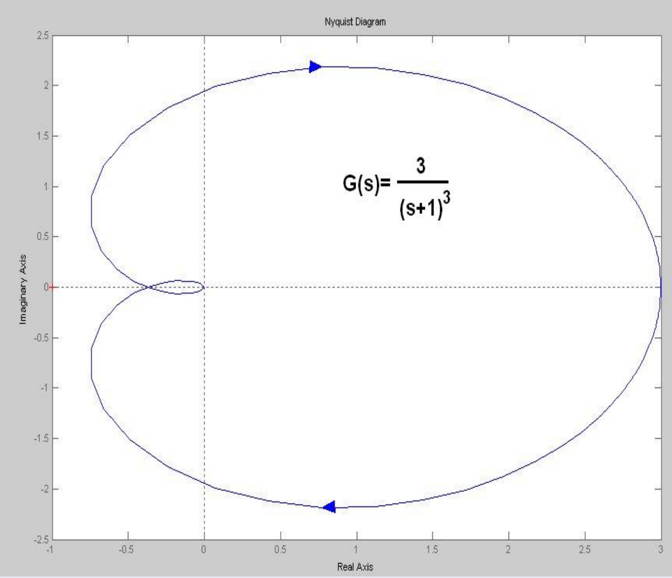
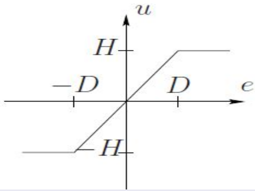
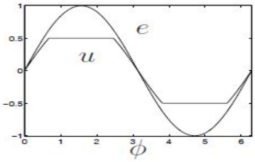


Example 8.4





Let $e = A\sin wt = A\sin\phi$ and $H = D$. For $\phi \in (0, \pi)$ we have

$$\begin{cases} A\sin\phi, & \phi \in (0, \phi_0) \cup (\pi - \phi_0, \pi) \\ D, & \phi \in (\phi_0, \pi - \phi_0) \end{cases}$$

where $\phi_0 = \arcsin\frac{D}{A}$.

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} u(\phi)\cos\phi d\phi = 0$$

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$$a_1 = \frac{1}{\pi}\int_0^{2\pi} u(\phi)\cos\phi d\phi = 0$$

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} u(\phi)\sin\phi d\phi$$

$$= \frac{4}{\pi}\int_0^{\frac{\pi}{2}} u(\phi)\sin\phi d\phi$$

$$= \frac{4A}{\pi}\int_0^{\phi_0} \sin^2\phi d\phi + \frac{4D}{\pi}\int_{\phi_0}^{\frac{\pi}{2}} \sin\phi d\phi$$

$$= \frac{A}{\pi}(2\phi_0 + \sin 2\phi_0)$$

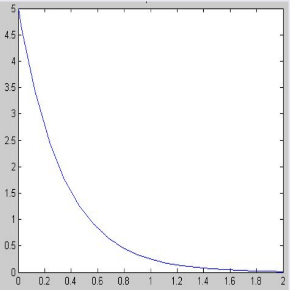
APPENDIX A
SOLVING IMPLICIT DIFFERENTIAL EQUATIONS BY MATLAB

Example 9.1

Consider the differential equation

$$\frac{dx}{dt} + 3x = 0, \quad x(0) = 5$$

It has the solution $x(t) = 5e^{-3t}$.



For every $A \in \mathbb{R}$, initial condition $x(0) = A$ results in a solution.

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For every $A \in \mathbb{R}$, initial condition $x(0) = A$ results in a solution. However, if we require additional condition $\dot{x}(0) = B$, for some $B \in \mathbb{R}$ we may or may not find a solution for the differential equation.

For the above example, the initial condition is $x(0) = 5$ which resulted in the solution $x(t) = 5e^{-3t}$. This automatically implies that $\dot{x}(0) = -15$. The reason is

$$\dot{x}(0) = -3 \times 5e^{-3t}\}_{t=0} = -15$$

Now assume that a differential equation solving software asks you provide $\dot{x}(0)$ as well as $x(0)$. Then we need to provide a consistent $(x(0), \dot{x}(0))$ pair.

Implicit differential equation solver of MATLAB requires redundantly many initial conditions for the solution process. Consistency of these initial conditions is the responsibility of the user.

Explicit differential equations have the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$$

On the other hand, implicit differential equations have the form

$$\mathbf{f}(\mathbf{x}, \dot{\mathbf{x}}, t) = 0$$

Example 9.2

$$\begin{aligned} \dot{x}_1 &= 2x_1 + x_1 x_2^2 + 1 \\ \dot{x}_2 &= x_1^2 + x_2 \end{aligned} \qquad \text{(Explicit DE)}$$

Example 9.3

$$\begin{aligned} \dot{x}_1 + \dot{x}_2 x_1 + x_2 + \dot{x}_2 &= 0 \\ x_1 x_2^3 + \dot{x}_1 + \dot{x}_2 x_1 &= 0 \end{aligned} \qquad \text{(Implicit DE)}$$

Example 9.4

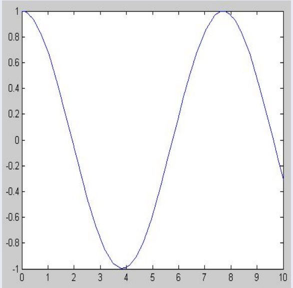
Consider

$$m\ddot{x} + kx = 0,\ x(0) = 1,\ \dot{x}(0) = 0$$

We may write it in state space form by defining $x_1 \stackrel{\triangle}{=} x$ and $x_2 \stackrel{\triangle}{=} \dot{x}$:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{k}{m}x_1 \end{aligned}$$

When $k = 2$ and $m = 3$, its solution $t \leftrightarrow x_1$ is



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Let us write the equation in the implicit form

$$\begin{aligned} \dot{x}_1 - x_2 &= 0 \\ m\dot{x}_2 + kx_1 &= 0 \end{aligned}$$

For a numerical solution we may use ode15i solver of MATLAB:

```
f=@(t,x,xd)[xd(1)-x(2); 3*xd(2)+2*x(1)];
x0=[1 0]; xd0=[5 3];
x0F=[1 1]; xd0F=[];
[x0 xd0]=decic(f,0,x0,x0F,xd0,xd0F);
r=ode15i(f,[0 10],x0,xd0);
plot(r.x, r.y(1,:))
```

It uses ode15i.
It is a one file program.
It uses MATLAB routine "decic" for generating a consistent $\dot{x}$.
$r.x$ and $r.y$ are time and solution vectors respectively.

Example 9.5

Consider

$$\begin{aligned} \ddot{x} + \ddot{y} + x &= 0 \\ \dot{x} + 3\ddot{y} + y &= 0 \end{aligned}$$

We want to write the above equations as a set of 1st order equations. The routine ode15i requires this. Define $x_1 \triangleq x$, $x_2 \triangleq \dot{x}$, $x_3 \triangleq y$, and $x_4 \triangleq \dot{y}$:

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_3 &= x_4 \\ \dot{x}_2 + \dot{x}_4 + x_1 &= 0 \\ x_2 + 3\dot{x}_4 + x_3 &= 0 \end{aligned}\right\}$$

Now we are done, but we can go further to write it in the explicit form.

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_3 &= x_4 \\ \dot{x}_2 + \dot{x}_4 &= -x_1 \\ 3\dot{x}_4 &= -x_2 - x_3 \end{aligned}\right\}$$

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$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

APPENDIX B

EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR NONLINEAR DIFFERENTIAL EQUATIONS

Definition 9.1

A function $\mathbf{f}(t, \mathbf{x})$ satisfies the Lipschitz condition at $(t_0, \mathbf{x}_0)$ if the inequality

$$||\mathbf{f}(t, \mathbf{x}) - \mathbf{f}(t, \mathbf{y})|| \leq L||\mathbf{x} - \mathbf{y}||$$

is satisfied for all $(t, \mathbf{x})$ and $(t, \mathbf{y})$ for the same fixed $L \in \mathbf{R}^+$ in the neighborhood of (t_0, x_0). Then we call the function $\mathbf{f}$ locally Lipschitz at $(t_0, \mathbf{x}_0)$.

The inequality may be written in the form

$$\frac{||\mathbf{f}(t, \mathbf{x}) - \mathbf{f}(t, \mathbf{y})||}{||\mathbf{x} - \mathbf{y}||} \leq L$$

which provides the intuition that the locally Lipschitz function $\mathbf{f}$ has a bounded slope in the neighborhood of $(t_0, \mathbf{x}_0)$.

Theorem 9.1

If $\mathbf{f}(t, \mathbf{x})$ is piecewise continuous in t and satisfies the Lipschitz condition

$$||\mathbf{f}(t, \mathbf{x}) - \mathbf{f}(t, \mathbf{y})|| \leq L||\mathbf{x} - \mathbf{y}||$$

in the neighborhood of $(t_0, \mathbf{x}_0)$, then there exits some $\delta > 0$ such that the differential equation

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}), \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

has a unique solution over $[t_0, t_0 + \delta]$.

Example 9.6

$f(x) = x^{\frac{1}{3}}$ is not locally Lipschitz at $x = 0$ since its slope $f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$ tends to ∞ as $x \to 0$.

Consider

$$\dot{x} = x^{\frac{1}{3}},\ x(0) = 0$$

Sufficient condition for existence and uniqueness of a solution, that is $x^{\frac{1}{3}}$ is locally Lipschitz, is not satisfied. Thus the differential equation may or may not have a unique solution. An analysis shows that this d.e. does not have a unique solution. The solutions

$$x(t) = (\frac{2t}{3})^{\frac{3}{2}}$$

and

$$x(t) = 0$$

both satisfy the differential equation and its initial condition.