Example 1

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$$

$$\frac{1}{s(s^2+1)} = \frac{A}{s}\frac{(s^2+1)}{(s^2+1)} + \frac{Bs+C}{s^2+1}\frac{s}{s}$$

$$\frac{1}{s(s^2+1)} = \frac{A(s^2+1)+(Bs+C)s}{s(s^2+1)}$$

$$1 = A(s^2+1)+(Bs+C)s$$

$$1 = (A+B)s^2 + Cs + A$$

$$\rightarrow A+B=0,\ C=0,\ A=1$$

$$\mathcal{L}^{-1}\{\frac{1}{s(s^2+1)}\} = \mathcal{L}^{-1}\{\frac{1}{s}\} - \mathcal{L}^{-1}\{\frac{s}{s^2+1}\} = 1-\cos t$$

Example 2

$$\frac{dy}{dt} - 2y = e^{5t},\ y(0) = 3$$

Take the Laplace transform of both sides. Let the Laplace transform of the unknown function y be Y which is also unknown meanwhile.

$$sY - y(0) - 2Y = \frac{1}{s-5} \to (s-2)Y - 3 = \frac{1}{s-5}$$

$$Y = \frac{3s-14}{(s-2)(s-5)} = \frac{A}{s-2} + \frac{B}{s-5}$$

To find A, multiply both sides by $(s-2)$ and evaluate at $s=2$:

$$\frac{3s-14}{(s-2)(s-5)} \times (s-2) = \frac{A}{s-2} \times (s-2) + \frac{B}{s-5} \times (s-2)$$

$$\left[\frac{3s-14}{(s-5)} = A + \frac{B}{s-5} \times (s-2)\right]_{s=2}$$

Example 2 (cont.)

$$\left[\frac{3s-14}{(s-5)} = A + \frac{B}{s-5} \times (s-2)\right]_{s=2}$$

$$\frac{3 \times 2 - 14}{(2-5)} = A + \frac{B}{2-5} \times (2-2) \rightarrow A = \frac{8}{3}$$

To find B, multiply both sides by $(s-5)$ and evaluate at $s=5$. This gives B as $\frac{1}{3}$. Thus

$$Y = \frac{3s-14}{(s-2)(s-5)} = \frac{\frac{8}{3}}{s-2} + \frac{\frac{1}{3}}{s-5} \leftrightarrow \frac{8}{3}e^{2t} + \frac{1}{3}e^{5t}$$

Example 3

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} - 8y = 0,\ y(0) = 3,\ y'(0) = 6$$

$$\{s^2Y - sy(0) - y'(0)\} - 2\{sY - y(0)\} - 8Y = 0$$

$$[s^2 - 2s - 8]Y - 3s = 0$$

$$Y = \frac{3s}{(s-4)(s+2)} = \frac{A}{s-4} + \frac{B}{s+2}$$

$$A = \left[\frac{3s}{(s-4)(s+2)} \times (s-4)\right]_{s=4} = 2$$

$$B = \left[\frac{3s}{(s-4)(s+2)} \times (s+2)\right]_{s=-2} = 1$$

$$Y = \frac{3s}{(s-4)(s+2)} = \frac{2}{s-4} + \frac{1}{s+2} \leftrightarrow 2e^{4t} + e^{-2t}$$

Example 4

$$\frac{d^2y}{dt^2} + y = e^{-2t}\sin t,\ y(0) = 0,\ y'(0) = 0$$

$$\{s^2Y - sy(0) - y'(0)\} + Y = \frac{1}{[(s+2)^2+1]}$$

$$\{s^2Y - s0 - 0\} + Y = \frac{1}{[(s+2)^2+1]}$$

$$Y = \frac{1}{(s^2+1)[(s+2)^2+1]} = \frac{As+B}{s^2+1} + \frac{Cs+D}{(s+2)^2+1}$$

$$\frac{1}{(s^2+1)[(s+2)^2+1]} = \frac{As+B}{s^2+1}\frac{(s+2)^2+1}{(s+2)^2+1} + \frac{Cs+D}{(s+2)^2+1}\frac{s^2+1}{s^2+1}$$

$$\frac{1}{(s^2+1)[(s+2)^2+1]} = \frac{(As+B)((s+2)^2+1)+(Cs+D)(s^2+1)}{(s^2+1)[(s+2)^2+1]}$$

Example 4 (cont.)

$$\frac{1}{(s^2+1)[(s+2)^2+1]} = \frac{(As+B)}{s^2+1}\frac{(s+2)^2+1}{(s+2)^2+1} + \frac{(Cs+D)}{(s+2)^2+1}\frac{s^2+1}{s^2+1}$$

$$\frac{1}{(s^2+1)[(s+2)^2+1]} = \frac{(As+B)((s+2)^2+1)+(Cs+D)(s^2+1)}{(s^2+1)[(s+2)^2+1]}$$

$$1 = (As+B)(s^2+4s+5)+(Cs+D)(s^2+1)$$

$$1 = (A+C)s^3+(4A+B+D)s^2+(5A+4B+C)s+(5B+D)$$

$$\left.\begin{array}{lcl} A+C & = & 0 \\ 4A+B+D & = & 0 \\ 5A+4B+C & = & 0 \\ 5B+D & = & 1 \end{array}\right\} A=\frac{-1}{8},\ B=\frac{1}{8},\ C=\frac{1}{8},\ D=\frac{3}{8}$$

Example 4 (cont.)

$$Y = \frac{\frac{-1}{8}s + \frac{1}{8}}{s^2+1} + \frac{\frac{1}{8}s + \frac{3}{8}}{(s+2)^2+1}$$

$$= \frac{\frac{-1}{8}s}{s^2+1} + \frac{\frac{1}{8}}{s^2+1} + \frac{\frac{1}{8}s}{(s+2)^2+1}$$

$$+\frac{\frac{2}{8}}{(s+2)^2+1} - \frac{\frac{2}{8}}{(s+2)^2+1} + \frac{\frac{3}{8}}{(s+2)^2+1}$$

$$= \frac{\frac{-1}{8}s}{s^2+1} + \frac{\frac{1}{8}}{s^2+1} + \frac{\frac{1}{8}(s+2)}{(s+2)^2+1} + \frac{\frac{1}{8}}{(s+2)^2+1}$$

$$y(t) = \frac{-1}{8}\cos t + \frac{1}{8}\sin t + \frac{1}{8}e^{-2t}\cos t + \frac{1}{8}e^{-2t}\sin t$$

Example 5

$$\frac{d^3y}{dt^3}+4\frac{d^2y}{dt^2}+5\frac{dy}{dt}+2y = 10\cos t,\ y(0)=0,\ y'(0)=0,\ y''(0)=3$$

$$\{s^3Y - s^2y(0) - sy'(0) - y''(0)\} + 4\{s^2Y - sy(0) - y'(0)\}$$
$$+5\{sY - y(0)\} + 2Y = 10\frac{s}{s^2+1}$$

$$\{s^3Y-s^2 0-s0-3\}+4\{s^2Y-s0-0\}+5\{sY-0\}+2Y = 10\frac{s}{s^2+1}$$

$$\{s^3Y - 3\} + 4\{s^2Y\} + 5\{sY\} + 2Y = 10\frac{s}{s^2+1}$$

$$Y = \frac{3s^2+10s+3}{(s^2+1)(s+1)^2(s+2)} = \frac{A}{s+2}+\frac{B}{s+1}+\frac{C}{(s+1)^2}+\frac{Ds+E}{s^2+1}$$
$$= \frac{-1}{s+2}+\frac{2}{s+1}-\frac{2}{(s+1)^2}-\frac{s}{s^2+1}+\frac{2}{s^2+1}$$
$$y(t) = -e^{-2t}+2e^{-t}-2te^{-t}-\cos t+2\sin t$$

Example 6

$$\begin{aligned} \frac{dx}{dt} - 6x + 3y &= 8e^t \\ \frac{dy}{dt} - 2x - y &= 4e^t \end{aligned}$$

$$x(0) = -1, \quad y(0) = 0$$

In Laplace domain :

$$\begin{aligned} sX + 1 - 6X + 3Y &= \frac{8}{s-1} \\ sY - 2X - Y &= \frac{4}{s-1} \end{aligned}$$

$$\begin{aligned} (s-6)X + 3Y &= \frac{-s+9}{s-1} \\ -2X + (s-1)Y &= \frac{4}{s-1} \end{aligned}$$

Example 6 (cont.)

$$\begin{array}{rcl} (s-6)X + 3Y & = & \frac{-s+9}{s-1} \\ -2X + (s-1)Y & = & \frac{4}{s-1} \end{array}$$

In matrix notation:

$$\begin{bmatrix} s-6 & 3 \\ -2 & s-1 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \frac{-s+9}{s-1} \\ \frac{4}{s-1} \end{bmatrix}$$

$$X = \frac{-s+7}{(s-1)(s-4)}, \; Y = \frac{2}{(s-1)(s-4)}$$

$$\leftrightarrow x(t) = -2e^t + e^{4t}, \; y(t) = \frac{-2}{3}e^t + \frac{2}{3}e^{4t}$$

Partial Fractions Decomposition

[The Laplace Transform: Theory and Applications, Joel L. Schiff]

Consider quotient of two polynomials

$$F(s) = \frac{P(s)}{Q(s)}$$

where the degree of Q is greater than that of P, and P and Q do not have common factors. Then F can be expressed as a finite sum of partial fractions.

(i) For each factor of the form $as + b$ of Q, there corresponds a partial fraction of the form

$$\frac{A}{as + b}$$

where A is a constant.

(ii) For each repeated factor of the form $(as + b)^n$ of Q there corresponds a partial fraction of the form

$$\frac{A_1}{as + b} + \frac{A_2}{(as + b)^2} + \cdots + \frac{A_{n-1}}{(as + b)^{n-1}} + \frac{A_n}{(as + b)^n}$$

where $A_1, A_2, ..., A_n$ are constants.

(iii) For every factor of the form $as^2 + bs + c$ of Q there corresponds a partial fraction of the form

$$\frac{As + B}{as^2 + bs + c}$$

where A and B are constants.

(iv) For every repeated factor of the form $(as^2 + bs + c)^n$ of Q there corresponds a partial fraction of the form

$$\frac{A_1 s + B_1}{as^2 + bs + c} + \frac{A_2 s + B_2}{(as^2 + bs + c)^2} + \cdots + \frac{A_{n-1} s + B_{n-1}}{(as^2 + bs + c)^{n-1}} + \frac{A_n s + B_n}{(as^2 + bs + c)^n}$$

where $A_1, A_2, ..., A_n, B_1, B_2, ..., B_n$ are constants.

Example 7

$$\frac{s^2}{(s+2)(s+3)(s+7)^4} = \frac{A_1}{s+2} + \frac{A_2}{s+3} + \frac{A_3}{s+7} + \frac{A_4}{(s+7)^2}$$
$$+ \frac{A_5}{(s+7)^3} + \frac{A_6}{(s+7)^4}$$

Example 8

$$\frac{4s}{(s+4)(s^2+3s+7)^3} = \frac{A}{s+4} + \frac{B_1 s + B_2}{(s^2+3s+7)}$$
$$+ \frac{B_3 s + B_4}{(s^2+3s+7)^2} + \frac{B_5 s + B_6}{(s^2+3s+7)^3}$$