Differential Equations[1]

A. Karamancıoğlu

Eskişehir Osmangazi University

January 9, 2024

[1] These slides are intended for educational use only; not for sale under any circumstances.

Notation

$$\dot{x},\ \frac{dx}{dt}:\ \text{first derivative of } x \text{ with respect to } t$$

$$\ddot{x},\ \frac{d^2x}{dt^2}:\ \text{second derivative of } x \text{ with respect to } t$$

$$x^{(n)},\ \frac{d^nx}{dt^n}:\ \text{n-th derivative of } x \text{ with respect to } t$$

Let f be a function of u and v.

$$f_u, \ \ \frac{\partial f}{\partial u} : \text{ partial derivative of } f \text{ with respect to } u$$

$$f_v, \ \ \frac{\partial f}{\partial v} : \text{ partial derivative of } f \text{ with respect to } v$$

Example 1

$$f(x) = x^3 + 2x \rightarrow \frac{df}{dx} = 3x^2 + 2$$

f is the **dependent variable**, and x is the **independent variable**.

Definitions and classifications

Definition 1

An equation involving derivatives of one or more dependent variables with respect to one or more independent variables is called a **differential equation**.
A differential equation involving ordinary derivatives of one or more dependent variables with respect to a single independent variable is called an **ordinary differential equation**.
A differential equation involving partial derivatives of one or more dependent variables with respect to a more than one independent variable is called a **partial differential equation**.

Example 2

$$\frac{d^2y}{dx^2} + xy(\frac{dy}{dx})^2 = 0 \quad (1)$$

$$\frac{d^4x}{dt^4} + 5\frac{d^2x}{dt^2} + 3x = \sin t \quad (2)$$

$$\frac{\partial v}{\partial s} + \frac{\partial v}{\partial t} = v \quad (3)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad (4)$$

The first and second differential equations are ordinary, the third and fourth differential equations are partial differential equations.

Definition 2

The order of the highest ordered derivative involved in a differential equation is called the **order of the differential equation**.

$$\frac{d^2y}{dx^2} + xy(\frac{dy}{dx})^2 = 0 \tag{5}$$

$$\frac{d^4x}{dt^4} + 5\frac{d^2x}{dt^2} + 3x = \sin t \tag{6}$$

$$\frac{\partial v}{\partial s} + \frac{\partial v}{\partial t} = v \tag{7}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \tag{8}$$

In the example the first is second order, the second is fourth order, the third is first order, the fourth is second order differential equations.

Definition 3

A **linear** ordinary differential equation of order n, in the dependent variable y and the independent variable x, is an equation that is in, or can be expressed in, the form

$$a_0(x)\frac{d^n y}{dx^n}+a_1(x)\frac{d^{n-1} y}{dx^{n-1}}+\ldots+a_{n-2}(x)\frac{d^2 y}{dx^2}+a_{n-1}(x)\frac{dy}{dx}+a_n(x)y=b(x)$$

where a_0 is not identically zero.

A function of x does not contain a dependent variable. Examples:

Functions of x: $x^2, \sin(x), x+1, 5, 0$

Not functions of x: $y, 3y, y^2, \frac{dy}{dx}, (\frac{dy}{dx})^2, x+y, xy$

Definition 4

A **nonlinear** ordinary differential equation is an ordinary differential equation that is not linear.

Example 3

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0 \ldots \text{ Linear} \quad (9)$$

$$\frac{d^4y}{dx^4} + x^2\frac{d^3y}{dx^3} + x^3\frac{dy}{dx} = xe^x \ldots \text{ Linear} \quad (10)$$

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y^2 = 0 \ldots \text{ Nonlinear} \quad (11)$$

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6yy = 0 \ldots \text{ Nonlinear}$$

$$\frac{d^2y}{dx^2} + 5(\frac{dy}{dx})^3 + 6y = 0 \ldots \text{ Nonlinear} \quad (12)$$

$$\frac{d^2y}{dx^2} + 5(\frac{dy}{dx})^2\frac{dy}{dx} + 6y = 0 \ldots \text{ Nonlinear}$$

$$\frac{d^2y}{dx^2} + 5y\frac{dy}{dx} + 6y = 0 \ldots \text{ Nonlinear} \quad (13)$$

$$\frac{d^2y}{dx^2} + 5y\frac{dy}{dx} + 6y = 0 \ldots \text{Nonlinear}$$

Normal Form

Definition 5

The normal form of a system of n differential equations in n unknown functions $x_1, x_2, \ldots, x_n$, is in the following form:

$$\left.\begin{aligned} \frac{dx_1}{dt} &= f_1(x_1, x_2, \ldots, x_n, t) \\ \frac{dx_2}{dt} &= f_2(x_1, x_2, \ldots, x_n, t) \\ &\vdots \\ \frac{dx_n}{dt} &= f_n(x_1, x_2, \ldots, x_n, t) \end{aligned}\right\} \tag{14}$$

Example 4

$$\begin{aligned} \dot{x}_1 &= x_1 + 3x_1x_2^2 + x_1 t \\ \dot{x}_2 &= x_2^3 + \sin x_1 + t^2 \\ \dot{x}_3 &= x_1x_2 + x_2x_3^2 \end{aligned}$$

Normal Form of a Linear System

Definition 6

The **normal form** of a linear system of n differential equations in n unknown functions $x_1, x_2, \ldots, x_n$, is in the following form:

$$\left.\begin{array}{l} \frac{dx_1}{dt} = a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + F_1(t) \\ \frac{dx_2}{dt} = a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + F_2(t) \\ \vdots \\ \frac{dx_n}{dt} = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + F_n(t) \end{array}\right\} \quad (15)$$

Example 5

$$\begin{aligned} \dot{x}_1 &= 2tx_1 + 3x_2 + 4x_3 + t^2 \\ \dot{x}_2 &= x_1 + 6x_3 + \frac{1}{t} \\ \dot{x}_3 &= 3t^2x_1 + (4+t)x_2 + (t+t^2)x_3 \end{aligned}$$

A single n-th order linear differential equation can be converted into a normal form. Consider

$$\frac{d^n x}{dt^n} + a_1(t)\frac{d^{n-1}x}{dt^{n-1}} + a_2(t)\frac{d^{n-2}x}{dt^{n-2}} + \cdots$$

$$\cdots + a_{n-2}(t)\frac{d^2 x}{dt^2} + a_{n-1}(t)\frac{dx}{dt} + a_n(t)x = F(t) \qquad (16)$$

$$\frac{d^n x}{dt^n} + a_1(t)\underbrace{\frac{d^{n-1}x}{dt^{n-1}}}_{x_n} + a_2(t)\underbrace{\frac{d^{n-2}x}{dt^{n-2}}}_{x_{n-1}} + \cdots$$

$$\cdots + a_{n-2}(t)\underbrace{\frac{d^2 x}{dt^2}}_{x_3} + a_{n-1}(t)\underbrace{\frac{dx}{dt}}_{x_2} + a_n(t)\underbrace{x}_{x_1} = F(t)$$

Notice that

$$\dot{x}_i = x_{i+1} \quad i = 1, \ldots, n-1$$

and

$$\dot{x}_n + a_1(t)x_n + a_2(t)x_{n-1} + \ldots + a_{n-1}(t)x_2 + a_n(t)x_1 = F(t)$$

$$\dot{x}_n = -a_n(t)x_1 - a_{n-1}(t)x_2 - \ldots - a_3(t)x_{n-2} - a_2(t)x_{n-1} - a_1(t)x_n + F(t)$$

$$\frac{d^n x}{dt^n} + a_1(t) \underbrace{\frac{d^{n-1}x}{dt^{n-1}}}_{x_n} + a_2(t) \underbrace{\frac{d^{n-2}x}{dt^{n-2}}}_{x_{n-1}} + \cdots$$

$$+ a_{n-2}(t) \underbrace{\frac{d^2x}{dt^2}}_{x_3} + a_{n-1}(t) \underbrace{\frac{dx}{dt}}_{x_2} + a_n(t) \underbrace{x}_{x_1} = F(t)$$

Using these definitions, the normal form equivalent of (16) is

$$\left.\begin{aligned}
\frac{dx_1}{dt} &= x_2 \\
\frac{dx_2}{dt} &= x_3 \\
&\vdots \\
\frac{dx_{n-1}}{dt} &= x_n \\
\frac{dx_n}{dt} &= -a_n(t)x_1 - a_{n-1}(t)x_2 - \cdots - a_2(t)x_{n-1} - a_1(t)x_n + F(t)
\end{aligned}\right\}$$

Example 6

$$\frac{d^4x}{dt^4} + e^{2t}\underbrace{\frac{d^3x}{dt^3}}_{x_4} + 7\underbrace{\frac{dx}{dt}}_{x_2} + (2+t^2)\underbrace{x}_{x_1} = t^2 \sin t$$

Using these definitions, the normal form equivalent of the above equation is

$$\left.\begin{aligned}
\frac{dx_1}{dt} &= x_2 \\
\frac{dx_2}{dt} &= x_3 \\
\frac{dx_3}{dt} &= x_4 \\
\frac{dx_4}{dt} &= -(2+t^2)x_1 - 7x_2 - e^{2t}x_4 + t^2 \sin t
\end{aligned}\right\}$$

Solution of a differential equation

Definition 7

Consider the n-th order ordinary differential equation

$$F[x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}] = 0 \qquad (17)$$

A solution of an ordinary differential equation (17) on interval I is a real function that satisfies the differential equation on the interval I.

Example 7

$$\underbrace{\frac{dy}{dx} + y\frac{d^2y}{dx^2} + 3x^2 + \frac{d^3y}{dx^3}\sin x}_{F[x,y,\frac{dy}{dx},\frac{d^2y}{dx^2},\frac{d^3y}{dx^3}]} = 0$$

$\therefore$ **Solution** is a real function.
$\therefore$ **Solution** is defined on some interval I.
$\therefore$ **Solution** satisfies the d.e. on I.
$\therefore$ **Solution** does not contain integration or differentiation in it.

A Solution Classification

Explicit Solutions and **Implicit** Solutions:
Explicit solution is a real function f defined on interval I such that it satisfies the ordinary differential equation on interval I when f is substituted for the dependent variable.
A relation $g(x, y) = 0$ is called an **implicit solution** of the ordinary differential equation on I if this relation defines at least one real function f of x on I such that this function is an explicit solution of (17) on this interval.
Both explicit and implicit solutions are called **solutions**.

Example 8

A function defined for all real x by

$$f(x) = 2\sin x + 3\cos x$$

is an explicit solution of the differential equation

$$\frac{d^2y}{dx^2} + y = 0$$

for all real x. First note that f is defined and has a second derivative on the entire real interval. Next observe that

$$f'(x) = 2\cos x - 3\sin x, \quad f''(x) = -2\sin x - 3\cos x$$

Substituting them in the differential equation we obtain

$$(-2\sin x - 3\cos x) + (2\sin x + 3\cos x) = 0$$

which holds for all real x. Thus the function f is an explicit solution of the differential equation.

The differential equation

$$\frac{d^2y}{dx^2} + y = 0$$

is satisfied by the function

$$f(x) = 3i \cos x$$

for all real x. However, in our context, we do not call it a solution; because we require a solution to be **real**.

Example 9

Consider the differential equation

$$x\frac{dy}{dx} - 2y = 0$$

The function $f(x) = x^2$ on the interval $I = (-\infty, \infty)$ is an explicit solution to the d.e. above. Substitute in the d.e.:

$$xf'(x) - 2f(x) = x \cdot 2x - 2 \cdot x^2 = 0$$

for all $x \in I$. Thus f is an explicit solution to the d.e. on the interval I.

Example 10

Is $f(x) = e^x - x$ on the interval $I = (-\infty, \infty)$ a solution to

$$\frac{dy}{dx} + y^2 = e^{2x} + (1 - 2x)e^x + x^2 - 1$$

Example 11

Consider the differential equation

$$x^2 \frac{d^2 y}{dx^2} - 2y = 0$$

and the solution candidate $f(x) = x^2 - x^{-1}$ on the interval $I = (0, \infty)$.

Note that $f'(x) = 2x + x^{-2}$ and $f''(x) = 2 - 2x^{-3}$. Substitute them in the d.e.:

$$x^2 \cdot (2 - 2x^{-3}) - 2(x^2 - x^{-1}) = 0$$

for all x on the interval I. It can be shown that this function is also a solution to the differential equation on the interval $(-\infty, 0)$.

Example 12

The relation

$$x^2 + y^2 - 25 = 0$$

is an implicit solution of the differential equation

$$x + y\frac{dy}{dx} = 0$$

on the interval I defined by $-5 < x < 5$. It defines two functions

$$f_1(x) = \sqrt{25 - x^2}$$

and

$$f_2(x) = -\sqrt{25 - x^2}$$

for all real x on I. It can easily be shown that each of these functions is an explicit solution for the differential equation on I. Note that if one of them is an explicit solution for the differential equation on I, it suffices for being an implicit solution.

$$x + y\frac{dy}{dx} = 0$$

Example 12 (cont.)

It can easily be shown that each of the functions f_1 and f_2 is an explicit solution for the differential equation on I. Note that if one of them is an explicit solution for the differential equation on $I: -5 < x < 5$ it suffices for being an implicit solution. Indeed, at least one of them satisfies the differential equation. For instance, substitute f_1 for y:

$$\left[x + y\frac{dy}{dx}\right]_{y=f_1} = 0$$

$$x + \sqrt{25 - x^2} \cdot \frac{-2x}{2\sqrt{25 - x^2}} = x - x = 0$$

Because the d.e. is satisfied by at least one of f_1 and f_2, the relation $x^2 + y^2 - 25 = 0$ is an implicit solution for the d.e.

Example 13

Consider the d.e.

$$\frac{dy}{dx} + \frac{1}{2y} = 0$$

The relation $y^2 + x - 3 = 0$ on the interval $(-\infty, 3)$ is an implicit solution to the d.e. above. Differentiate throughout:

$$2y\frac{dy}{dx} + 1 = 0$$

$$\rightarrow \frac{dy}{dx} + \frac{1}{2y} = 0$$

Solution generated the d.e.! Thus the relation $y^2 + x - 3 = 0$ on the interval $(-\infty, 3)$ is an implicit solution to the given d.e.

Example 14

Consider the relation $xy^3 - xy^3 \sin x = 1$ and solve it for y for later use:

$$xy^3(1 - \sin x) = 1 \to y^3 = \frac{1}{x(1 - \sin x)}$$

$$\to y = \left[\frac{1}{x(1 - \sin x)}\right]^{\frac{1}{3}} = [x(1 - \sin x)]^{\frac{-1}{3}}$$

Differentiate this:

$$\frac{dy}{dx} = -\frac{1}{3}[x(1 - \sin x)]^{\frac{-4}{3}}[x(-\cos x) + (1 - \sin x)]$$

$$= \frac{x\cos x + \sin x - 1}{3[x(1 - \sin x)]^{\frac{4}{3}}} = \frac{x\cos x + \sin x - 1}{3[x(1 - \sin x)]}\frac{1}{[x(1 - \sin x)]^{\frac{1}{3}}}$$

$$= \frac{x\cos x + \sin x - 1}{3[x(1 - \sin x)]} \cdot y$$

Example 14 (cont.)

$$xy^3 - xy^3 \sin x = 1 \leftrightarrow \frac{dy}{dx} = \frac{x \cos x + \sin x - 1}{3[x(1 - \sin x)]} \cdot y$$

Thus the relation

$$xy^3 - xy^3 \sin x = 1$$

is an implicit solution to the d.e.

$$\frac{dy}{dx} = \frac{x \cos x + \sin x - 1}{3[x(1 - \sin x)]} \cdot y$$

Initial value problems

Problem Find a solution of the differential equation

$$\frac{dy}{dx} = 2x \tag{18}$$

such that at $x = 1$ this solution equals 4.
Equivalently Solve

$$\frac{dy}{dx} = 2x, \quad y(1) = 4$$

$$\frac{dy}{dx} = 2x \qquad \text{(cf. 18)}$$

The solution $y(x) = x^2 + c$ satisfies (18) for an arbitrary constant c.

The other condition $y(1) = 4$ implies

$$\underbrace{y(x)}_{4} = \underbrace{x^2}_{1^2} + c$$

That is, $4 = 1^2 + c$ implies $c = 3$. Thus initial value problem has the solution

$$y(x) = x^2 + 3$$

The condition in addition to the differential equation (18) is called **boundary condition**. If the boundary conditions relate to one x value, the problem is called the **initial value problem**. If the conditions relate to two different x values, the problem is called a (two point) **boundary value problem**.

Example 15

$$\frac{d^2y}{dx^2} + y = 0, \quad y(1) = 3, \; y'(1) = -4$$

Since the boundary conditions are given at one x value the problem is an initial value problem.

Example 16

$$\frac{d^2y}{dx^2} + y = 0, \quad y(0) = 1, \; y(2) = 5$$

Boundary conditions are given at two different x values; the problem is a boundary value problem.

Existence of a unique solution

Theorem 1

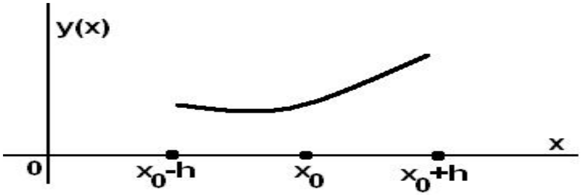
Consider the differential equation

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0 \tag{19}$$

where
1) the function f is a continuous function of x and y in some domain D of xy-plane, and
2) the partial derivative $\frac{\partial f}{\partial y}$ is also a continuous function of x and y in D; and
3) let (x_0, y_0) be a point in D. Then there exists a unique solution of the differential equation (19) defined on some interval $|x - x_0| < h$ where h is sufficiently small.

Theorem 1 (cont.)

Then there exists a unique solution of the differential equation (19) defined on some interval $|x - x_0| < h$ *where h is sufficiently small.*



Note that this is a sufficiency theorem. $A \to B$ does not mean A is necessary for B to hold true.

Example 17

Consider the initial value problem

$$\frac{dy}{dx} = x^2 + y^2, \quad y(1) = 3$$

Let us apply the existence theorem where $f(x,y) = x^2 + y^2$, $\frac{\partial f}{\partial y} = 2y$. Both functions f and $\frac{\partial f}{\partial y}$ are continuous in every domain D of the xy-plane. The point $(1,3)$ is in the domain D.

Thus the differential equation has a unique solution defined in the neighborhood of $x = 1$.

A first order linear differential equation in the form

$$\frac{dy}{dx} + p(x)y = g(x), \; y(x_0) = y_0$$

is a special case of the one we considered:

$$\frac{dy}{dx} = f(x, y), \;\; y(x_0) = y_0 \qquad \text{(cf. 19)}$$

Example 18

Consider

$$(t^2 - 9)\dot{y} + 2y = \ln|20 - 4t|, \; y(4) = -3$$

In the standard form:

$$\dot{y} = -\frac{2}{(t^2 - 9)}y + \frac{\ln|20 - 4t|}{(t^2 - 9)}, \; y(4) = -3$$

$$\dot{y} = -\frac{2}{(t^2-9)}y + \frac{\ln|20-4t|}{(t^2-9)}, \; y(4) = -3$$

Example 18 (cont.)

comparing to the expression

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0 \qquad \text{(cf. 19)}$$

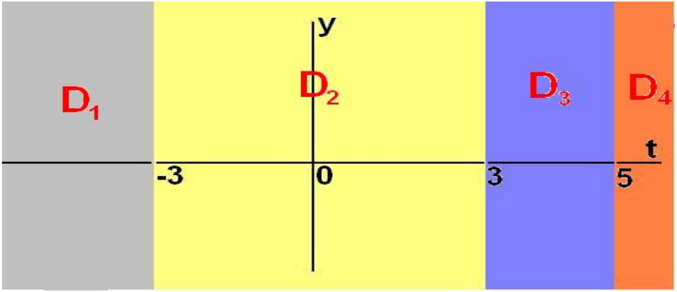
we have

$$f(t,y) = -\frac{2}{(t^2-9)}y + \frac{\ln|20-4t|}{(t^2-9)}$$

f has discontinuities at $t = -3, +3, 5$. Discontinuities of $\frac{\partial f}{\partial y}$ are at $t = -3, +3$. The continuous interval of y is $(-\infty, \infty)$, and continuous intervals of t are

$$(-\infty, -3), (-3, 3), (3, 5), (5, \infty)$$

Example 18 (cont.)



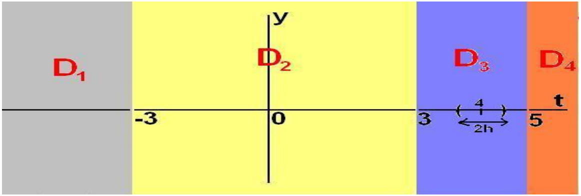
The first two hypotheses of the theorem are satisfied by the following domains:

$$\underbrace{(-\infty,-3)\times(-\infty,\infty)}_{D_1},\underbrace{(-3,3)\times(-\infty,\infty)}_{D_2},$$

$$\underbrace{(3,5)\times(-\infty,\infty)}_{D_3},\underbrace{(5,\infty)\times(-\infty,\infty)}_{D_4}$$

The initial condition $y(4)=-3$, corresponding to the pair $(4,-3)$ in the third hypothesis, is in the domain D_3.

Example 18 (cont.)



Thus, the differential equation

$$(t^2 - 9)\dot{y} + 2y = \ln|20 - 4t|, \; y(4) = -3$$

satisfies the hypotheses of the existence and uniqueness theorem in domain D_3. Therefore, it has a unique solution defined for $|t - 4| < h$ for some h.

We will see in the sequel that the sufficient existence conditions are simpler for the linear differential equations.

Exercise

1) Show that $y(x) = 4e^{2x} + 2e^{-3x}$ is a solution of the initial value problem

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0; \; y(0) = 6, \; y'(0) = 2$$

2) Do the following problems have unique solutions?

a)

$$\frac{dy}{dx} = x^2 \sin y, \; y(1) = -2$$

b)

$$\frac{dy}{dx} = \frac{y^2}{x-2}, \; y(1) = 0$$

Exact differential equations

The first order differential equations to be studied may be expressed in either the derivative form

$$\frac{dy}{dx} = f(x, y)$$

or the differential form

$$M(x, y)dx + N(x, y)dy = 0$$

An equation in one of these forms may readily be written in the other form. For example

$$\frac{dy}{dx} = \frac{x^2 + y^2}{x - y} \leftrightarrow (x^2 + y^2)dx + (y - x)dy = 0$$

$$(\sin(x) + y)dx + (x + 3y)dy = 0 \leftrightarrow \frac{dy}{dx} = -\frac{\sin(x) + y}{x + 3y}$$

Definition 8

Let F be a function of two real variables such that F has continuous first order partial derivatives in a domain D. The **total differential** dF of the function F is defined by the formula

$$dF(x,y) = \frac{\partial F(x,y)}{\partial x}dx + \frac{\partial F(x,y)}{\partial y}dy$$

for all $(x,y) \in D$.

Example 19

Consider

$$F(x,y) = xy^2 + 2x^3y$$

for all real (x,y). Then

$$dF(x,y) = (y^2 + 6x^2y)dx + (2xy + 2x^3)dy$$

Definition 9

The expression

$$M(x,y)dx + N(x,y)dy \tag{20}$$

is called **exact differential** in a domain D if there exists a function F of two variables such that this expression equals the total differential $dF(x,y)$ for all $(x,y) \in D$. That is the expression (20) is an exact differential in D if there exists a function F such that

$$\frac{\partial F(x,y)}{\partial x} = M(x,y) \text{ and } \frac{\partial F(x,y)}{\partial y} = N(x,y)$$

for all $(x,y) \in D$.

If $M(x,y)dx + N(x,y)dy$ is an exact differential then $M(x,y)dx + N(x,y)dy = 0$ is called an exact differential equation.

Example 20

The differential equation

$$y^2dx + 2xydy = 0$$

is an exact differential equation since $y^2dx + 2xydy$ is an exact differential. Consider $F(x, y) = xy^2$:

$$\frac{\partial F(x,y)}{\partial x} = y^2 \text{ and } \frac{\partial F(x,y)}{\partial y} = 2xy$$

Test for exactness

Theorem 2

Consider the differential equation

$$M(x,y)dx + N(x,y)dy = 0 \tag{21}$$

where M and N have continuous first partial derivatives at all points (x,y) in a rectangular domain D.
Exactness of the differential equation (21) in D is equivalent to

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$

for all $(x,y) \in D$

Example 21

Is the following d.e. exact?

$$(y^2 - 2x)dx + (2xy + 1)dy = 0$$

Yes, because

$$\frac{\partial M}{\partial y} = 2y; \quad \frac{\partial N}{\partial x} = 2y$$

Justification of exactness test

Exactness of $M(x, y)dx + N(x, y)dy = 0$ implies

$$\frac{\partial F(x, y)}{\partial x} = M(x, y); \ \frac{\partial F(x, y)}{\partial y} = N(x, y)$$

for some $F(x, y)$. Because order in differentiation does not matter, the rhs can be written as

$$\frac{\partial^2 F(x, y)}{\partial x \partial y} = \frac{\partial M(x, y)}{\partial y}; \ \frac{\partial^2 F(x, y)}{\partial y \partial x} = \frac{\partial N(x, y)}{\partial x}$$

This leads to

$$\frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x}$$

which is the exactness condition.

Theorem 3

Suppose the differential equation $M(x, y)dx + N(x, y)dy = 0$ is exact in a rectangular domain D. Then a one parameter family of solutions of this differential equation is given by $F(x, y) = c$ where F is a function such that

$$\frac{\partial F(x, y)}{\partial x} = M(x, y) \text{ and } \frac{\partial F(x, y)}{\partial y} = N(x, y)$$

for all $(x, y) \in D$ and c is an arbitrary constant.

Justification

Thus,

$$\frac{\partial F(x,y)}{\partial x}dx + \frac{\partial F(x,y)}{\partial y}dy = 0$$

is the same as

$$dF(x,y) = 0$$

which is possible if

$$F(x,y) = c$$

where c is an arbitrary constant. Namely

$$\frac{\partial F(x,y)}{\partial x}dx + \frac{\partial F(x,y)}{\partial y}dy = 0 \to F(x,y) = c$$

Example 22

$$(3x^2 + 4xy)dx + (2x^2 + 2y)dy = 0$$

is exact since

$$\frac{\partial M(x,y)}{\partial y} = 4x = \frac{\partial N(x,y)}{\partial x}$$

for all real (x, y). Thus we must find F such that

$$\underbrace{\frac{\partial F(x,y)}{\partial x} = 3x^2 + 4xy}_{Property 1} \text{ and } \underbrace{\frac{\partial F(x,y)}{\partial y} = 2x^2 + 2y}_{Property 2}$$

First property implies

$$F(x,y) = \int (3x^2 + 4xy)\partial x + \phi(y)$$

$$= x^3 + 2x^2 y + \phi(y)$$

Example 22 (cont.)

$$F(x,y) = x^3 + 2x^2y + \phi(y) \qquad \text{(by Property 1)}$$

$$\rightarrow \frac{\partial F(x,y)}{\partial y} = 2x^2 + \frac{d\phi(y)}{dy}$$

But 2nd property requires

$$\frac{\partial F(x,y)}{\partial y} = N(x,y) = 2x^2 + 2y$$

Thus

$$2x^2 + 2y = 2x^2 + \frac{d\phi(y)}{dy}$$

or

$$2y = \frac{d\phi(y)}{dy} \rightarrow \phi(y) = y^2 + c_0$$

Hence

$$F(x,y) = x^3 + 2x^2y + y^2 + c_0$$

Example 22 (cont.)

$$F(x, y) = x^3 + 2x^2y + y^2 + c_0$$

One parameter family of solutions:

$$x^3 + 2x^2y + y^2 + c_0 = c_1$$

or

$$x^3 + 2x^2y + y^2 = c$$

For a verification, compare total differentials of both sides:

$$d(x^3 + 2x^2y + y^2) = d(c)$$

$$(3x^2 + 4xy)dx + (2x^2 + 2y)dy = 0$$

We obtained the original equation; thus solution is verified.

Example 22 (cont.)

For another verification way, write the given differential equation in derivative form:

$$(3x^2 + 4xy)dx + (2x^2 + 2y)dy = 0 \to \frac{dy}{dx} = -\frac{3x^2 + 4xy}{2x^2 + 2y}$$

Solve the implicit solution $x^3 + 2x^2y + y^2 = c$ for y to generate an explicit solution:

$$y^2 + \underbrace{2x^2}_{B} y + \underbrace{x^3 - c}_{C} = 0$$

$$y^2 + By + C = 0 \to y_{1,2} = -\frac{B}{2} \pm \sqrt{\left[\frac{-B}{2}\right]^2 - C} = -x^2 \pm \sqrt{x^4 - x^3 + c}$$

One can show that at least one of $y_{1,2}$ satisfies the given differential equation; this is another verification of that the solution is correct.

Example 23

Solve the initial value problem

$$(2x\cos y + 3x^2y)dx + (x^3 - x^2\sin y - y)dy = 0, \quad y(0) = 2$$

The equation is exact:

$$\frac{\partial M(x,y)}{\partial y} = -2x\sin y + 3x^2 = \frac{\partial N(x,y)}{\partial x}$$

for all real (x, y). We must find F such that

$$\frac{\partial F(x,y)}{\partial x} = M(x,y) = 2x\cos y + 3x^2y \qquad \text{(P1)}$$

$$\frac{\partial F(x,y)}{\partial y} = N(x,y) = x^3 - x^2\sin y - y \qquad \text{(P2)}$$

$$\frac{\partial F(x,y)}{\partial x} = M(x,y) = 2x\cos y + 3x^2 y \tag{P1}$$

$$\frac{\partial F(x,y)}{\partial y} = N(x,y) = x^3 - x^2 \sin y - y \tag{P2}$$

Example 23 (cont.)

P1 implies

$$F(x,y) = \int (2x\cos y + 3x^2 y)\partial x + \phi(y) = x^2 \cos y + x^3 y + \phi(y) \tag{22}$$

$$\frac{\partial F(x,y)}{\partial y} = x^3 - x^2 \sin y + \frac{d\phi(y)}{dy}$$

However, P2 implies

$$\frac{\partial F(x,y)}{\partial y} = x^3 - x^2 \sin y - y$$

The $\frac{\partial F(x,y)}{\partial y}$ terms implied by P1 and P2 must be equal:

Example 23 (cont.)

The $\frac{\partial F(x,y)}{\partial y}$ terms implied by P1 and P2 must be equal:

$$x^3 - x^2 \sin y + \frac{d\phi(y)}{dy} = x^3 - x^2 \sin y - y$$

$$\rightarrow \frac{d\phi(y)}{dy} = -y$$

$$\rightarrow \phi(y) = -\frac{y^2}{2} + c_0$$

Recall that P1 has implied $F(x, y)$ which depends on an arbitrary function of y:

$$F(x, y) = x^2 \cos y + x^3 y + \phi(y) \tag{22}$$

Because $\phi(y)$ is resolved, we can write

$$F(x, y) = x^2 \cos y + x^3 y - y^2/2 + c_0$$

Example 23 (cont.)

Family of solutions:

$$F(x,y) = c_1 \to x^2 \cos y + x^3 y - y^2/2 + c_0 = c_1$$

$$x^2 \cos y + x^3 y - \frac{y^2}{2} = c$$

Apply the initial conditions: $y = 2$ at $x = 0$.

$$0^2 \times \cos 2 + 0^3 \times 2 - \frac{2^2}{2} = c$$

We find $c = -2$. Thus the solution is:

$$x^2 \cos y + x^3 y - \frac{y^2}{2} = -2$$

Example 24

Consider the differential equation

$$(3x^2y^3 - 5x^4)dx + (y + 3x^3y^2)dy = 0 \tag{23}$$

We have

$$M(x, y) = 3x^2y^3 - 5x^4, \quad N(x, y) = y + 3x^3y^2$$

Test for exactness:

$$\frac{\partial M(x, y)}{\partial y} = 9x^2y^2 = \frac{\partial N(x, y)}{\partial x}$$

The d.e. is exact!

$$(3x^2y^3 - 5x^4)dx + (y + 3x^3y^2)dy = 0 \qquad \text{(cf. 23)}$$

Example 24 (cont.)

$$F(x,y) = \int M(x,y)\partial x = \int (3x^2y^3 - 5x^4)\partial x = x^3y^3 - x^5 + \phi(y)$$

Next we are going to find $\phi(y)$:

$$\left.\begin{array}{l} \frac{\partial F(x,y)}{\partial y} = y + 3x^3y^2 \\ \frac{\partial F(x,y)}{\partial y} = 3x^3y^2 + \frac{d\phi}{dy} \end{array}\right\} \rightarrow \frac{d\phi}{dy} = y \rightarrow \phi(y) = \frac{y^2}{2} + c_1$$

Let us put ϕ in place: $F(x,y) = x^3y^3 - x^5 + \frac{y^2}{2} + c_1$
The solution is therefore

$$x^3y^3 - x^5 + \frac{y^2}{2} = c$$

$$(3x^2y^3 - 5x^4)dx + (y + 3x^3y^2)dy = 0 \qquad \text{(cf. 23)}$$
$$x^3y^3 - x^5 + \frac{y^2}{2} = c$$

Example 24 (cont.)

A verification

Total diferentiate both sides of the solution:

$$d(x^3y^3 - x^5 + \frac{y^2}{2}) = d(c)$$

$$(3x^2y^3 - 5x^4)dx + (y + 3x^3y^2)dy = 0$$

A successful verification!

Integrating factors

If the differential equation

$$M(x,y)dx + N(x,y)dy = 0 \tag{24}$$

is not exact in a domain D but the differential equation

$$\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0$$

is exact in D, then $\mu(x,y)$ is called an **integrating factor** of the differential equation (24).

Example 25

$$(3y + 4xy^2)dx + (2x + 3x^2y)dy = 0$$

is not exact. $\mu(x,y) = x^2y$ works as an integrating factor for this equation.

Multiplication of a nonexact differential equation by an integrating factor thus transforms the nonexact equation into an exact one. We refer to this resulting exact equation as *essentially equivalent* to the original. This essentially equivalent exact equation has the same one parameter family of solutions as the nonexact original. However, the multiplication of the original equation by the integrating factor may result in either

1) the loss of one or more solutions of the original, or
2) the gain of one or more functions which are solutions of the new equation but not of the original, or
3) both of these phenomena.

We should check to determine whether any solutions may have been lost or gained.

Exercises

Check whether the following are exact or not. If exact, solve them.

$$(3x + 2y)dx + (2x + y)dy = 0$$

$$(y^2 + 3)dx + (2xy - 4)dy = 0$$

$$(2xy + 1)dx + (x^2 + 4y)dy = 0$$

Solve the initial value problem

$$(2xy - 3)dx + (x^2 + 4y)dy = 0, \quad y(1) = 2$$

$$(3x^2y^2 - y^3 + 2x)dx + (2x^3y - 3xy^2 + 1)dy = 0, \quad y(-2) = 1$$

Separable differential equations

Definition 10

An equation of the form

$$F(x)G(y)dx + f(x)g(y)dy = 0 \tag{25}$$

is called a separable equation.

Multiply (25) by the integrating factor $\frac{1}{f(x)G(y)}$:

$$\frac{F(x)}{f(x)}dx + \frac{g(y)}{G(y)}dy = 0 \tag{26}$$

Multiply (25) by the integrating factor

$$\frac{F(x)}{f(x)}dx + \frac{g(y)}{G(y)}dy = 0 \qquad \text{(cf. 26)}$$

This equation is exact since

$$\frac{\partial\left(\frac{F(x)}{f(x)}\right)}{\partial y} = 0 = \frac{\partial\left(\frac{g(y)}{G(y)}\right)}{\partial x}$$

Denoting $\frac{F(x)}{f(x)}$ by $M(x)$ and $\frac{g(y)}{G(y)}$ by $N(y)$, Equation (26) takes the form

$$M(x)dx + N(y)dy = 0$$

Since M is function of x only, and N is function of y only, a one parameter family of solutions is

$$\int M(x)dx + \int N(y)dy = c$$

where c is the arbitrary constant.

$$F(x)G(y)dx + f(x)g(y)dy = 0 \qquad \text{(cf. 25)}$$

Consider the original equation (25) in the following form:

$$f(x)g(y)\frac{dy}{dx} + F(x)G(y) = 0 \qquad (27)$$

If there exists a real number $y = y_0$ such that $G(y_0) = 0$ then (27) reduces to

$$f(x)g(y)\frac{dy}{dx} = 0$$

which has a constant solution $y = y_0$. We next should investigate whether the constant solution $y = y_0$ of the original equation is lost or not in the process of multiplying by the integrating factor.

Example 26

$$(x-4)y^4dx - x^3(y^2-3)dy = 0$$

The equation above is separable.
Multiply throughout by the integrating factor $\frac{1}{x^3y^4}$:

$$\frac{x-4}{x^3}dx - \frac{y^2-3}{y^4}dy = 0$$

or

$$(x^{-2}-4x^{-3})dx - (y^{-2}-3y^{-4})dy = 0$$

Integrating we obtain the solutions

$$\frac{-1}{x} + \frac{2}{x^2} + \frac{1}{y} - \frac{1}{y^3} = c$$

where c is any arbitrary constant.

Original equation	:	$(x-4)y^4dx - x^3(y^2-3)dy = 0$
Essentially equivalent equation	:	$\frac{x-4}{x^3}dx - \frac{y^2-3}{y^4}dy = 0$
Soln. of essentially equiv. d.e.	:	$\frac{-1}{x} + \frac{2}{x^2} + \frac{1}{y} - \frac{1}{y^3} = c$

Example 26 (cont.)

In multiplying by the integrating factor $\frac{1}{f(x)G(y)} = \frac{1}{x^3y^4}$ in the separation process, we assumed that $x^3 \neq 0$ and $y^4 \neq 0$. We now consider the solution $y = 0$ of $G(y) = y^4 = 0$. It is not a member of the one parameter family of solutions which we obtained. However, writing the original differential equation of the problem in the derivative form

$$\frac{dy}{dx} = \frac{(x-4)y^4}{x^3(y^2-3)}$$

it is obvious that $y = 0$ is a solution of the original equation. We conclude that it is a solution which was lost in the separation process.

Example 27

Consider

$$\frac{dy}{dt} = \frac{1 + \cos t}{1 + 3y^2}$$

We can write it as

$$(1 + 3y^2)dy = (1 + \cos t)dt$$

Integrating throughout yields the solution:

$$y + y^3 = t + \sin t + c$$

Example 28

Consider

$$\frac{dy}{dt} = -y\frac{1+2t^2}{t}, \; y(1) = 2$$

We can write it as $\frac{dy}{y} = -\frac{1+2t^2}{t}dt$

$$\int \frac{dy}{y} = -\int \frac{1+2t^2}{t}dt$$

$$\int \frac{dy}{y} = -\int \frac{dt}{t} - \int 2t dt$$

$$\ln y = -\ln(t) - t^2 + c$$

$$y = e^{-\ln t - t^2 + c} = e^{-\ln t} e^{-t^2} \underbrace{e^c}_{A} = \frac{A}{t} e^{-t^2}$$

At $t = 1$ we have $y = 2$. So, $2 = Ae^{-1} \rightarrow A = 2e^1$. Therefore, the solution is

$$y(t) = \frac{2e^1}{t} e^{-t^2} = \frac{2}{t} e^{1-t^2}$$

Homogeneous differential equations

The first order differential equation

$$M(x, y)dx + N(x, y)dy = 0$$

is said to be homogeneous if, when written in derivative form

$$\frac{dy}{dx} = f(x, y)$$

there exists a function g such that $f(x, y)$ can be expressed in the form $g(v)$ where $v = \frac{y}{x}$

Example 29

The differential equation

$$(x^2 - 3y^2)dx + 2xydy = 0$$

is homogeneous. This equation can be written as

$$\frac{dy}{dx} = \frac{3y^2 - x^2}{2xy} = \frac{3}{2}v - \frac{1}{2}\frac{1}{v}$$

where $v := \frac{y}{x}$.

A function F is called homogeneous of degree n if $F(tx, ty) = t^n F(x, y)$.

Theorem 4

If

$$M(x, y)dx + N(x, y)dy = 0 \tag{28}$$

is a homogeneous equation, then the change of variables $y = vx$ transforms (28) into a separable equation in the variables v and x.

Proof

Homogeneity implies $\frac{dy}{dx} = g(\frac{y}{x})$ for some g. Let $y = vx$, then

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \to v + x\frac{dv}{dx} = g(v) \to [v - g(v)]dx + xdv = 0$$

$$\frac{dv}{v - g(v)} + \frac{dx}{x} = 0$$

Integrate throughout:

$$\int \frac{dv}{v - g(v)} + \int \frac{dx}{x} = c$$

where c is an arbitrary constant. Define $F(v) \triangleq \int \frac{dv}{v-g(v)}$ then the solution of the original equation is

$$F(\frac{y}{x}) + \ln|x| = c$$

Example 30

Consider the differential equation

$$(x^2 - 3y^2)dx + 2xydy = 0$$

We have already seen that this is homogeneous. Write this in the form

$$\frac{dy}{dx} = \frac{-x}{2y} + \frac{3y}{2x} = \frac{-1}{2v} + \frac{3}{2}v$$

Transform $y = vx$ results $\frac{dy}{dx} = v + x\frac{dv}{dx}$, so we have

$$v + x\frac{dv}{dx} = \frac{-1}{2v} + \frac{3v}{2} \rightarrow \frac{2v}{v^2 - 1}dv = \frac{dx}{x}$$

Integration gives:

$$\ln|v^2 - 1| = \ln|x| + \ln|c| \rightarrow \ln|v^2 - 1| = \ln|x||c|$$

$$\rightarrow |v^2 - 1| = |cx| \rightarrow |\frac{y^2}{x^2} - 1| = |cx|$$

Example 30 (cont.)

In the previous page we had

$$\ln|v^2-1| = \ln|x| + \ln|\mathbf{c}| \to \ln|v^2-1| = \ln|x||c|$$

$$\to |v^2-1| = |cx| \to |\frac{y^2}{x^2} - 1| = |cx|$$

Now let us write the arbitrary constant as c:

$$\ln|v^2-1| = \ln|x| + \mathbf{c}$$

$$\to e^{\ln|v^2-1|} = e^{\ln|x|+c}$$

$$\to |v^2-1| = e^{\ln|x|}e^{c}$$

$$\to |\frac{y^2}{x^2} - 1| = e^{c}x$$

Example 31

Consider the differential equation

$$\frac{dy}{dx} = \frac{y(x-y)}{x^2} \tag{29}$$

This can be written as

$$\frac{dy}{dx} = \frac{y}{x} - (\frac{y}{x})^2$$

It is homogeneous! Using the transformation $v = \frac{y}{x}$:

$$v + x\frac{dv}{dx} = v - v^2$$

$$v + x\frac{dv}{dx} = v - v^2$$

Example 31 (cont.)

$$x\frac{dv}{dx} = -v^2$$

$$-\frac{1}{v^2}dv = \frac{1}{x}dx$$

$$\int -\frac{1}{v^2}dv = \int \frac{1}{x}dx$$

$$\frac{1}{v} = \ln x + c \to \frac{1}{v} = \ln x + \ln k \to \frac{1}{v} = \ln kx$$

$$v = \frac{1}{\ln kx} \to \frac{y}{x} = \frac{1}{\ln kx}$$

$$y = \frac{x}{\ln kx}$$

Linear differential equations

Definition 11

A first order ordinary differential equation is linear in the dependent variable y and the independent variable x if it is, or can be, written in the form

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{30}$$

Note that:
If $P(x) = 0$, then direct integration gives the solution:
$y(x) = \int Q(x)dx$
If $Q(x) = 0$, then the equation is separable.

$$\frac{dy}{dx} + P(x)y = Q(x) \qquad \text{(cf. 30)}$$

Equation above can be written in the form

$$[P(x)y - Q(x)]dx + dy = 0 \qquad (31)$$

This has the form $M(x, y)dx + N(x, y)dy = 0$. Lets check the exactness:

$$\frac{\partial M(x, y)}{\partial y} = P(x) \text{ and } \frac{\partial N(x, y)}{\partial x} = 0$$

Equation (31) is not exact unless $P(x) = 0$, in which case Equation (30) becomes trivially simple. Let us proceed with the general case $P(x) \neq 0$.

$$[P(x)y - Q(x)]dx + dy = 0 \qquad \text{(cf. 31)}$$

Multiply equation (31) by $\mu(x)$ to obtain

$$[\mu(x)P(x)y - \mu(x)Q(x)]dx + \mu(x)dy = 0$$

Now the equation is exact iff:

$$\frac{\partial[\mu(x)P(x)y - \mu(x)Q(x)]}{\partial y} = \frac{\partial \mu(x)}{\partial x}$$

This condition reduces to

$$\mu(x)P(x) = \frac{d\mu(x)}{dx}$$

$$\mu(x)P(x) = \frac{d\mu(x)}{dx}$$

This can be written as a differential equation

$$\frac{d\mu}{\mu} = P(x)dx$$

$$\rightarrow \ln|\mu| = \int P(x)dx$$

$$\rightarrow \mu = e^{\int P(x)dx}$$

where it is clear that $\mu > 0$.

Thus

$$\mu = e^{\int P(x)dx} \tag{32}$$

is the integrating factor for (30).
Recall

$$\frac{dy}{dx} + P(x)y = Q(x) \qquad \text{(cf. 30)}$$

Multiply (30) throughout by the integrating factor:

$$e^{\int P(x)dx}\frac{dy}{dx} + e^{\int P(x)dx}P(x)y = e^{\int P(x)dx}Q(x) \tag{33}$$

This is equivalent to

$$\frac{d}{dx}[e^{\int P(x)dx}y] = e^{\int P(x)dx}Q(x) \tag{34}$$

$$\frac{d}{dx}[e^{\int P(x)dx}y] = e^{\int P(x)dx}Q(x) \qquad \text{(cf. 34)}$$

This results in

$$e^{\int P(x)dx}y = \int e^{\int P(x)dx}Q(x)dx + c \tag{35}$$

$$y = e^{-\int P(x)dx}[\int e^{\int P(x)dx}Q(x)dx + c] \tag{36}$$

Example 32

$$\frac{dy}{dx} + \frac{2x+1}{x}y = e^{-2x} \tag{37}$$

Here $P(x) = \frac{2x+1}{x}$ and the integrating factor is

$$e^{\int \frac{2x+1}{x}dx} = e^{2x+\ln|x|} = e^{2x}e^{\ln|x|} = xe^{2x}$$

Multiply (37) by the integrating factor

$$xe^{2x}\frac{dy}{dx} + xe^{2x}\frac{2x+1}{x}y = x$$

or

$$\frac{d}{dx}(xe^{2x}y) = x$$

$$\frac{d}{dx}(xe^{2x}y) = x$$

Example 32 (cont.)

Integrate throughout

$$xe^{2x}y = \frac{x^2}{2} + c$$

$$y = e^{-2x}\frac{x}{2} + \frac{c}{x}e^{-2x}$$

where c is arbitrary constant.

In the last example we calculated the integrating factor as

$$e^{\int \frac{2x+1}{x}dx} = e^{2x+\ln|x|} = e^{2x}e^{\ln|x|} = xe^{2x}$$

We could have calculated as

$$e^{\int \frac{2x+1}{x}dx} = e^{2x+\ln|x|+c} = e^{2x}e^{\ln|x|}e^{c} \triangleq Kxe^{2x}$$

for an arbitrary constant c.
Note that if μ is an integrating factor then so is $K\mu$ for any positive $K \in \mathbb{R}$.

Consider the 1st order linear ordinary de

$$\frac{dy}{dx} + P(x)y = Q(x) \qquad \text{(cf. 30)}$$

where $P(x)$ and $Q(x)$ are continuous in $a \leq x \leq b$. It has been shown that it has a solution by a construction. The closed form solution constructed is

$$y(x) = e^{-\int P(x)dx}[\int e^{\int P(x)dx}Q(x)dx + c]$$

If (30) comes with an initial condition $y(x_0) = y_0$, does it make this solution unique? We discuss this, following an example where the D.E. (30) comes with initial condition $y(x_0) = y_0$.

Let the previous example together with the initial conditions be

$$\frac{dy}{dx} + \frac{2x+1}{x}y = e^{-2x}, \; y(1) = 2$$

Then the solution

$$y(x) = e^{-2x}\frac{x}{2} + \frac{c}{x}e^{-2x}$$

is evaluated at the initial condition:

$$y(1) = e^{-2\cdot 1}\frac{1}{2} + \frac{c}{1}e^{-2\cdot 1} = 2$$

$$\rightarrow e^{-2}\frac{1}{2} + ce^{-2} = 2$$

$$c = 2e^2 - \frac{1}{2}$$

Thus, the solution becomes

$$y(x) = e^{-2x}\frac{x}{2} + \frac{2e^2 - \frac{1}{2}}{x}e^{-2x}$$

Is the solution $y(x) = e^{-2x}\frac{x}{2} + \frac{2e^2 - \frac{1}{2}}{x}e^{-2x}$ unique to the i.v.p.

$$\frac{dy}{dx} + \frac{2x+1}{x}y = e^{-2x},\ y(1) = 2$$

Consider general 1st order ivp:

$$\frac{dy}{dx} + P(x)y = Q(x),\ y(x_0) = y_0$$

Let us assume that we have two solutions y_1 and y_2 to the d.e. above. If there is a unique solution to the de, then y_1 and y_2 cannot be two different functions, they must be equal. Consequently, if w is defined by $w(x) := y_1(x) - y_2(x)$ for all $a \leq x \leq b$, then $w(x) = 0$ for all $a \leq x \leq b$.

$$\frac{dy}{dx} + P(x)y = Q(x), \; y(x_0) = y_0$$

For this differential equation we defined $w(x) := y_1(x) - y_2(x)$ where y_1 and y_2 are two solutions to the d.e. If there is a unique solution to the de, then y_1 and y_2 cannot be two different functions, they must be equal. This makes $w = 0$ for all $a \leq x \leq b$. We will show that $w(x) = 0$ for all $a \leq x \leq b$.

We, below, show that $w(x)$ satisfies the homogeneous part of the 1st order linear differential equation

$$\frac{dw}{dx} + P(x)w = 0 \qquad (*)$$

$$\begin{aligned} \frac{d(y_1 - y_2)}{dx} + P(x)(y_1 - y_2) &= \left[\frac{dy_1}{dx} + P(x)y_1\right] - \left[\frac{dy_2}{dx} + P(x)y_2\right] \\ &= Q(x) - Q(x) \\ &= 0 \end{aligned}$$

Multiply both sides of (*) by $e^{\int_{x_0}^{x} P(s)ds}$:

$$e^{\int_{x_0}^{x} P(s)ds} \frac{dw}{dx} + e^{\int_{x_0}^{x} P(s)ds} P(x)w = 0$$

$$\left(e^{\int_{x_0}^{x} P(s)ds} w(x)\right)' = 0$$

$$\left(e^{\int_{x_0}^{x} P(s)ds} w(x)\right)' = 0$$

$$e^{\int_{x_0}^{x} P(s)ds} w(x) = C$$

$$\to w(x) = Ce^{-\int_{x_0}^{x} P(s)ds}$$

Noting

$$w(x_0) = y_1(x_0) - y_2(x_0) = y_0 - y_0 = 0$$

implies $C = 0$. Thus

$$w(x) = 0, \; a \le x \le b,$$

which implies

$$y_1(x) - y_2(x) = 0, \; a \le x \le b$$

and

$$y_1(x) = y_2(x), \; a \le x \le b$$

Example 33

Solve the DE

$$x\frac{dy}{dx} = y + 2x^3$$

In the standard form, this d.e. is

$$\frac{dy}{dx} - \frac{1}{x}y = 2x^2$$

Integrating factor is

$$e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$$

Multiply the d.e. throughout by the integrating factor

$$\frac{1}{x}\frac{dy}{dx} - \frac{1}{x}\frac{1}{x}y = \frac{1}{x}2x^2$$

$$\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = 2x$$

$$\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = 2x$$

Example 33 (cont.)

$$\frac{d}{dx}\left(\frac{1}{x}y\right) = 2x$$

Integrate both sides:

$$\frac{1}{x}y = x^2 + c$$

$$y = x^3 + cx$$

where c is an arbitrary constant.

Example 34

Solve the DE

$$\frac{dy}{dx} = 5y - 3$$

Ans.

$$y(x) = \frac{3}{5} + ce^{5x}$$

where c is an arbitrary constant.

Bernouilli differential equations

Definition 12

An equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \tag{38}$$

is called a Bernouilli differential equation

Clearly, for $n = 0$ and $n = 1$, the equation is linear.

Theorem 5

Excluding the cases $n = 0$ and $n = 1$, the transformation $v = y^{1-n}$ reduces the Bernouilli equation to a linear equation in v.

Proof

Multiply the Bernouilli equation by y^{-n} to obtain

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \rightsquigarrow y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x) \qquad (39)$$

Let $v = y^{1-n}$, then

$$\frac{dv}{dx} = (1-n)y^{-n}\frac{dy}{dx} \to \frac{1}{1-n}\frac{dv}{dx} = y^{-n}\frac{dy}{dx}$$

Now the (39) becomes

$$\frac{1}{1-n}\frac{dv}{dx} + P(x)v = Q(x)$$

$$\frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x)$$

Letting $P_1(x) := (1-n)P(x)$ and $Q_1(x) := (1-n)Q(x)$ the differential equation can be written as

$$\frac{dv}{dx} + P_1(x)v = Q_1(x)$$

which is linear in v.

Example 35

$$\frac{dy}{dx} - \frac{2}{x}y = -x^2y^2$$

Multiply the above Bernouilli d.e. by y^{-2}:

$$\frac{dy}{dx}y^{-2} - \frac{2}{x}y^{-1} = -x^2$$

Transformation $v = y^{1-n}$ corresponds $v = y^{-1}$, and $\frac{dv}{dx} = -\frac{\dot{y}}{y^2} = -\frac{dy}{dx}y^{-2}$. Using the transformation the d.e. becomes:

$$-\frac{dv}{dx} - \frac{2}{x}v = -x^2$$

or, equivalently,

$$\frac{dv}{dx} + \frac{2}{x}v = x^2$$

$$\frac{dv}{dx} + \frac{2}{x}v = x^2$$

Example 35 (cont.)

The first order d.e. has the solution

$$v = \frac{1}{5}x^3 + cx^{-2}$$

Recalling $v = y^{-1}$ yields

$$y = \frac{1}{\frac{1}{5}x^3 + cx^{-2}}$$

where c is an arbitrary constant.

Riccati differential equations

Definition 13

A Riccati differential equation is an ordinary differential equation that has the form

$$\frac{dy}{dx} = q_0(x) + q_1(x)y + q_2(x)y^2 \tag{40}$$

Theorem 6

The Riccati equation can always be reduced to a second order linear ODE.

Here we assume that q_2 is nonzero, otherwise (40) is a linear differential equation. We also assume that $q_0 \neq 0$, otherwise (40) becomes a Bernouilli differential equation.

$$\frac{dy}{dx} = q_0(x) + q_1(x)y + q_2(x)y^2 \qquad \text{(cf. 40)}$$

Use the transform

$$v = yq_2$$

then

$$\dot{v} = \dot{y}q_2 + y\dot{q}_2 = (q_0 + q_1 y + q_2 y^2)q_2 + v\frac{\dot{q}_2}{q_2} = q_0 q_2 + (q_1 + \frac{\dot{q}_2}{q_2})v + v^2$$

$$\dot{v} = q_0 q_2 + (q_1 + \frac{\dot{q}_2}{q_2})v + v^2$$

Define $Q := q_0 q_2$ and $P := q_1 + \frac{\dot{q}_2}{q_2}$ we can write

$$\dot{v} = v^2 + P(x)v + Q(x)$$

$$\dot{v} = v^2 + P(x)v + Q(x)$$

$$v^2 - \dot{v} = -P(x)v - Q(x)$$

Now use

$$v = -\frac{\dot{u}}{u}$$

This implies

$$\dot{v} = -(\frac{\dot{u}}{u})' = -(\dot{u} \times \frac{1}{u})' = -(\frac{\ddot{u}}{u}) + (\frac{\dot{u}}{u})^2 = -(\frac{\ddot{u}}{u}) + v^2$$

so that

$$\frac{\ddot{u}}{u} = v^2 - \dot{v} = -Q - Pv = -Q + P\frac{\dot{u}}{u}$$

$$\frac{\ddot{u}}{u} = -Q + P\frac{\dot{u}}{u}$$

and hence

$$\ddot{u} - P\dot{u} + Qu = 0.$$

Q.E.D.

$$\frac{dy}{dx} = q_0(x) + q_1(x)y + q_2(x)y^2 \qquad \text{(cf. 40)}$$

Theorem 7

If any solution $u(x)$ of the Riccati equation (40) is known, then substitution of $y = u + \frac{1}{z}$ will transform (40) into a linear 1st order equation in z.

Proof If u is a solution of the Riccati equation then

$$\frac{du}{dx} = q_0(x) + q_1(x)u + q_2(x)u^2 \tag{41}$$

The transform $y = u + \frac{1}{z}$ implies

$$\frac{dy}{dx} = \frac{d}{dx}(u + \frac{1}{z}) = \frac{du}{dx} - \frac{1}{z^2}\frac{dz}{dx} \tag{42}$$

$$\frac{dy}{dx} = q_0(x) + q_1(x)y + q_2(x)y^2 \qquad \text{(cf. 40)}$$

$$\frac{dy}{dx} = \frac{d}{dx}(u + \frac{1}{z}) = \frac{du}{dx} - \frac{1}{z^2}\frac{dz}{dx} \qquad \text{(cf. 42)}$$

Substitute $y = u + \frac{1}{z}$ and (42) in the Riccati equation (40):

$$\begin{aligned} \frac{du}{dx} - \frac{1}{z^2}\frac{dz}{dx} &= q_2(u + \tfrac{1}{z})^2 + q_1(u + \tfrac{1}{z}) + q_0 \\ &= \underbrace{(q_2 u^2 + q_1 u + q_0)}_{\text{equals } \frac{du}{dx} \text{ by } (41)} + (\tfrac{2u}{z}q_2 + \tfrac{1}{z^2}q_2 + \tfrac{1}{z}q_1) \end{aligned}$$

$$-\frac{1}{z^2}\frac{dz}{dx} = \frac{2u}{z}q_2 + \frac{1}{z^2}q_2 + \frac{1}{z}q_1$$

$$\frac{dz}{dx} = -2uzq_2 - q_2 - zq_1 = -(2uq_2 + q_1)z - q_2$$

A linear 1st order differential equation in z!

Example 36

Consider the Riccati equation

$$\frac{dy}{dx} + y = xy^2 - \frac{1}{x^2} \tag{43}$$

$y = \frac{1}{x}$ is a particular solution to (43). We want to find the other solution. Use the transform

$$y = \frac{1}{x} + \frac{1}{z}$$

then we have

$$y' = -\frac{z'}{z^2} - \frac{1}{x^2}$$

Substitute y and y' in (43):

$$-\frac{z'}{z^2} - \frac{1}{x^2} + \frac{1}{x} + \frac{1}{z} = x(\frac{1}{x} + \frac{1}{z})^2 - \frac{1}{x^2}$$

$$-\frac{z'}{z^2}-\frac{1}{x^2}+\frac{1}{x}+\frac{1}{z}=x(\frac{1}{x}+\frac{1}{z})^2-\frac{1}{x^2}$$

Example 36 (cont.)

$$-\frac{z'}{z^2}-\frac{1}{x^2}+\frac{1}{x}+\frac{1}{z}=x(\frac{1}{x^2}+\frac{1}{z^2}+\frac{2}{xz})-\frac{1}{x^2}$$

$$-\frac{z'}{z^2}=x(\frac{1}{z^2}+\frac{1}{xz})$$

$$z'+z=-x$$

A 1st order linear de! Its solution is

$$z=1-x+ce^{-x}$$

Noting that $y=\frac{1}{x}+\frac{1}{z}$, the solution to (43) is

$$y=\frac{1}{x}+\frac{1}{1-x+ce^{-x}}$$

Example 37

Consider the Riccati equation

$$x\frac{dy}{dx} - 3y + y^2 = 4x^2 - 4x$$

Obviously $u(x) = 2x$ is a particular solution of this differential equation. From this we can obtain a 1st order linear differential equation in z.

Example 38

The Riccati equation

$$\frac{dy}{dx} + y^2 = \frac{2}{x^2}$$

has a particular solution $u = \frac{2}{x}$. Let us find the other solution.Use the transformation

$$y = \frac{2}{x} + \frac{1}{z}$$

This leads to

$$\frac{dy}{dx} = -\frac{2}{x^2} - \frac{1}{z^2}\frac{dz}{dx}$$

Substitute in the de:

$$-\frac{2}{x^2} - \frac{1}{z^2}\frac{dz}{dx} + \left(\frac{2}{x} + \frac{1}{z}\right)^2 = \frac{2}{x^2}$$

$$-\frac{2}{x^2}-\frac{1}{z^2}\frac{dz}{dx}+\left(\frac{2}{x}+\frac{1}{z}\right)^2=\frac{2}{x^2}$$

Example 38 (cont.)

$$\cancel{-\frac{2}{x^2}}-\frac{1}{z^2}\frac{dz}{dx}+\left(\cancel{\frac{4}{x^2}}+\frac{1}{z^2}+\frac{4}{xz}\right)=\cancel{\frac{2}{x^2}}$$

$$-\frac{1}{z^2}\frac{dz}{dx}+\left(\frac{1}{z^2}+\frac{4}{xz}\right)=0\rightarrow\frac{dz}{dx}-\frac{4}{x}z=1$$

This is a linear equation.Its solution is

$$z(x)=-\frac{x}{3}+cx^4$$

Recalling the transformation $y=\frac{2}{x}+\frac{1}{z}$ the solution we seek is

$$y=\frac{2}{x}+\frac{1}{-\frac{x}{3}+cx^4}$$

Orthogonal trajectories

Let

$$F(x, y, c) = 0 \tag{44}$$

be a given one parameter family of curves in xy-plane. A curve that intersects curves of the family (44) at right angles is called an orthogonal trajectory of the given family.

Example 39

Consider the family of curves $x^2 + y^2 = c^2$. Each straight line passing through the origin $y = kx$ is an orthogonal trajectory of the given family of circles.

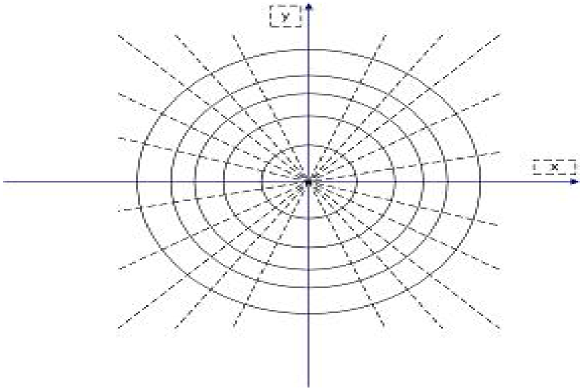


Figure: Orthogonal trajectories for $x^2 + y^2 = c$

How to find orthogonal trajectories

Step 1. Differentiate $F(x, y, c) = 0$ with respect to x to obtain

$$\frac{dy}{dx} = f(x, y) \tag{45}$$

Step 2. Solutions of $\frac{dy}{dx} = \frac{-1}{f(x,y)}$ are the orthogonal trajectories.

Reasoning

Step 1. Differentiate $F(x, y, c) = 0$ with respect to x to obtain

$$\frac{dy}{dx} = f(x, y) \qquad \text{(cf. 45)}$$

Step 2. Solutions of $\frac{dy}{dx} = \frac{-1}{f(x,y)}$ are the orthogonal trajectories.

In $F(x, y, c) = 0$ the slope of the curve passing through the point (x, y) is $\frac{dy}{dx}$, which is $f(x, y)$. However, the slope of the curves passing through (x, y) having right angle to $F(x, y, c) = 0$ curves are $\frac{-1}{f(x,y)}$.

Caution. In step 1 finding the differential equation (45) of the given family, be sure to eliminate the parameter c during the process.

Example 40

$F(x, y, c) = 0$ is given by $x^2 + y^2 - c^2 = 0$. Differentiation gives

$$2x + 2y\frac{dy}{dx} = 0 \to \frac{dy}{dx} = \underbrace{\frac{-x}{y}}_{f(x,y)}$$

We are looking for the orthogonal trajectories, so we must solve

$$\frac{dy}{dx} = \underbrace{\frac{y}{x}}_{\frac{-1}{f(x,y)}}$$

or

$$\frac{dy}{dx} = \frac{y}{x} \to \frac{dy}{y} = \frac{dx}{x} \to \ln y = \ln x + \ln k \to \ln y = \ln kx \to y = kx$$

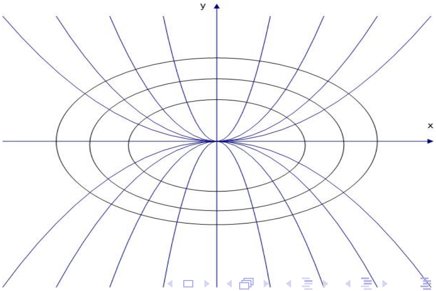
Example 41

Find the orthogonal trajectories of the family of parabolas $y = cx^2$.

$$y = cx^2 \rightarrow \frac{dy}{dx} = \underbrace{c}_{\frac{y}{x^2}} 2x \rightarrow \frac{dy}{dx} = 2\frac{y}{x}$$

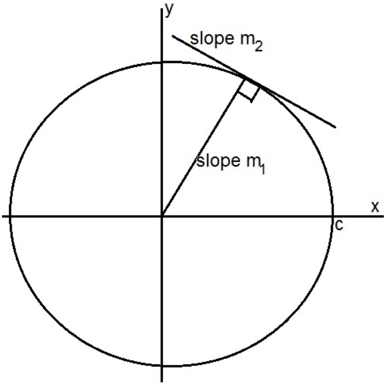
Orthogonal trajectory finding requires solving $\frac{dy}{dx} = \frac{-x}{2y}$

$$2ydy = -xdx \rightarrow y^2 = -\frac{x^2}{2} + c \rightarrow x^2 + 2y^2 = k^2$$



Example 42

(A proof of Pythagorean theorem) A line segment from origin to a point (x, y) on circle has slope $m_1 = \frac{y}{x}$. Circle's slope m_2 at (x, y) satisfies $m_1 \times m_2 = -1$. Thus, circle's slope at (x, y) is $-\frac{x}{y}$.

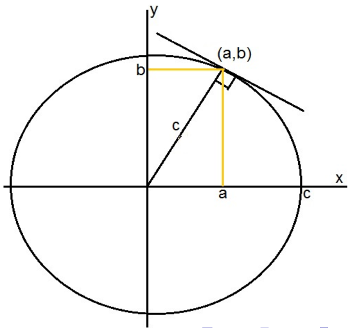


Example 42 (cont.)

Knowing circle's slope at (x, y), its d. e. can be written as

$$\frac{dy}{dx} = -\frac{x}{y}, \quad y(0) = c$$

Its solution is $y^2 + x^2 = c^2$. Let the curve pass through the point (a, b), then it satisfies the relationship $a^2 + b^2 = c^2$. This proves the Pythagorean theorem for the right triangle with smaller side lengths a and b, and hypothenuse c.



Oblique trajectories

Definition 14

Let

$$F(x, y, c) = 0 \tag{46}$$

be a given one parameter family of curves in xy-plane. A curve that intersects curves of the family (46) at a constant angle $\alpha \neq 90^0$ is called an oblique trajectory of the given family.

Differential equation corresponding to (46) is

$$\frac{dy}{dx} = f(x, y) \tag{47}$$

Differential equation corresponding to (46) is

$$\frac{dy}{dx} = f(x, y) \qquad \text{(cf. 47)}$$

Then the curve of family (46) passing through the point (x, y) has slope $f(x, y)$ at this point, and its tangent line has angle of inclination $\tan^{-1}[f(x, y)]$. The tangent line of an oblique trajectory that intersects this curve at the angle α will thus have an inclination $\tan^{-1}[f(x, y)] + \alpha$ at the point (x, y). Hence the slope of the oblique trajectory is given by

$$\tan\{\tan^{-1}[f(x, y)] + \alpha\} = \frac{f(x, y) + \tan\alpha}{1 - f(x, y)\tan\alpha}$$

Thus the differential equation of such a family of oblique trajectories is given by

$$\frac{dy}{dx} = \frac{f(x, y) + \tan\alpha}{1 - f(x, y)\tan\alpha} \qquad (48)$$

Example 43

Find the family of oblique trajectories that intersect the family of straight lines $y = cx$ at angle 45^0.

$$y = cx \to \frac{dy}{dx} = c \to \frac{dy}{dx} = \frac{y}{x}$$

In

$$\frac{dy}{dx} = \frac{f(x,y) + \tan\alpha}{1 - f(x,y)\tan\alpha} \qquad \text{(cf. 48)}$$

use $f(x,y) = \frac{y}{x}$ and $\tan\alpha = 1$:

$$\frac{dy}{dx} = \frac{\frac{y}{x} + 1}{1 - \frac{y}{x}1} = \frac{x+y}{x-y}$$

This is a homogeneous differential equation Let $y = vx$:

$$v + x\frac{dv}{dx} = \frac{1+v}{1-v}$$

$$v + x\frac{dv}{dx} = \frac{1+v}{1-v}$$

Example 43 (cont.)

After simplification

$$\frac{(v-1)dv}{v^2+1} = \frac{-dx}{x}$$

Integrating

$$\frac{1}{2}\ln(v^2+1) - \tan^{-1}(v) = -\ln|x| - \ln|c|$$

$$\ln(v^2+1) - 2\tan^{-1}(v) = -2\ln|x| - 2\ln|c|$$

$$\ln(v^2+1) - 2\tan^{-1}(v) = -\ln|x|^2 - \ln|c|^2$$

$$\ln c^2x^2(v^2+1) - 2\tan^{-1} v = 0$$

$$\ln c^2(x^2+y^2) - 2\tan^{-1}\frac{y}{x} = 0$$

More on the Existence and Uniqueness of the Solutions

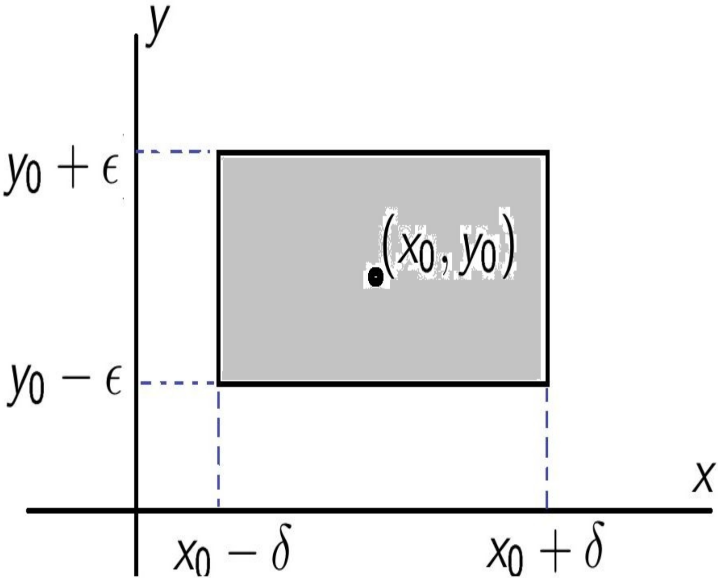
Sufficient Conditions

Consider the first order d.e. in the form

$$\dot{y}(x) = f(x, y),\ y(x_0) = y_0 \tag{49}$$

Suppose that $f(x, y)$ is continuous (**Condition 1**) with respect to x and y in the region $\{(x, y) :\ x_0 - \delta \leq x \leq x_0 + \delta, y_0 - \epsilon \leq y \leq y_0 + \epsilon\}$ for some $\delta > 0, \epsilon > 0$. Then the diff. equation (49) **has a solution** defined in a neighborhood of x_0.

Suppose that both $f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous (**Condition 2**) with respect to x and y in the region $\{(x, y) :\ x_0 - \delta \leq x \leq x_0 + \delta, y_0 - \epsilon \leq y \leq y_0 + \epsilon\}$ for some $\delta > 0, \epsilon > 0$. Then the diff. eqn. (49) **has a unique solution** defined in a neighborhood of x_0.



$$\{(x, y) : \ x_0 - \delta \leq x \leq x_0 + \delta, y_0 - \epsilon \leq y \leq y_0 + \epsilon\}$$

Example 44

$$\dot{y} = 1 + y^2, \; y(0) = 0$$

Condition 1: $f(x, y) = 1 + y^2$ is continuous for all (x, y) on the xy-plane. Therefore, there exists a solution for every initial condition pair (x_0, y_0)

Condition 2: Besides $f(x, y) = 1 + y^2$, the function $\frac{\partial f}{\partial y} = 2y$ is also continuous for all (x, y) on the xy-plane. Therefore, there exists a unique solution for every initial condition pair (x_0, y_0)

Indeed, by sepation of variables, the d.e. becomes

$$\frac{dy}{y^2 + 1} = dx$$

This d.e. has solutions in the form:

$$\tan^{-1} y = x + c \rightarrow y(x) = \tan(x + c)$$

where c is arbitrary constant.

Example 44 (cont.)

In the solution $y(x) = \tan(x + c)$ use the i.c. $y(0) = 0$
$0 = \tan(0 + c) \to \tan(c) = 0 \to c = 0, \pm\pi, \pm 2\pi, \dots$
Noting that tan function is π periodic (i.e., $\tan(x) = \tan(x + k\pi), k = 0, \pm 1, \pm 2, \dots$), choosing any of $0, \pm\pi, \pm 2\pi, \dots$ wouldn't make any difference. Choosing $c = 0$, the unique solution is, therefore,

$$y(x) = \tan(x)$$

This solution is defined in the neighborhood of $x = 0$, more specifically, this solution is defined in the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

Example 45

$$\dot{y} = x - y + 1, \; y(1) = 3$$

Condition 1: $f(x,y) = x - y + 1$ is continuous for all (x,y) on the xy-plane. Therefore, there exists a solution for every initial condition pair (x_0, y_0)
Condition 2: $f(x,y) = x - y + 1$ and $\frac{\partial f}{\partial y} = -1$ are continuous for all (x,y) on the xy-plane. Therefore, there is a unique solution for every initial condition pair (x_0, y_0)
Indeed, the explicit solution

$$y(x) = x + ce^{-x}$$

with the i.c. $y(1) = 3$ yields

$$y(x) = x + 2e^{1-x}$$

shows this.

Example 46

Consider

$$\dot{y} = \frac{2y}{x}, \; y(x_0) = y_0$$

Condition 1: $f(x, y) = \frac{2y}{x}$ is continuous for (x, y) except $x = 0$.
$\therefore$ There is a solution when $x \neq 0$. There is ambiguity when $(x, y) = (0, 0)$. This will be resolved.
Condition 2: $f(x, y)$ and $\frac{\partial f}{\partial y}(x, y) = \frac{2}{x}$ are continuous when $x \neq 0$.
$\therefore$ There is a unique solution when $x \neq 0$.
By separation of variables we obtain the solutions:

$$y(x) = cx^2$$

where c is arbitrary constant. There are infinitely many solutions!
All of them pass through $(0, 0)$.
No solution passes through $(0, y_0)$ with $y_0 \neq 0$.

Example 46 (cont.)

By separation of variables we obtain the solutions:

$$y(x) = cx^2$$

where c is arbitrary constant. There are infinitely many solutions! All of the pass through $(0, 0)$.
No solution passes through $(0, y_0)$ with $y_0 \neq 0$.

Obviously there are solutions when $x \neq 0$. Besides, there are solutions at $x = 0$, There are infinitely many solutions passing through $(0, 0)$.

There is a **unique solutions** when $x \neq 0$, however, there is no unique solution when $x = 0$.

The existence and uniqueness theorem is a sufficiency theorem. The following theorem enlarges the class of differential equations having unique solutions.

Theorem Suppose $f(x, y)$ is continuous in the open set S and is locally Lipschitz in y in S. Let $(x_0, y_0) \in S$. Then, the initial value problem

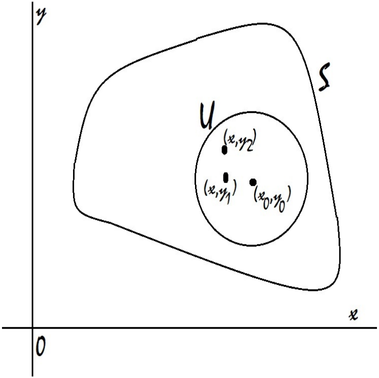
$$\dot{y} = f(x, y), \quad y(x_0) = y_0$$

has a unique solution defined in an interval containing x_0.

Definition Let $f(x, y)$ be a continuous function defined in the open set S. We say that f is **locally Lipschitz** in y if for each $(x_0, y_0) \in S$, there is a neighborhood U of (x_0, y_0) such that there is a constant $K > 0$, called **Lipschitz constant**, satisfying

$$|f(x, y_1) - f(x, y_2)| \leq K|y_1 - y_2|$$

for every $(x, y_1), (x, y_2) \in U$.



Definition Let $f(x, y)$ be a continuous function defined in the open set S. We say that f is **locally Lipschitz** in y if for each $(x_0, y_0) \in S$, there is a neighborhood U of (x_0, y_0) such that there is a constant $K > 0$, called **Lipschitz constant**, satisfying

$$|f(x, y_1) - f(x, y_2)| \leq K|y_1 - y_2|$$

for every $(x, y_1), (x, y_2) \in U$.

$$|f(x, y_1) - f(x, y_2)| \leq K|y_1 - y_2| \qquad \text{(Lipschitz Condition)}$$

Example 47

$$\dot{y} = y^2, \; y(t_0) = y_0$$

The function $f(y) = y^2$ is continuous, and it is locally Lipschitz, for instance, in the region $-2y_0 \leq y \leq 2y_0$ Note that for every y_1, y_2 in this region, $|y_1 + y_2| \leq 4y_0$. Thus

$$|f(t, y_2) - f(t, y_1)| = |y_2^2 - y_1^2| = |y_1 + y_2| \cdot |y_2 - y_1| \leq 4y_0|y_2 - y_1|$$

Lipschitz condition is satisfied for $K = 4y_0$. Therefore, the d.e. has a unique solution in the neighborhood of t_0. Indeed, it is

$$y(t) = \frac{1}{\frac{1}{y_0} - (t - t_0)} = \frac{y_0}{1 - y_0(t - t_0)}$$

$$\dot{y} = y^2, \; y(t_0) = y_0$$

$$y(t) = \frac{1}{\frac{1}{y_0} - (t - t_0)} = \frac{y_0}{1 - y_0(t - t_0)}$$

Example 47 (cont.)

If $t_0 = 0$ the solution becomes

$$y(t) = \frac{1}{\frac{1}{y_0} - t}$$

Thus, for example, if $y_0 > 0$, the solution only exists so long as

$$\frac{1}{y_0} - t > 0, \text{ or } t < \frac{1}{y_0}.$$

$$|f(x,y_1) - f(x,y_2)| \leq K|y_1 - y_2| \qquad \text{(Lipschitz Condition)}$$

Example 48

The function $\sqrt{|y|} : \Re \to \Re_{\geq 0}$ is NOT Lipschitz in $-0.1 \leq y \leq 0.1$.

Proof Assume that there exists K such that

$$|f(t,y_1) - f(t,y_2)| = \left|\sqrt{|y_1|} - \sqrt{|y_2|}\right| \leq K|y_1 - y_2|, \quad \text{for all } y_1, y_2 \in \mathcal{R}$$

Let $y_2 = \frac{1}{n^2}$, $n \in \mathcal{N}$ and $y_1 = 0$ then

$$|\sqrt{|\frac{1}{n^2}|} - \sqrt{|0|}| \leq K|\frac{1}{n^2} - 0|$$

$$\to \frac{1}{n} \leq \frac{K}{n^2} \to n \leq K$$

which contradicts with the fact that K is finite.

Example 49

$f(t, y) = \sqrt{y}$ is NOT locally Lipschitz in an interval around $y_0 = 0$. Thus, it is not guaranteed to have a unique solution for the d.e.

$$\frac{dy}{dt} = \sqrt{y}, \; y(t_0) = y_0$$

Indeed, both

$$y(t) = 0 \text{ and } y(t) = \frac{t^2}{4}$$

satisfy the d.e. $\dot{y} = \sqrt{y}, \; y(0) = 0.$

Solving higher order linear differential equations

Definition 15

A linear ordinary differential equation of order n in the dependent variable y and the independent variable x is an equation that is in, or can be expressed in, the form

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = F(x) \quad (50)$$

where a_0 is not identically zero.

The righthand member $F(x)$ is called the nonhomogeneous term. If F is identically zero Equation (50) reduces to

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = 0 \quad (51)$$

and is then called homogeneous.

When $n = 2$, a linear ordinary differential equation in the dependent variable y and in the independent variable x has the form

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = F(x)$$

where a_0 is not identically zero. Corresponding 2nd order homegeneous linear differential equation is

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0$$

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = F(x) \qquad \text{(cf. 50)}$$

Theorem 8

Consider the n-th order linear differential equation given by Equation (50) where $a_0,\ a_1, \ldots, a_n$ *and* F *are continuous real functions on a real interval* $a \le x \le b$ *and that* $a_0(x) \neq 0$ *for any* x *on* $a \le x \le b$*. Let* x_0 *be any point on the interval* $a \le x \le b$*, and let* $c_0,\ c_1, \ldots, c_{n-1}$ *be n arbitrary real constants. Then there exists a unique solution of Equation (50) such that*

$$f(x_0) = c_0,\ f'(x_0) = c_1, \ldots, f^{(n-1)}(x_0) = c_{n-1}$$

and this solution is defined over the entire interval $a \le x \le b$*.*

Example 50

Consider the initial value problem

$$\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + x^3y = e^x, \quad y(1) = 2, y'(1) = -5$$

In the interval $-\infty < x < \infty$ the hypotheses of Theorem 8 are satisfied, so the equation has a unique solution in this interval.

In the analysis of the linear ordinary differential equation

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = F(x) \quad \text{(cf. 50)}$$

unless otherwise is stated, we shall assume that $a_0,\ a_1, \ldots, a_n$ and F are continuous real functions on a real interval $a \leq x \leq b$. Not stating the usual assumptions each time, we keep the analysis more concise.

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = 0 \qquad \text{(cf. 51)}$$

Corollary 1

Let f be a solution of the n-th order homogeneous linear differential equation given by Equation (51) such that

$$f(x_0) = 0,\ f'(x_0) = 0, \ldots, f^{(n-1)}(x_0) = 0$$

where x_0 is a point of the interval $a \le x \le b$ in which the coefficients $a_0,\ a_1, \ldots, a_n$ are all continuous and $a_0(x) \neq 0$. Then $f(x) = 0$ for all $x \in [a, b]$.

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = 0 \qquad \text{(cf. 51)}$$

Theorem 9

For a homogeneous linear differential equation, (a) the sum of the solutions is also a solution and (b) a constant multiple of a solution is also a solution.

Proof Consider

$$\alpha\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + \gamma x = 0 \qquad (52)$$

where α, β and γ are functions of t.

Let the functions x_1 and x_2 be solutions to (52). Then

$$\alpha\frac{d^2x_1}{dt^2} + \beta\frac{dx_1}{dt} + \gamma x_1 = 0 \text{ and } \alpha\frac{d^2x_2}{dt^2} + \beta\frac{dx_2}{dt} + \gamma x_2 = 0$$

We wish to prove that $x_1 + x_2$ is also a solution, that is

$$\alpha\frac{d^2(x_1+x_2)}{dt^2} + \beta\frac{d(x_1+x_2)}{dt} + \gamma(x_1+x_2) \stackrel{?}{=} 0$$

$$\alpha\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + \gamma x = 0 \qquad \text{(cf. 52)}$$

$$\alpha\frac{d^2x_1}{dt^2} + \beta\frac{dx_1}{dt} + \gamma x_1 = 0 \text{ and } \alpha\frac{d^2x_2}{dt^2} + \beta\frac{dx_2}{dt} + \gamma x_2 = 0$$

$$\alpha\frac{d^2(x_1+x_2)}{dt^2} + \beta\frac{d(x_1+x_2)}{dt} + \gamma(x_1+x_2) \stackrel{?}{=} 0$$

Using the basic property of the derivatives:

$$\alpha\frac{d^2(x_1)}{dt^2} + \alpha\frac{d^2(x_2)}{dt^2} + \beta\frac{d(x_1)}{dt} + \beta\frac{d(x_2)}{dt} + \gamma x_1 + \gamma x_2 \stackrel{?}{=} 0$$

$$\alpha\frac{d^2(x_1)}{dt^2} + \beta\frac{d(x_1)}{dt} + \gamma x_1 + \alpha\frac{d^2(x_2)}{dt^2} + \beta\frac{d(x_2)}{dt} + \gamma x_2 = 0 + 0 = 0$$

$$\alpha \frac{d^2x}{dt^2} + \beta \frac{dx}{dt} + \gamma x = 0 \qquad \text{(cf. 52)}$$

Likewise, we wish to show that if x satisfies (52) then kx also satisfies it for any constant k.

$$\alpha \frac{d^2(kx)}{dt^2} + \beta \frac{d(kx)}{dt} + \gamma(kx) = \alpha k \frac{d^2(x)}{dt^2} + \beta k \frac{d(x)}{dt} + k\gamma x$$

$$= k(\alpha \frac{d^2(x)}{dt^2} + \beta \frac{d(x)}{dt} + \gamma x) = k \cdot 0 = 0$$

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = 0 \qquad \text{(cf. 51)}$$

Theorem 10

Let $f_1, f_2, \ldots, f_m$ be any m solutions of the homogeneous linear differential equation (51). Then $c_1 f_1 + c_2 f_2 + \cdots + c_m f_m$ is also a solution of (51), where $c_1, \ldots, c_m$ are m arbitrary constants.

Definition 16

If $f_1, f_2, \ldots, f_m$ are m given functions, and $c_1, c_2, \ldots, c_m$ are m constants then the expression $c_1 f_1 + c_2 f_2 + \cdots + c_m f_m$ is called **a linear combination** of $f_1, f_2, \ldots, f_m$.

Theorem 11

(Restated) For the homogeneous linear differential equation (51): Any linear combination of its solutions is also its solution.

Example 51

$\sin x$ and $\cos x$ are solutions of

$$\frac{d^2y}{dx^2} + y = 0$$

By the theorem $5\sin x + 6\cos x$ is also a solution of the equation.

Definition 17

The n functions $f_1, f_2, \ldots, f_n$ are called **linearly dependent** on $a \leq x \leq b$ if there exist constants $c_1, c_2, \ldots, c_n$, not all zero, such that

$$c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0$$

for all x such that $a \leq x \leq b$.

Example 52

Are the functions $f_1(x) = x,\ f_2(x) = x^2,\ f_3(x) = x^2 + 2x\ f_4(x) = 3$ linearly dependent on $0 \leq x \leq 10$?

$$c_1 x + c_2 x^2 + c_3(x^2 + 2x) + c_4 3 = 0,\ \forall x \in [0, 10]$$

In addition to zero the solution $c_1 = 0, c_2 = 0, c_3 = 0, c_4 = 0$ we have a nonzero solution $c_1 = 2, c_2 = 1, c_3 = -1, c_4 = 0$.
$\therefore$ This group of functions is linearly dependent.

In particular two functions f_1 and f_2 are linearly dependent on $a \le x \le b$ if there exist constants c_1, c_2, not both zero, such that

$$c_1 f_1 + c_2 f_2 = 0$$

for all x such that $a \le x \le b$.

Example 53

x and $2x$ are linearly dependent on the interval $0 \le x \le 1$, since there exist constants c_1, c_2, not both zero, such that

$$c_1 \cdot x + c_2 \cdot 2x = 0 \qquad (53)$$

for all x on the interval $0 \le x \le 1$. For instance, $c_1 = 2, c_2 = -1$

Notice that we found constants c_1 and c_2 that work for all x in the given interval $0 \le x \le 1$. If they worked for some x values only then we wouldn't say that the functions are linearly dependent. The next example illustrates this idea:

Example 54

Consider the functions $\cos x$, $\cos 2x$, and $\cos 3x$ on the interval $-\pi \leq x \leq \pi$. Form the linear dependence equation

$$c_1 \cos x + c_2 \cos 2x + c_3 \cos 3x = 0, \ -\pi \leq x \leq \pi \qquad (54)$$

When $x = 0$ this equation holds for $c_1 = 1, c_2 = 1$ and $c_3 = -2$. But this does not make this set linearly dependent. For linear dependency on $-\pi \leq x \leq \pi$, the constants c_1, c_2 and c_3 must work for ALL x on the interval $-\pi \leq x \leq \pi$. Notice that, for instance, when $x = \frac{\pi}{2}$, the above c_1, c_2, c_3 don't satisfy Equation (54). Thus, the functions $\cos x$, $\cos 2x$, and $\cos 3x$ on the interval $-\pi \leq x \leq \pi$ are not linearly dependent.

Definition 18

The n functions $f_1, f_2, \ldots, f_n$ are called **linearly independent** on the interval $a \leq x \leq b$ if they are not linearly dependent there.

Example 55

Are $f_1(t) = 2t$ and $f_2(t) = t^2$ linearly dependent on $0 \leq t \leq 2$? If we can find constants c_1 and c_2, not both zero, such that

$$c_1 2t + c_2 t^2 = 0, \; 0 \leq t \leq 2 \tag{55}$$

holds, then f_1 and f_2 are linearly dependent.
Suppose for some c_1 and c_2, not both zero, Equation (55) is satisfied. Then it must hold particularly at $t = 0.5$ and $t = 1$:

$$c_1 + 0.25c_2 = 0$$
$$2c_1 + c_2 = 0$$

These two equations imply $c_1 = c_2 = 0$, that is, there even does not exist c_1, c_2, not both zero, when considering only two points $t = 0.5$ and $t = 1$. So, if we cannot do it for only two points, then how can we do it for these two points plus infinitely many?
$\therefore$ This group of functions is **linearly independent**.

Alternative analysis of the previous example

Example 56

Are $f_1(t) = 2t$ and $f_2(t) = t^2$ linearly dependent on $0 \leq t \leq 2$? If we can find constants c_1 and c_2, not both zero, such that

$$c_1 2t + c_2 t^2 = 0, \; 0 \leq t \leq 2 \qquad \text{(cf. 55)}$$

holds, then f_1 and f_2 are linearly dependent.
Note that if (55) holds on $0 \leq t \leq 2$, then so does its derivative:

$$c_1 \cdot 2 + c_2 \cdot 2t = 0, \; 0 \leq t \leq 2$$

This implies $c_1 = -c_2 t$. Substitute this in (55):
$-c_2 t \cdot 2t + c_2 t^2 = 0, \; 0 \leq t \leq 2 \to -c_2 t^2 = 0, \; 0 \leq t \leq 2. \to$ $c_2 = 0$.. Use this in (55):
$c_1 \cdot 2t = 0, \; 0 \leq t \leq 2, \to c_1 = 0.$
We have only one solution $c_1 = c_2 = 0$,
$\therefore$ the set of functions $\{f_1, f_2\}$ is linearly independent.

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = 0 \qquad \text{(cf. 51)}$$

Theorem 12

The n-th order homogeneous linear differential equation (51) always possesses n solutions that are linearly independent. Further, if $f_1, f_2, \ldots, f_n$ are n linearly independent solutions of (51), then every solution f of (51) can be expressed as a linear combination

$$c_1 f_1 + c_2 f_2 + \cdots + c_n f_n$$

of these n linearly independent solutions by proper choice of the constants $c_1, c_2, \ldots, c_n$.

Example 57

$\sin x$ and $\cos x$ are solutions of

$$\frac{d^2y}{dx^2} + y = 0 \tag{56}$$

for all x, $-\infty < x < \infty$. Further one can show that these two solutions are linearly independent. Now suppose f is any solution of (56), then by the theorem f can be expressed as a linear combination $c_1 \sin x + c_2 \cos x$ of the two linearly independent solutions $\sin x$ and $\cos x$ by proper choices of c_1 and c_2.

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = 0 \qquad \text{(cf. 51)}$$

Definition 19

If $f_1, f_2, \ldots, f_n$ are n linearly independent solutions of the n-th order homogeneous linear differential equation (51) on $a \leq x \leq b$, then the set $f_1, f_2, \ldots, f_n$ is called **a fundamental set of solutions** of (51) and the function

$$f(x) = c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x), \quad a \leq x \leq b$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants, is called **a general solution** of (51) on $a \leq x \leq b$.

Example 58

$\sin x$ and $\cos x$ are linearly independent solutions of

$$\frac{d^2y}{dx^2} + y = 0 \tag{57}$$

for all x, $-\infty < x < \infty$. So, $\{\sin x, \cos x\}$ is a fundamental set of solutions for the differential equations (57). Thus $c_1 \sin x + c_2 \cos x$ is a general solution for (57). One can verify that $3 \sin x$ and $2 \sin x + \cos x$ are linearly independent solutions of (57). Therefore, $\{3 \sin x, 2 \sin x + \cos x\}$ is another fundamental set of solutions for (57). This implies that $k_1 3 \sin x + k_2(2 \sin x + \cos x)$ is also a general solution for (57). The two general solution expressions represent the same set, that is, if y is element of $c_1 \sin x + c_2 \cos x$ for some c_1, c_2 then it is also element of $k_1 3 \sin x + k_2(2 \sin x + \cos x)$ for some k_1, k_2, and vice versa. That is, expressing the general solution is not unique.

Definition 20

Let $f_1, f_2, \ldots, f_n$ be n real functions each of which has an $(n-1)$st derivative on a real interval $a \leq x \leq b$. The determinant

$$W(f_1, f_2, \ldots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \cdots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

is called **Wronskian** of these n functions.

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = 0 \qquad \text{(cf. 51)}$$

Theorem 13

The n solutions $f_1, f_2, \ldots, f_n$ of the n-th order homogeneous linear differential equation (51) are linearly independent on $a \leq x \leq b$ if and only if the Wronskian of $f_1, f_2, \ldots, f_n$ is different from zero for some x on the interval $a \leq x \leq b$.

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = 0 \qquad \text{(cf. 51)}$$

Theorem 14

The Wronskian of n solutions $f_1, f_2, \ldots, f_n$ of equation (51) is either identically zero on $a \leq x \leq b$ or else is never zero on $a \leq x \leq b$.

Example 59

Let us show that $\sin x$ and $\cos x$ are linearly independent for all real x:

$$W(\sin x, \cos x) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -\sin^2 x - \cos^2 x = -1 \neq 0$$

Example 60

The solutions e^x, e^{-x}, and e^{2x} of

$$\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} - \frac{dy}{dx} + 2y = 0$$

are linearly independent on every real interval:

$$W(e^x, e^{-x}, e^{2x}) = \begin{vmatrix} e^x & e^{-x} & e^{2x} \\ e^x & -e^{-x} & 2e^{2x} \\ e^x & e^{-x} & 4e^{2x} \end{vmatrix} = -6e^{2x} \neq 0$$

for all real x.

The general solution to the d.e. is, therefore,

$$y(x) = c_1 e^x + c_2 e^{-x} + c_3 e^{2x}$$

Example 61

Do the solutions e^x, e^{-x}, and $e^x + e^{-x}$ of

$$\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} - \frac{dy}{dx} + 2y = 0$$

linearly independent on every real interval? Can we write the general solution to the d.e. as

$$y(x) = c_1 e^x + c_2 e^{-x} + c_3(e^x + e^{-x})$$

Example 62

Are the functions $\sin x$ and $|\sin x|$ linearly independent on (a) $0 \le x \le \pi$ (b) $0 \le x \le 2\pi$ (c) $0 \le x \le 4\pi$

Properties of linear differential equations

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = F(x) \qquad \text{(cf. 50)}$$

Theorem 15

Let v be any solution of the given n-th order nonhomogeneous linear differential equation (50). Let u be any solution of the corresponding homogeneous equation. Then $u + v$ is also a solution of the given nonhomogeneous linear differential equation (50).

Example 63

$y(x) = x$ is a solution of the nonhomogeneous differential equation $\frac{d^2 y}{dx^2} + y = x$ and that $y(x) = \sin x$ is a solution of the corresponding homogeneous differential equation $\frac{d^2 y}{dx^2} + y = 0$. By the theorem, the sum $y(x) = x + \sin x$ is also a solution of the nonhomogeneous equation.

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = F(x) \qquad \text{(cf. 50)}$$

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = 0 \qquad \text{(cf. 51)}$$

Theorem 16

Let y_p be a given solution of the n-th order nonhomogeneous linear differential equation (50) involving no arbitrary constants. Let $y_c = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$ be the general solution of the corresponding homogeneous equation (51). Then every solution ϕ of the n-th order nonhomogeneous linear differential equation (50) can be expressed in the form

$$y_c + y_p$$

that is

$$c_1 y_1 + c_2 y_2 + \cdots + c_n y_n + y_p$$

for suitable choice of n arbitrary constants $c_1, c_2, \ldots, c_n$.

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = F(x) \qquad \text{(cf. 50)}$$

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = 0 \qquad \text{(cf. 51)}$$

Definition 21

Consider the n-th order nonhomogeneous linear differential equation (50) and the corresponding homogeneous equation (51). The general solution of (51) is called the **complementary function** of (50). We shall denote this by y_c. Any particular solution of (50) involving no arbitrary constants, denoted by y_p, is called a **particular integral** of (50). The solution $y_c + y_p$ is called the **general solution** of (50).

Example 64

Consider

$$\frac{d^2y}{dx^2} + y = x \qquad \text{(NonHomLin)}$$

This has a particular solution

$$y_p(x) = x$$

Corresponding homogenenous equation

$$\frac{d^2y}{dx^2} + y = 0$$

has a general solution

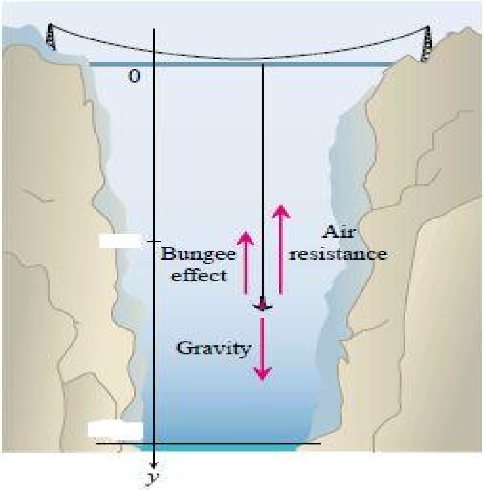
$$y_c(x) = c_1 \sin x + c_2 \cos x$$

Thus a general solution of (NonHomLin) is:

$$y(x) = c_1 \sin x + c_2 \cos x \; + x$$

Example 65

Example 65 (cont.)



Example 65 (cont.)

A man $1.8m$ tall and weighing $80kg$ bungee jumps off a bridge over a river.

The bridge is $200m$ above the water surface and the unstretched bungee cord is $30m$ long.

The spring constant of the bungee cord is $K_s = 11N/m$, meaning that, when the cord is stretched, it resists the stretching with a force of 11 newtons per meter of stretch.

Example 65 (cont.)

When the man jumps off the bridge he goes into free fall until the bungee cord is extended to its full unstretched length.

This occurs when the man's feet are at $30m$ below the bridge.

His initial velocity and position are zero. His acceleration is $9.8m/s^2$ until he reaches 30 m below the bridge.

Example 65 (cont.)

His position is the integral of his velocity and his velocity is the integral of his acceleration.

So, during the initial free-fall time, his velocity is $9.8 \times t$ m/s, where t is time in seconds and his position is $4.9 \times t^2$ m below the bridge.

Solving for the time of full unstretched bungee-cord extension we get $2.47s$. At that time his velocity is 24.25 meters per second, straight down. At this point the analysis changes because the bungee cord starts having an effect.

Example 65 (cont.)

There are two forces on the man:

1. The downward pull of gravity mg where m is the man's mass and g is the acceleration caused by the earth's gravity

2. The upward pull of the bungee cord $K_s(y(t) - 30)$ where $y(t)$ is the vertical position of the man below the bridge as a function of time.

Example 65 (cont.)

Then, using the principle that force equals mass times acceleration and the fact that acceleration is the second derivative of position, we can write

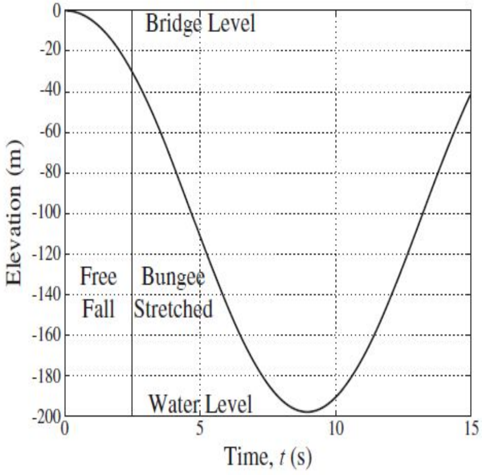
$$mg - K_s(y(t) - 30) = m\ddot{y}(t)$$

or

$$m\ddot{y}(t) + K_s y(t) = mg + 30K_s$$

This is a second-order, linear, constant-coefficient, inhomogeneous, ordinary differential equation. Its total solution is the sum of its homogeneous solution and its particular solution.

Example 65 (cont.)



An order reduction technique

$$x^3 + 6x^2 + 11x + 6 = 0$$

It is hard to solve 3rd degree polynomial equations. For the example above, knowing that one of its roots is at $x = -3$, the others can be found by a simple formula:

$$\frac{x^3 + 6x^2 + 11x + 6}{x + 3} = x^2 + 3x + 2$$

It is simple to deal with a second degree polynomial.

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = 0 \qquad \text{(cf. 51)}$$

Theorem 17

Let f be a nontrivial solution of the n-th order homogeneous linear differential equation given by Equation (51). Then the transformation $y = f(x)v$ reduces Equation (51) to an $(n-1)$st order homogeneous linear differential equation in the dependent variable $w = \frac{dv}{dx}$.

An illustration on the 2nd order differential equation

Suppose f is a known nontrivial solution of the second order homogeneous linear differential equation

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0 \tag{58}$$

Let a solution to the equation above be

$$y = f(x)v \tag{59}$$

where f is the known solution of (58) and v is a function of x that will be determined.

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0 \qquad \text{(cf. 58)}$$

$$y = f(x)v \to \qquad \text{(cf. 59)}$$

$$\frac{dy}{dx} = f(x)\frac{dv}{dx} + f'(x)v \qquad (60)$$

$$\frac{d^2y}{dx^2} = f(x)\frac{d^2v}{dx^2} + 2f'(x)\frac{dv}{dx} + f''(x)v \qquad (61)$$

Substituting (59), (60), and (61) in (58) we obtain

$$a_0(x)[f(x)\frac{d^2v}{dx^2} + 2f'(x)\frac{dv}{dx} + f''(x)v]$$

$$+a_1(x)[f(x)\frac{dv}{dx} + f'(x)v] + a_2(x)f(x)v = 0$$

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0 \qquad \text{(cf. 58)}$$

$$a_0(x)[f(x)\frac{d^2v}{dx^2} + 2f'(x)\frac{dv}{dx} + f''(x)v]$$

$$+a_1(x)[f(x)\frac{dv}{dx} + f'(x)v] + a_2(x)f(x)v = 0$$

$$a_0(x)f(x)\frac{d^2v}{dx^2} + [2a_0(x)f'(x) + a_1(x)f(x)]\frac{dv}{dx}$$
$$+[a_0(x)f''(x) + a_1(x)f'(x) + a_2(x)f(x)]v = 0$$

Since f is a solution of (58), the coefficient of v is zero, and so that the last equation reduces to

$$a_0(x)f(x)\frac{d^2v}{dx^2} + [2a_0(x)f'(x) + a_1(x)f(x)]\frac{dv}{dx} = 0$$

$$a_0(x)f(x)\frac{d^2v}{dx^2} + [2a_0(x)f'(x) + a_1(x)f(x)]\frac{dv}{dx} = 0$$

Letting $w := \frac{dv}{dx}$, this becomes

$$a_0(x)f(x)\frac{dw}{dx} + [2a_0(x)f'(x) + a_1(x)f(x)]w = 0$$

This is a first order homogeneous linear differential equation in the dependent variable w. The equation is separable, thus by the assumptions $f(x) \neq 0$ and $a_0(x) \neq 0$, we may write

$$\frac{dw}{w} = -[2\frac{f'(x)}{f(x)} + \frac{a_1(x)}{a_0(x)}]dx$$

$$\frac{dw}{w} = -[2\frac{f'(x)}{f(x)} + \frac{a_1(x)}{a_0(x)}]dx$$

Integrating we obtain

$$\ln|w| = -\ln[f(x)]^2 - \int \frac{a_1(x)}{a_0(x)}dx + \ln|c|$$

$$\ln\frac{|w|[f(x)]^2}{c} = -\int \frac{a_1(x)}{a_0(x)}dx \rightarrow \frac{|w|[f(x)]^2}{c} = e^{-\int \frac{a_1(x)}{a_0(x)}dx}$$

$$w = \frac{ce^{-\int \frac{a_1(x)}{a_0(x)}dx}}{[f(x)]^2} \rightarrow v = \int \frac{ce^{-\int \frac{a_1(x)}{a_0(x)}dx}}{[f(x)]^2}dx$$

$$\rightarrow y(x) = f(x)\int \frac{ce^{-\int \frac{a_1(x)}{a_0(x)}dx}}{[f(x)]^2}dx$$

It can be shown that the new solution and f are linearly independent. Thus the linear combination $c_1 f + c_2 fv$ is the general solution of (58).

Example 66

$y(x) = x$ is a solution of

$$(x^2+1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0 \tag{62}$$

Find a linearly independent solution by reducing the order.
Let $y = vx$, then $\frac{dy}{dx} = x\frac{dv}{dx} + v$ and $\frac{d^2y}{dx^2} = x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}$. Substitute them in (62):

$$(x^2+1)(x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}) - 2x(x\frac{dv}{dx} + v) + 2xv = 0$$

or

$$x(x^2+1)\frac{d^2v}{dx^2} + 2\frac{dv}{dx} = 0$$

Example 66 (cont.)

$$x(x^2+1)\frac{d^2v}{dx^2}+2\frac{dv}{dx}=0$$

Letting $w := \frac{dv}{dx}$ we obtain

$$x(x^2+1)\frac{dw}{dx}+2w=0$$

$$\frac{dw}{w}=\frac{-2}{x(x^2+1)}dx$$

$$\frac{dw}{w}=(\frac{-2}{x}+\frac{2x}{x^2+1})dx$$

$$\ln|w|=-2\ln|x|+\ln(x^2+1)+\ln|c|\rightarrow$$

$$\ln|w|=-\ln x^2+\ln(x^2+1)+\ln|c|\rightarrow$$

$$\ln|w|=\ln\frac{c(x^2+1)}{x^2}$$

$$\ln|w| = \ln\frac{c(x^2+1)}{x^2}$$

Example 66 (cont.)

$$w = \frac{c(x^2+1)}{x^2}$$

Use $\frac{dv}{dx} = w$:

$$v(x) = c\left[x - \frac{1}{x}\right]$$

$$y(x) = cx\left[x - \frac{1}{x}\right] = c(x^2 - 1)$$

Example 67

The 2nd order de

$$t^2\frac{d^2y}{dt^2} + 2t\frac{dy}{dt} - 2y = 0 \tag{63}$$

has a known solution $y_1(t) = t$. Find the other linearly independent solution by the reduction of order method.

$$y_2 = tv,\ \dot{y}_2 = v + t\dot{v},\ \ddot{y}_2 = 2\dot{v} + t\ddot{v}$$

Plug $y_2, \dot{y}_2$ and $\ddot{y}_2$ into the differential equation to obtain

$$t^2(2\dot{v} + t\ddot{v}) + 2t(\cancel{v} + t\dot{v}) - \cancel{2(tv)} = 0$$

$$t^3\ddot{v} + 4t^2\dot{v} = 0$$

$$t^3\ddot{v} + 4t^2\dot{v} = 0$$

Example 67 (cont.)

Define $w := \dot{v}$ to obtain

$$t^3\dot{w} + 4t^2 w = 0$$

This is a 1st order linear differential equation. Its solution is

$$w(t) = ct^{-4}$$

where c is an arbitrary constant. Obtain v from w :

$$v = \int w dt = \int ct^{-4} dt = -\frac{1}{3}ct^{-3} + k$$

$$v = -\frac{1}{3}ct^{-3} + k$$

$$v = -\frac{1}{3}ct^{-3} + k$$

Example 67 (cont.)

$$y_2(t) = tv = t(-\frac{1}{3}ct^{-3} + k) = -\frac{1}{3}ct^{-2} + kt$$

General solution to the d.e. (63) is

$$y(t) = C_1 t + C_2 \left(-\frac{1}{3}ct^{-2} + kt\right)$$

$$y(t) = (C_1 + C_2 k)t - \frac{1}{3}C_2 ct^{-2}$$

$$y(t) = K_1 t + K_2 t^{-2}$$

where K_1 and K_2 are arbitrary constants.

Theorem 18

Let x_1 and x_2 respectively be the solutions of

$$\alpha \frac{d^2x}{dt^2} + \beta \frac{dx}{dt} + \gamma x = f_1(t) \tag{64}$$

and

$$\alpha \frac{d^2x}{dt^2} + \beta \frac{dx}{dt} + \gamma x = f_2(t) \tag{65}$$

where α, β and γ are functions of t. Then $x_1 + x_2$ is a solution of

$$\alpha \frac{d^2x}{dt^2} + \beta \frac{dx}{dt} + \gamma x = f_1(t) + f_2(t) \tag{66}$$

Proof

$$\alpha \frac{d^2(x_1 + x_2)}{dt^2} + \beta \frac{d(x_1 + x_2)}{dt} + \gamma(x_1 + x_2) = \cdots\cdots$$

x_1 solves

$$\alpha \frac{d^2 x}{dt^2} + \beta \frac{dx}{dt} + \gamma x = f_1(t)$$

x_2 solves

$$\alpha \frac{d^2 x}{dt^2} + \beta \frac{dx}{dt} + \gamma x = f_2(t)$$

$$\alpha \frac{d^2(x_1 + x_2)}{dt^2} + \beta \frac{d(x_1 + x_2)}{dt} + \gamma(x_1 + x_2)$$

$$= \alpha \frac{d^2 x_1}{dt^2} + \alpha \frac{d^2 x_2}{dt^2} + \beta \frac{dx_1}{dt} + \beta \frac{dx_2}{dt} + \gamma x_1 + \gamma x_2$$

$$= \underbrace{\alpha \frac{d^2 x_1}{dt^2} + \beta \frac{dx_1}{dt} + \gamma x_1}_{f_1(t)} + \underbrace{\alpha \frac{d^2 x_2}{dt^2} + \beta \frac{dx_2}{dt} + \gamma x_2}_{f_2(t)} = f_1(t) + f_2(t)$$

Indeed, knowing solutions corresponding to f_1 and f_2 we get the solution corresponding to the forcing function $f_1 + f_2$.

Superposition principle

Theorem 19

Let f_1 be a solution of

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = F_1(x)$$

Let f_2 be a solution of

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = F_2(x)$$

Then $k_1 f_1 + k_2 f_2$ is a solution of

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = k_1 F_1(x) + k_2 F_2(x)$$

where k_1 and k_2 are arbitrary constants.

Example 68

Find a particular solution of

$$\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = 4e^{2x} + 5e^{4x} \tag{67}$$

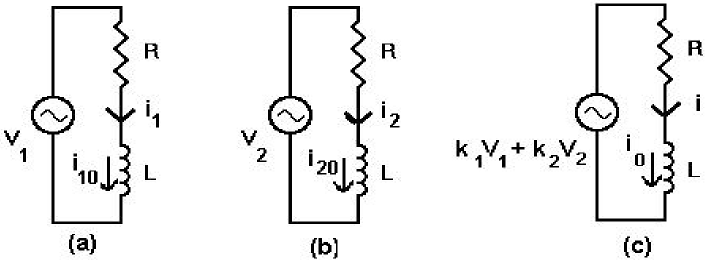
Noting that

$$\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = 4e^{2x}$$

has a particular solution $2e^{2x}$, and

$$\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = 5e^{4x}$$

has a particular solution $5xe^{4x}$, superposition principle implies that the d.e. (67) has a particular solution $y_p(x) = 2e^{2x} + 5xe^{4x}$.



Let $i_{10} = i_{20} = 0$. And let v_1 result in the current i_1, and v_2 result in the current i_2. Then for the input $k_1 v_1 + k_2 v_2$ with $i_0 = 0$, the current i will be $k_1 i_1 + k_2 i_2$.

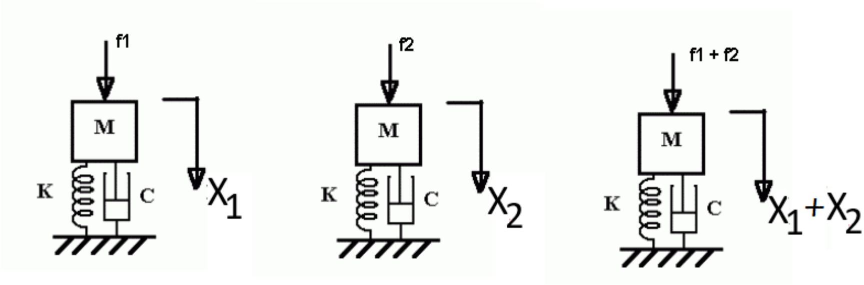


Figure: Mass-damper-spring system with inputs

Homogeneous linear differential equations with constant coefficients

Preliminaries
Quadratic formula If

$$ax^2 + bx + c = 0 \tag{68}$$

then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{69}$$

Cubic formula If

$$x^3 + px^2 + qx + r = 0 \tag{70}$$

then use the transformation

$$x = u - \frac{p}{3} \tag{71}$$

to obtain

$$u^3 + (q - \frac{p^2}{3})u + (r - \frac{pq}{3} + \frac{2p^3}{27}) = 0$$

$$u^3 + au + b = 0 \tag{72}$$

where $a \triangleq q - \frac{p^2}{3}$ and $b \triangleq r - \frac{pq}{3} + \frac{2p^3}{27}$. For the solution of (72) evaluate

$$A = \sqrt[3]{-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}$$

$$B = \sqrt[3]{-\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}$$

$$A = \sqrt[3]{-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}$$

$$B = \sqrt[3]{-\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}$$

$$u^3 + au + b = 0 \qquad \text{(cf. 72)}$$

The roots of (72) are:

$u = A + B$

$u = -\frac{1}{2}(A + B) + \sqrt{-\frac{3}{4}}(A - B)$

$u = -\frac{1}{2}(A + B) - \sqrt{-\frac{3}{4}}(A - B)$

Consider

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = 0 \qquad (73)$$

where $a_0, a_1, \ldots, a_n$ are real constants. Consider the solution candidate:

$$y = e^{mx}$$

Then we have:

$$\frac{dy}{dx} = me^{mx}, \quad \frac{d^2 y}{dx^2} = m^2 e^{mx}, \quad \ldots, \frac{d^n y}{dx^n} = m^n e^{mx}$$

Substitute in (73):

$$a_0 m^n e^{mx} + a_1 m^{n-1} e^{mx} + \cdots + a_{n-1} m e^{mx} + a_n e^{mx} = 0$$

or

$$e^{mx}(a_0 m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_n) = 0$$

Since $e^{mx} \neq 0$, for the satisfaction of the equation we must have

$$a_0 m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_n = 0 \tag{74}$$

This equation is called **auxiliary equation** or the **characteristic equation** of the given differential equations (73).

Coefficients of the auxiliary equation

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = 0 \qquad \text{(cf. 73)}$$

$$a_0 m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_n = 0 \qquad \text{(cf. 74)}$$

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = 0 \qquad \text{(cf. 73)}$$

$$a_0 m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_n = 0 \qquad \text{(cf. 74)}$$

Theorem 20

Consider the n-th order homogeneous linear differential equation (73) with constant coefficients. If the auxiliary equation (74) has the n real-distinct roots $m_1, m_2, \ldots, m_n$ *then the general solution of (73) is*

$$y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$$

where $c_1, c_2, \ldots, c_n$ *are arbitrary constants.*

Example 69

Consider

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0.$$

The auxiliary equation is

$$m^2 - 3m + 2 = 0$$

Hence $m_1 = 1$ and $m_2 = 2$. The roots are real and distinct. Thus e^x and e^{2x} are solutions. The general solution is then

$$y(x) = c_1 e^x + c_2 e^{2x}$$

where c_1, c_2 are arbitrary constants.

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = 0 \qquad \text{(cf. 73)}$$

$$a_0 m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_n = 0 \qquad \text{(cf. 74)}$$

Theorem 21

Consider the n-th order homogeneous linear differential equation (73) with constant coefficients. If the auxiliary equation (74) has the real root m occurring k times, then the part of the general solution of (73) corresponding to this k-fold repeated root is

$$(c_1 + c_2 x + c_3 x^2 + \cdots + c_k x^{k-1}) e^{mx}$$

where $c_1, c_2, \ldots, c_k$ are arbitrary constants.

Example 70

Find the general solution of

$$\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 18y = 0.$$

The auxiliary equation

$$m^3 - 4m^2 - 3m + 18 = 0$$

has the roots $3, 3, -2$. The general solution is then

$$y(x) = (c_1 + c_2x)e^{3x} + c_3e^{-2x}$$

where c_1, c_2, c_3 are arbitrary constants.

Example 71

Let a constant coefficient homogeneous linear differential in the independent variable x have the characteristic equation

$$(m-4)^3(m-2)^2(m-5)=0$$

The general solution is

$$(c_1+c_2x+c_3x^2)e^{4x}+(c_4+c_5x)e^{2x}+c_6e^{5x}$$

where $c_1, c_2, \ldots, c_6$ are arbitrary constants.

Theorem 22

Consider the n-th order homogeneous linear differential equation (73) with constant coefficients. If the auxiliary equation (74) has the conjugate complex roots $a + bi$ and $a - bi$, neither repeated, then the corresponding part of the general solution of (73) may be written as

$$y(x) = e^{ax}(c_1 \sin(bx) + c_2 \cos(bx))$$

where c_1, c_2 are arbitrary constants.

If, however, $a + bi$ and $a - bi$ are each k-fold roots of the auxiliary equation (74) then the corresponding part of the general solution of (73) may be written as

$$y(x) = e^{ax}[(c_1 + c_2 x + \cdots + c_k x^{k-1}) \sin(bx)$$

$$+(c_{k+1} + c_{k+2} x + \cdots + c_{2k} x^{k-1}) \cos(bx)]$$

where $c_1, c_2, \ldots, c_{2k}$ are arbitrary constants.

Example 72

$$\frac{d^2y}{dx^2} + y = 0 \rightarrow m^2 + 1 = 0 \rightarrow m = 0 \pm i$$

$$y(x) = e^{0x}[c_1 \sin(1 \cdot x) + c_2 \cos(1 \cdot x)] = [c_1 \sin x + c_2 \cos x]$$

where c_1, c_2 are arbitrary constants.

Example 73

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = 0 \rightarrow m = 3 \pm 4i$$
$$y(x) = e^{3x}[c_1 \sin(4x) + c_2 \cos(4x)]$$

where c_1, c_2 are arbitrary constants.

Example 74

Let a constant coefficient homogeneous linear differential in the independent variable x have the characteristic equation

$$(m - 4 - i3)^3 (m - 4 + i3)^3 (m - 5) = 0$$

The general solution is

$$e^{4x}[(c_1 + c_2 x + c_3 x^2) \sin 3x + (c_4 + c_5 x + c_6 x^2) \cos 3x] + c_7 e^{5x}$$

where $c_1, c_2, \ldots, c_7$ are arbitrary constants.

Example 75

Solve the initial value problem

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = 0 \quad y(0) = -3, \; y'(0) = -1$$

Its general solution is

$$y(x) = e^{3x}[c_1 \sin(4x) + c_2 \cos(4x)]$$

where c_1, c_2 are arbitrary constants. From this we find:

$$\frac{dy(x)}{dx} = e^{3x}[(3c_1 - 4c_2)\sin 4x + (4c_1 + 3c_2)\cos 4x]$$

Example 75 (cont.)

$$y(x) = e^{3x}[c_1 \sin(4x) + c_2 \cos(4x)]$$

$$\frac{dy(x)}{dx} = e^{3x}[(3c_1 - 4c_2)\sin 4x + (4c_1 + 3c_2)\cos 4x]$$

Apply the initial conditions:

$$-3 = e^{3\cdot 0}[c_1 \sin(4\cdot 0) + c_2 \cos(4\cdot 0)] \to c_2 = -3$$

$$-1 = e^{3\cdot 0}[(3c_1 - 4c_2)\sin(4\cdot 0) + (4c_1 + 3c_2)\cos(4\cdot 0)]$$

$$\to 4c_1 + 3c_2 = -1 \to c_1 = 2$$

The solution is

$$y(x) = e^{3x}[2\sin(4x) - 3\cos(4x)]$$

Example 76

Consider

$$\ddot{y} - 4\dot{y} + 13y = 0$$

Its auxiliary polynomial equation is $m^2 - 4m + 13 = 0$; its roots are $2 \pm i3$. Its two roots are distinct. So, can we write the solution as

$$y(t) = c_1 e^{(2+3i)t} + c_2 e^{(2-3i)t}$$

The above expression satisfies the d.e. However, it is not a real function; it has both real and imaginary components. This expression can be freed from the imaginary components, so that it becomes a solution which is a linear combination of real functions only.

Example 76 (cont.)

$$y(t) = c_1 e^{(2+3i)t} + c_2 e^{(2-3i)t}$$

Use Euler's identity $e^{it} = \cos t + i \sin t$ in the above equation.

$$y(t) = c_1 e^{2t}[\cos 3t + i \sin 3t] + c_2 e^{2t}[\cos 3t - i \sin 3t]$$

$$y(t) = e^{2t}[(c_1 + c_2)\cos 3t + i(c_1 - c_2)\sin 3t]$$

This expression satisfies the d.e. for every c_1 and c_2; so take $c_1 = \frac{1}{2}$ and $c_2 = \frac{1}{2}$. This gives a solution:

$$y_1 = e^{2t}\cos 3t$$

Taking $c_1 = \frac{1}{2i}$ and $c_2 = -\frac{1}{2i}$ gives another solution

$$y_2 = e^{2t}\sin 3t$$

Example 76 (cont.)

$$y(t) = e^{2t}[(c_1 + c_2)\cos 3t + i(c_1 - c_2)\sin 3t]$$

Take $c_1 = \frac{1}{2}$ and $c_2 = \frac{1}{2}$. This gives a solution:

$$y_1(t) = e^{2t}\cos 3t$$

Taking $c_1 = \frac{1}{2i}$ and $c_2 = -\frac{1}{2i}$ gives another solution:

$$y_2(t) = e^{2t}\sin 3t$$

Because y_1 and y_2 are linearly independent, general solution can be written as

$$y_g(t) = c_1 y_1 + c_2 y_2 = c_1 e^{2t}\cos 3t + c_2 e^{2t}\sin 3t$$

$$y_g(t) = e^{2t}[c_1\cos 3t + c_2\sin 3t]$$

Every term in the solution above is now real.

Undetermined coefficients method

Consider

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 2e^{4x}$$

A solution candidate for this system is $y_p(x) = Ae^{4x}$. Hope that for some value of A, this candidate satisfies the differential equation. Substitute the candidate and its derivatives

$$\rightarrow y_p'(x) = 4Ae^{4x}, \quad y_p''(x) = 16Ae^{4x}$$

in the differential equation:

$$16Ae^{4x} - 2(4Ae^{4x}) - 3(Ae^{4x}) = 2e^{4x}$$

Simplification yields: $A = \frac{2}{5} \rightarrow y_p(x) = \frac{2}{5}e^{4x}$.

Now consider

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 2e^{3x}$$

Let this time the particular solution be $y_p(x) = Ae^{3x}$. Substitute this and its derivatives in the differential equation:

$$9Ae^{3x} - 2(3Ae^{3x}) - 3(Ae^{3x}) = 2e^{3x}$$

This results in:

$$0 = 2e^{3x}$$

This equality does not hold. Therefore, this candidate does not work for any A. The reason that $y_p(x) = Ae^{3x}$ does not work is that e^{3x} is also the solution of the homogeneous part. Now try: $y_p(x) = Axe^{3x}$. Substitute this and its derivatives in the differential equation to find that $A = \frac{1}{2}$. Thus $y_p(x) = \frac{1}{2}xe^{3x}$ is the solution.

Definition 22

UC functions are x^n, where n is a nonnegative integer, e^{ax}, $\sin(bx + c)$, $\cos(bx + c)$ and finite product of these four types.

Example 77

$$x^3,\ e^{3x},\ \sin(2x),\ e^x \sin(2x + \frac{\pi}{2}),\ e^x x^3 \cos(4x)$$

Definition 23

Given a UC function f, its **UC set** is standardized set of linearly independent functions whose linear combinations are f and its all derivatives.

For some UC functions, corresponding UC sets are shown in Table 1.

Example 78

The UC function $f(x) = x^5$ has the derivatives $x^5, 5x^4, 20x^3, 60x^2, 120x, 120, 0$. The UC set of x^5 is $S = \{x^5, x^4, x^3, x^2, x, 1\}$. Notice that, constant multiples or linear combinations of the linearly independent functions $x^5, x^4, x^3, x^2, x, 1$ yield f and its all successive derivatives.

Example 79

Given $f(x) = \sin 2x$, we use the UC set from the table as $\{\sin 2x, \cos 2x\}$. Note that, derivatives of $f(x)$ are
$f'(x) = 2\cos 2x, f''(x) = -4\sin 2x,$
$f'''(x) = -8\cos 2x, f^{(4)}(x) = 16\sin 2x, \ldots$
which are multiples of either $\sin 2x$ or $\cos 2x$.

Example 80

Given $f(x) = e^{ax}$, we use the UC set from the table as $S = \{e^{ax}\}$.
Note that, derivatives of $f(x)$ are:
$\dot{f}(x) = ae^{ax}, \ddot{f}(x) = a^2 e^{ax}, \dots, f^{(n)}(x) = a^n e^{ax}$.
These are all multiples of e^{ax}.

Example 81

Let $f(x) = x^3$ and $g(x) = \cos 2x$, then
$h(x) = f(x)g(x) = x^3 \cos 2x$. UC set of x^3 is $S_1 = \{x^3, x^2, x, 1\}$,
UC set of $\cos 2x$ is $S_2 = \{\cos 2x, \sin 2x\}$.
Then we have UC set of $x^3 \cos 2x$ as
$S = \{x^3 \cos 2x, x^3 \sin 2x, x^2 \cos 2x, x^2 \sin 2x, x \cos 2x, x \sin 2x,$
$\cos 2x, \sin 2x\}$.

For some UC functions, the UC sets are presented in Table 1.

UC function	UC set
x^n	$\{x^n, x^{n-1}, x^{n-2}, \ldots, x, 1\}$
e^{ax}	$\{e^{ax}\}$
$\sin(bx+c)$	$\{\sin(bx+c), \cos(bx+c)\}$
$\cos(bx+c)$	$\{\sin(bx+c), \cos(bx+c)\}$
$x^n e^{ax}$	$\{x^n e^{ax}, x^{n-1} e^{ax}, \ldots, xe^{ax}, e^{ax}\}$
$x^n \sin(bx+c)$	$\{x^n \sin(bx+c), x^n \cos(bx+c), x^{n-1} \sin(bx+c),$ $x^{n-1} \cos(bx+c), \ldots, x\sin(bx+c), x\cos(bx+c)$ $\sin(bx+c), \cos(bx+c)\}$
$x^n \cos(bx+c)$	$\{x^n \sin(bx+c), x^n \cos(bx+c), x^{n-1} \sin(bx+c),$ $x^{n-1} \cos(bx+c), \ldots, x\sin(bx+c), x\cos(bx+c)$ $\sin(bx+c), \cos(bx+c)\}$
$e^{ax}\sin(bx+c)$	$\{e^{ax}\sin(bx+c), e^{ax}\cos(bx+c)\}$
$e^{ax}\cos(bx+c)$	$\{e^{ax}\sin(bx+c), e^{ax}\cos(bx+c)\}$

Table: 1 Some UC functions and their UC sets

Problem statement

We want to find a particular solution of

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = F(x)$$

where F is a finite linear combination of UC functions $u_1, u_2, \ldots, u_m$:

$$F = k_1 u_1 + k_2 u_2 + \cdots + k_m u_m$$

Undetermined coefficients method

$$a_0\frac{d^n y}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_{n-1}\frac{dy}{dx}+a_n y=k_1u_1+k_2u_2+\cdots+k_mu_m$$

1. Obtain UC sets $S_1, S_2, \ldots, S_m$ for the UC functions $u_1, u_2, \ldots, u_m$ as in Table 1.
2. If $S_i \subseteq S_j$ for some $i, j \in \{1, 2, \ldots, m\}$, then omit S_i from further consideration. This step is not applicable for the problems with only one UC set.
3. Consider the UC sets remaining after step 2. If any element of any set S_i is a solution for the homogeneous part, then multiply S_i by the lowest integer power of x so that the resulting set S_i' does not contain solution of homogeneous part anymore. If any set is revised, then omit its original form from further consideration.
4. Multiply elements of the sets by undetermined coefficients and add them up. This sum is a valid particular solution candidate. Substitute the candidate in the differential equation and solve it for the undetermined coefficients.

Example 82

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = x^2e^x$$

Let us find the general solution of the homogeneous part. Homogeneous part of the d.e. is as follows:

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$$

Its characteristic equation is $m^2 - 3m + 2 = 0$. This has the roots 1 and 2, therefore, the general solution is:

$$y_c(x) = c_1e^x + c_2e^{2x}$$

Step 1
UC set of x^2e^x is $S = \{x^2e^x, xe^x, e^x\}$.
Step 2
Since we have only one UC set, this step is no applicable.

Example 82 (cont.)

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = x^2e^x$$

$$y_c(x) = c_1e^x + c_2e^{2x}$$

UC set of x^2e^x is $S = \{x^2e^x, xe^x, e^x\}$.

Step 3

e^x is a member of y_c, therefore we multiply S by x.

$$S' = \{x^3e^x, x^2e^x, xe^x\}$$

Multiplication by x^2, or x^3 also result in a set that does not contain a solution of homogeneous part. But the algorithm says "Multiply it by the lowest integer power of x"

Step 4

A particular solution candidate is:

$$y_p(x) = Ax^3e^x + Bx^2e^x + Cxe^x$$

Example 82 (cont.)

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = x^2e^x$$

Step 4

A particular solution candidate is:

$$y_p(x) = Ax^3e^x + Bx^2e^x + Cxe^x$$

$$\dot{y}_p(x) = Ax^3e^x + (3A + B)x^2e^x + (2B + C)xe^x + Ce^x$$

$$\ddot{y}_p(x) = Ax^3e^x + (6A + B)x^2e^x + (6A + 4B + C)xe^x + (2B + 2C)e^x$$

Substitute $y_p, \dot{y}_p, \ddot{y}_p$ in the d.e.:

$$Ax^3e^x + (6A + B)x^2e^x + (6A + 4B + C)xe^x + (2B + 2C)e^x$$

$$-3(Ax^3e^x + (3A + B)x^2e^x + (2B + C)xe^x + Ce^x)$$

$$+2(Ax^3e^x + Bx^2e^x + Cxe^x) = x^2e^x$$

Example 82 (cont.)

Substitute $y_p, \dot{y}_p, \ddot{y}_p$ in the d.e.:

$$Ax^3e^x + (6A+B)x^2e^x + (6A+4B+C)xe^x + (2B+2C)e^x$$

$$-3(Ax^3e^x + (3A+B)x^2e^x + (2B+C)xe^x + Ce^x)$$

$$+2(Ax^3e^x + Bx^2e^x + Cxe^x) = x^2e^x$$

Equate coefficients of x^3e^x:

$$A - 3A + 2A = 0 \to 0 = 0$$

Equate coefficients of x^2e^x:

$$(6A+B) - 3(3A+B) + 2B = 1 \to -3A = 1 \to A = -\frac{1}{3}$$

Equate coefficients of xe^x:

$$(6A+4B+C) - 3(2B+C) + 2C = 0 \to -2B = 2 \to B = -1$$

Example 82 (cont.)

$$Ax^3e^x + (6A+B)x^2e^x + (6A+4B+C)xe^x + (2B+2C)e^x$$
$$-3(Ax^3e^x + (3A+B)x^2e^x + (2B+C)xe^x + Ce^x)$$
$$+2(Ax^3e^x + Bx^2e^x + Cxe^x) = x^2e^x$$

Equate coefficients of e^x:

$$2B + 2C - 3C = 0 \to C = -2$$

Thus, the particular solution is:

$$y_p(x) = Ax^3e^x + Bx^2e^x + Cxe^x\Big|_{A=-1/3, B=-1, C=-2}$$

$$\to y_p(x) = -\frac{1}{3}x^3e^x - x^2e^x - 2xe^x$$

Therefore, the general solution is

$$y(x) = y_p(x) + y_c(x) = -\frac{1}{3}x^3e^x - x^2e^x - 2xe^x + c_1e^x + c_2e^{2x}$$

Example 83

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x^2e^x$$

$$y_c(x) = c_1e^x + c_2xe^x$$

Step 1 UC set of x^2e^x is $S = \{x^2e^x, xe^x, e^x\}$.
Step 2 Because we have only one UC set, this step is not applicable to this problem.
Step 3 e^x is a member of y_c, however, if we multiply S by x the resulting set will contain xe^x which is also member of y_c. Hence, we multiply the set by x^2.

$$S' = \{x^4e^x, x^3e^x, x^2e^x\}$$

Step 4 A particular solution candidate is:

$$y_p(x) = Ax^4e^x + Bx^3e^x + Cx^2e^x$$

Example 84

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 2e^x - 10\sin x$$
$$y_c(x) = c_1 e^{3x} + c_2 e^{-x}$$

Step 1 UC sets: $S_1 = \{e^x\}$, $S_2 = \{\sin x, \cos x\}$
Step 2 Note that neither of these sets is identical with nor included in the other, hence both are retained.
Step 3 None of the functions e^x, $\sin x$, $\cos x$ in either of these sets is a solution of the corresponding homogeneous equation. Hence neither sets needs to be revised.
Step 4 Form the linear combination:

$$y_p(x) = Ae^x + B\sin x + C\cos x$$

Substitute this and its derivatives in the differential equation to obtain $A = -\frac{1}{2}$, $B = 2$, and $C = -1$.

Example 85

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 2x^2 + e^x + 2xe^x + 4e^{3x}$$

$$y_c(x) = c_1 e^{2x} + c_2 e^x$$

Step 1
UC sets: $S_1 = \{x^2, x, 1\}$, $S_2 = \{e^x\}$, $S_3 = \{xe^x, e^x\}$,$S_4 = \{e^{3x}\}$
Step 2 $S_2 \subset S_3 \to$ Delete the set S_2.
Now we have the sets S_1, S_3 and S_4 remaining.
Step 3 e^x of S_3 is a member of y_c. Multiply S_3 by x:

$$S_3' = \{x^2e^x, xe^x\}$$

Now we have S_1, S_3' and S_4 to consider.

Example 86

Step 1

UC sets: $S_1 = \{x^2, x, 1\}$, $S_2 = \{e^x\}$, $S_3 = \{xe^x, e^x\}$, $S_4 = \{e^{3x}\}$

Step 2 $S_2 \subset S_3 \rightarrow$ Delete the set S_2.

Now we have the sets S_1, S_3 and S_4 remaining.

Step 3 e^x of S_3 is a member of y_c. Multiply S_3 by x:

$$S_3' = \{x^2e^x, xe^x\}$$

Now we have S_1, S_3' and S_4 to consider.

Step 4

Form the linear combination by using the members of S_1, S_3', and S_4:

$$y_p(x) = Ax^2 + Bx + C + Dx^2e^x + Exe^x + Fe^{3x}$$

Substitute this and its derivatives in the differential equation to obtain

$$y_p(x) = x^2 + 3x + \frac{7}{2} - x^2e^x - 3xe^x + 2e^{3x}$$

Example 87

$$\frac{d^4y}{dx^4} + \frac{d^2y}{dx^2} = 3x^2 + 4\sin x - 2\cos x$$

$$y_c(x) = c_1 + c_2 x + c_3 \sin x + c_4 \cos x$$

Step 1 UC sets: $S_1 = \{x^2, x, 1\}$, $S_2 = \{\sin x, \cos x\}$, $S_3 = \{\sin x, \cos x\}$
Step 2 S_2 and S_3 are identical; delete the set S_3.
Step 3 Multiply S_1 by x^2. The revised set is $S_1' = \{x^4, x^3, x^2\}$.
Multiply S_2 by x. The revised set is $S_2' = \{x\sin x, x\cos x\}$
Step 4 Form the linear combination by using the members of S_1' and S_2':

$$y_p(x) = Ax^4 + Bx^3 + Cx^2 + Dx\sin x + Ex\cos x$$

Substitute this and its derivatives in the differential equation to obtain

$$y_p(x) = \frac{1}{4}x^4 - 3x^2 + x\sin x + 2x\cos x$$

The underlying idea behind the undetermined coefficients

Proving validity of the UC algorithm is involved, so we just highlight the main idea behind it. Consider

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_2 \frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = f(x) \quad (75)$$

The lefthand side may be viewed as a linear operator operating on the dependent variable:

$$\left[a_n \frac{d^n}{dx^n} + a_{n-1} \frac{d^{n-1}}{dx^{n-1}} + \cdots + a_2 \frac{d^2}{dx^2} + a_1 \frac{d}{dx} + a_0 \right] y = f(x)$$

If y is a polynomial of degree n, the operation by a linear operator results in a polynomial of degree n or less. So, linear operator preserves the polynomial type; polynomial stays as a polynomial. Besides, it does not increase the degree.

$$L(\cdot) \triangleq \left[a_n \frac{d^n}{dx^n} + a_{n-1} \frac{d^{n-1}}{dx^{n-1}} + \cdots + a_2 \frac{d^2}{dx^2} + a_1 \frac{d}{dx} + a_0\right](\cdot)$$

The linear operator above
(i) multiplies the argument by a real constant,
(ii) differentiates the argument k times, $k = 1, 2, \ldots, n$,
(iii) takes a linear combination of the terms generated in steps (i) and (ii).

All of the properties above preserve the polynomials, that is, if the argument is a polynomial then what linear operation yields is also a polynomial. Besides, the degree is not increased.

Regarding

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_2 \frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = f(x) \text{ (cf. 75)}$$

if the righthand side term is a polynomial of n-th degree, it is viable to use a polynomial solution candidate of degree n in the simplest case where complementary function y_c of (75) has no polynomial term in it. If it has a polynomial term in y_c, then it requires increase in solution candidates degree.
Linear operators preserve not only polynomials but also the other UC functions. Therefore, the idea used for the polynomial righthand sides can be extended for the other UC functions.

Variation of parameters method

Consider

$$\ddot{y}(x) + P(x)\dot{y}(x) + Q(x)y(x) = f(x) \tag{76}$$

We want to find a particular solution in cases where undetermined coefficients method cannot be applied to produce y_p.
Suppose

$$y_c = c_1 y_1 + c_2 y_2$$

is a known general solution to

$$\ddot{y}(x) + P(x)\dot{y}(x) + Q(x)y(x) = 0. \tag{77}$$

Then it is possible to find a y_p of the form

$$y_p = Ay_1 + By_2$$

where A and B are some functions of x to be determined (at the present moment they are unknowns).

We need to substitute this form of y_p in (76) and try to find A and B. To do this, we need to find $\dot{y}_p$ and $\ddot{y}_p$.

$$y_p = Ay_1 + By_2 \rightarrow \dot{y}_p = A\dot{y}_1 + \dot{A}y_1 + B\dot{y}_2 + \dot{B}y_2$$

To avoid dealing with second derivatives of A and B we will look for A and B satisfying the following condition:

$$\dot{A}y_1 + \dot{B}y_2 = 0 \tag{78}$$

Now we need to find a solution that satisfies both (76) and (78). We shall see that imposing an additional condition would not cause any additional trouble in finding a solution.

$$\rightarrow \dot{y}_p = A\dot{y}_1 + B\dot{y}_2$$

Thus

$$\ddot{y}_p = A\ddot{y}_1 + \dot{A}\dot{y}_1 + B\ddot{y}_2 + \dot{B}\dot{y}_2$$

We substitute them in (76):

$$\underbrace{A\ddot{y}_1} + \dot{A}\dot{y}_1 + \overbrace{B\ddot{y}_2} + \dot{B}\dot{y}_2 + \underbrace{PA\dot{y}_1}$$

$$+ \overbrace{PB\dot{y}_2} + \underbrace{QAy_1} + \overbrace{QBy_2} = f \tag{79}$$

Recall that each of y_1 and y_2 is a solution to the d.e.'s homogeneous part:

$$\ddot{y}(x) + P(x)\dot{y}(x) + Q(x)y(x) = 0. \qquad \text{(cf. 77)}$$

Thus, the sum of the underbraced terms $A(\ddot{y}_1 + P\dot{y}_1 + Qy_1)$ equals zero. The sum of the overbraced terms above $B(\ddot{y}_2 + P\dot{y}_2 + Qy_2)$ also equals zero. Thus (79) becomes

$$\dot{A}\dot{y}_1 + \dot{B}\dot{y}_2 = f \tag{80}$$

To find A and B we need to solve (78) and (80):

$$\dot{A}y_1 + \dot{B}y_2 = 0$$

$$\dot{A}\dot{y}_1 + \dot{B}\dot{y}_2 = f$$

In matrix notation

$$\begin{bmatrix} y_1 & y_2 \\ \dot{y}_1 & \dot{y}_2 \end{bmatrix} \begin{bmatrix} \dot{A} \\ \dot{B} \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}$$

In matrix notation

$$\begin{bmatrix} y_1 & y_2 \\ \dot{y}_1 & \dot{y}_2 \end{bmatrix} \begin{bmatrix} \dot{A} \\ \dot{B} \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}$$

Cramer's rule may be used:

$$\dot{A} = \frac{\begin{vmatrix} 0 & y_2 \\ f & \dot{y}_2 \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ \dot{y}_1 & \dot{y}_2 \end{vmatrix}} = \frac{-y_2 f}{W(y_1, y_2)} \to A = \int \frac{-y_2 f}{W(y_1, y_2)} dx$$

$$\dot{B} = \frac{\begin{vmatrix} y_1 & 0 \\ \dot{y}_1 & f \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ \dot{y}_1 & \dot{y}_2 \end{vmatrix}} = \frac{y_1 f}{W(y_1, y_2)} \to B = \int \frac{y_1 f}{W(y_1, y_2)} dx$$

Since the particular solution has the form

$$y_p = A(x)y_1 + B(x)y_2$$

we have

$$y_p = -y_1 \int \frac{y_2 f}{W(y_1, y_2)} dx + y_2 \int \frac{y_1 f}{W(y_1, y_2)} dx$$

Example 88

Determine the general solution for

$$\frac{d^2y}{dx^2} + y = \tan x$$

$$y_c(x) = c_1 \cos x + c_2 \sin x \to y_p(x) = A(x)\cos x + B(x)\sin x$$

$$\dot{A}(x) = \frac{\begin{vmatrix} 0 & \sin x \\ \tan x & \cos x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \cos x - \sec x$$

$$\to A(x) = \sin x - \ln|\sec x + \tan x| + c_3$$

$$\dot{B}(x) = \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & \tan x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \sin x \to B(x) = -\cos x + c_4$$

Example 88 (cont.)

$$y_p(x) = A(x)\cos x + B(x)\sin x$$

$\to y_p(x) = \cos x(\sin x - \ln|\sec x + \tan x| + c_3) + \sin x(-\cos x + c_4)$

Particular solution, by definition, is free of arbitrary constants. So take $c_3 = 0$ and $c_4 = 0$:

$$y_p(x) = \cos x(\sin x - \ln|\sec x + \tan x|) + \sin x(-\cos x)$$

Thus the general solution to the differential equation is

$$y(x) = c_1 \sin x + c_2 \cos x + \cos x(\sin x - \ln|\sec x + \tan x|) + \sin x(-\cos x)$$

Note that without having $c_3 = 0$ and $c_4 = 0$ we would have

$$y(x) = c_1 \sin x + c_2 \cos x + \cos x(\sin x - \ln|\sec x + \tan x|) + \sin x(-\cos x) + \underbrace{c_3 \cos x + c_4 \sin x}_{\textit{redundancies}}$$

Example 89

Consider the differential equation

$$\ddot{y} - 2\dot{y} - 3y = xe^{-x}$$

One may solve it by undetermined coefficients method. We solve it by the variation of parameters method. The homogeneous part has the general solution:

$$y_c(x) = c_1 e^{-x} + c_2 e^{3x}$$

The particular solution will have the form:

$$y_p = A(x)y_1 + B(x)y_2$$

or more explicitly

$$y_p = -y_1 \int \frac{y_2 f}{W(y_1, y_2)} dx + y_2 \int \frac{y_1 f}{W(y_1, y_2)} dx$$

Example 89 (cont.)

$$y_c(x) = c_1 \underbrace{e^{-x}}_{y_1} + c_2 \underbrace{e^{3x}}_{y_2}$$

$$y_p = y_1 \cdot \underbrace{-\int \frac{y_2 f}{W(y_1, y_2)} dx}_{A(x)} + y_2 \cdot \underbrace{\int \frac{y_1 f}{W(y_1, y_2)} dx}_{B(x)}$$

$$A(x) = -\int \frac{y_2 f}{W(y_1, y_2)} dx = -\int \frac{e^{3x} x e^{-x}}{\begin{vmatrix} e^{-x} & e^{3x} \\ -e^{-x} & 3e^{3x} \end{vmatrix}} dx$$

$$= -\int \frac{xe^{2x}}{4e^{2x}} dx = -\int \frac{x}{4} dx = -\frac{x^2}{8}$$

Example 89 (cont.)

$$y_c(x) = c_1 \underbrace{e^{-x}}_{y_1} + c_2 \underbrace{e^{3x}}_{y_2}$$

$$y_p = y_1 \cdot \underbrace{-\int \frac{y_2 f}{W(y_1, y_2)} dx}_{A(x)} + y_2 \cdot \underbrace{\int \frac{y_1 f}{W(y_1, y_2)} dx}_{B(x)}$$

$$B(x) = \int \frac{y_1 f}{W(y_1, y_2)} dx = \int \frac{e^{-x} x e^{-x}}{4 e^{2x}} dx = \int \frac{x e^{-4x}}{4} dx$$

$$= -\frac{x}{16} e^{-4x} - \frac{1}{64} e^{-4x}$$

Example 89 (cont.)

$$y_c(x) = c_1 \underbrace{e^{-x}}_{y_1} + c_2 \underbrace{e^{3x}}_{y_2}$$

$$y_p = y_1 A(x) + y_2 B(x)$$

$$A(x) = -\frac{x^2}{8}$$

$$B(x) = -\frac{x}{16} e^{-4x} - \frac{1}{64} e^{-4x}$$

Thus

$$y_p(x) = -\frac{x^2}{8} e^{-x} + e^{3x}\left(-\frac{x}{16} e^{-4x} - \frac{1}{64} e^{-4x}\right)$$

$$y_p(x) = -\frac{x^2}{8} e^{-x} + e^{-x}\left(-\frac{x}{16} - \frac{1}{64}\right)$$

General Solution:

$$y(x) = y_c(x) + y_p(x) = c_1 e^{-x} + c_2 e^{3x} - \frac{x^2}{8} e^{-x} + e^{-x}\left(-\frac{x}{16} - \frac{1}{64}\right)$$

First order case

Consider

$$\dot{y} + P(x)y = f(x) \tag{81}$$

Suppose y_1 is a nonzero solution to

$$\dot{y} + P(x)y = 0 \tag{82}$$

Look for $y_p = A(x)y_1$. Substitute in (81) yields:

$$\underbrace{A\dot{y}_1} + \dot{A}y_1 + \underbrace{PAy_1} = f$$

Since y_1 is a solution to (82) the sum of the underbraced terms, i.e., $A(\dot{y}_1 + Py_1)$ equals zero, so

$$\dot{A}y_1 = f \to \dot{A} = \frac{f}{y_1} \to A = \int \frac{f}{y_1} dx \to y_p = y_1 \int \frac{f}{y_1} dx$$

Example 90

Third and higher order cases Consider

$$\frac{d^3y}{dt^3} + \frac{dy}{dt} = \sec t \tag{83}$$

Its complementary solution is

$$y_c(t) = c_1 + c_2 \sin t + c_3 \cos t$$

Particular solution has the form

$$y_p(t) = A(t) + B(t) \sin t + C(t) \cos t$$

Substituting this in the d.e. (83) and imposing appropriate conditions on $A(t), B(t)$, and $C(t)$, one obtains the particular solution y_p.
Particular solution form is generalized for higher order differential equations in a straightforward manner.

The Cauchy-Euler equation

Definition 24

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} x \frac{dy}{dx} + a_n y = F(x)$$

with $a_0, \ldots, a_n$ constants, is called the n-th order Cauchy-Euler equation.

Theorem 23

The transformation $x = e^t$ reduces the equation

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} x \frac{dy}{dx} + a_n y = F(x)$$

to a linear differential equation with constant coefficients.

We shall show it for the second order differential equation

$$a_0 x^2 \frac{d^2y}{dx^2} + a_1 x \frac{dy}{dx} + a_2 y = F(x)$$

Letting $x = e^t$ assuming $x > 0$, we have $t = \ln x$. Then

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \cdot \frac{1}{x} \rightarrow x \frac{dy}{dx} = \frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{1}{x} \frac{d}{dx}(\frac{dy}{dt}) + \frac{dy}{dt} \frac{d}{dx} \frac{1}{x} = \frac{1}{x}(\frac{d^2y}{dt^2} \frac{dt}{dx}) - \frac{1}{x^2} \frac{dy}{dt}$$

$$= \frac{1}{x^2}(\frac{d^2y}{dt^2} - \frac{dy}{dt}) \rightarrow x^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$$

Note that

$$\frac{d}{dx}(u) = \frac{du}{dt} \frac{dt}{dx} \rightarrow \frac{d}{dx}(\frac{dy}{dt}) = \frac{d^2y}{dt^2} \frac{dt}{dx}$$

$$a_0 x^2 \frac{d^2y}{dx^2} + a_1 x \frac{dy}{dx} + a_2 y = F(x)$$

Substituting in the differential equation

$$a_0(\frac{d^2y}{dt^2} - \frac{dy}{dt}) + a_1 \frac{dy}{dt} + a_2 y = F(e^t)$$

or

$$a_0 \frac{d^2y}{dt^2} + (a_1 - a_0)\frac{dy}{dt} + a_2 y = F(e^t)$$

Compare to:

$$a_0 x^2 \frac{d^2y}{dx^2} + a_1 x \frac{dy}{dx} + a_2 y = F(x)$$

Remark

1. The leading coefficient $a_0 x^n = 0$ for $x = 0$, therefore, $x = 0$ is not included in the domain. We take the domain as $x > 0$.
2. If the domain is $x < 0$, then the correct transformation is $x = -e^t$.

Example 91

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = x^3$$

Let $x = e^t$, assume $x > 0$. Noting that $a_0 = 1,\ a_1 = -2,\ a_2 = 2$, we obtain

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = e^{3t}$$

The general solution will be

$$y(t) = c_1 e^t + c_2 e^{2t} + \frac{1}{2}e^{3t}$$

In terms of the original independent variable x:

$$y(x) = c_1 x + c_2 x^2 + \frac{1}{2}x^3$$

Power series solutions

Consider a second order homogeneous linear differential equation

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0 \tag{84}$$

or in a normalized form

$$\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0 \tag{85}$$

where $P_1(x) = \frac{a_1(x)}{a_0(x)}$ and $P_2(x) = \frac{a_2(x)}{a_0(x)}$. Assume that Equation (84) does not have a solution expressible as a finite linear combination of known elementary functions. Assume that it has a solution in the form of infinite series:

$$c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x - x_0)^n \tag{86}$$

where $c_0, c_1, \ldots$ are constants. (86) is known as power series in $(x - x_0)$.

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0 \qquad \text{(cf. 84)}$$

$$\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0 \qquad \text{(cf. 85)}$$

Definition 25

A function f is said to be analytic at x_0 if its Taylor series about x_0

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

exists and converges to $f(x)$ for all x in some interval including x_0.

Definition 26

The point x_0 is called **an ordinary point** of the differential equation (84) if both of the functions P_1 and P_2 in the equivalent normalized equation (85) are analytic at x_0. If either (or both) of the functions is not analytic at x_0, then x_0 is called **a singular point** of the differential equation (84).

Example 92

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2+2)y = 0$$

Here $P_1(x) = x$ and $P_2(x) = x^2 + 2$. Both functions are analytic everywhere. Thus all the points are ordinary points.

Example 93

$$(x-1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \frac{1}{x}y = 0$$

or equivalently,

$$\frac{d^2y}{dx^2} + \frac{x}{(x-1)}\frac{dy}{dx} + \frac{1}{x(x-1)}y = 0$$

Here $P_1(x) = \frac{x}{(x-1)}$ and $P_2(x) = \frac{1}{x(x-1)}$. P_1 is analytic everywhere except at $x = 1$. P_2 is analytic everywhere except at $x = 0$ and $x = 1$. Thus $x = 0$ and $x = 1$ are singular points of the differential equation.

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0 \qquad \text{(cf. 84)}$$

Theorem 24

Hypothesis:

The point x_0 is an ordinary point of the differential equation (84).

Conclusion:

The differential equation (84) has two nontrivial linearly independent power series solutions of the form

$$\sum_{n=0}^{\infty} c_n(x - x_0)^n$$

and these power series converge in some interval $|x - x_0| < R$ (where $R > 0$) about x_0.

The method of solution

Assume that the solution y is

$$y = c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x - x_0)^n$$

Then

$$\frac{dy}{dx} = c_1 + 2c_2(x - x_0) + 3c_3(x - x_0)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x - x_0)^{n-1}$$

$$\frac{d^2y}{dx^2} = 2c_2 + 6c_3(x - x_0) + 12c_4(x - x_0)^2 + \cdots = \sum_{n=2}^{\infty} n(n-1)c_n(x - x_0)^{n-2}$$

We substitute y and its derivatives in the differential equation. We then simplify the resulting equation

$$K_0 + K_1(x - x_0) + K_2(x - x_0)^2 + \cdots = 0 \tag{87}$$

In order that this equation be valid for all x in the interval of convergence $|x - x_0| < R$, we must set

$$K_0 = K_1 = K_2 = \cdots = 0$$

Example 94

Consider

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2+2)y = 0$$

We want to find power series solution of this equation about $x_0 = 0$. Solution has the form: $y = \sum_{n=0}^{\infty} c_n (x - x_0)^n$
Equivalently, $y = \sum_{n=0}^{\infty} c_n x^n$. This implies:

$$\frac{dy}{dx} = \sum_{n=1}^{\infty} n c_n x^{n-1}, \quad \frac{d^2y}{dx^2} = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

Substituting in the differential equation we obtain

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + x\sum_{n=1}^{\infty} n c_n x^{n-1} + x^2 \sum_{n=0}^{\infty} c_n x^n + 2\sum_{n=0}^{\infty} c_n x^n = 0$$

Example 94 (cont.)

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + x \sum_{n=1}^{\infty} n c_n x^{n-1} + x^2 \sum_{n=0}^{\infty} c_n x^n + 2 \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} n c_n x^{n} + \sum_{n=0}^{\infty} c_n x^{n+2} + 2 \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{1} + \underbrace{\sum_{n=1}^{\infty} n c_n x^{n}}_{2} + \underbrace{\sum_{n=0}^{\infty} c_n x^{n+2}}_{3} + \underbrace{2 \sum_{n=0}^{\infty} c_n x^n}_{4} = 0 \tag{88}$$

Consider the first term and use $n = m + 2$ transformation

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} = \sum_{m=0}^{\infty} (m+2)(m+1)c_{m+2} x^m = \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n$$

Example 94 (cont.)

Consider the third term and use $n = m - 2$ transformation

$$\sum_{n=0}^{\infty} c_n x^{n+2} = \sum_{m=2}^{\infty} c_{m-2} x^m = \sum_{n=2}^{\infty} c_{n-2} x^n$$

Now Equation (88) becomes

$$\underbrace{\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n}_{1} + \underbrace{\sum_{n=1}^{\infty} n c_n x^n}_{2} + \underbrace{\sum_{n=2}^{\infty} c_{n-2} x^n}_{3} + 2\underbrace{\sum_{n=0}^{\infty} c_n x^n}_{4} = 0 \tag{89}$$

Example 94 (cont.)

Now Equation (88) becomes

$$\underbrace{\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n}_{1} + \underbrace{\sum_{n=1}^{\infty} nc_n x^n}_{2} + \underbrace{\sum_{n=2}^{\infty} c_{n-2}x^n}_{3} + \underbrace{2\sum_{n=0}^{\infty} c_n x^n}_{4} = 0 \tag{90}$$

Obtain useful appearances of the terms:

$$\text{1st term: } 2c_2 + 6c_3 x + \sum_{n=2}^{\infty}(n+2)(n+1)c_{n+2}x^n$$

$$\text{2nd term: } c_1 x + \sum_{n=2}^{\infty} nc_n x^n$$

$$\text{4th term: } 2c_0 + 2c_1 x + 2\sum_{n=2}^{\infty} c_n x^n$$

Example 94 (cont.)

Now Equation (90) becomes

$$2c_2 + 6c_3 x + \sum_{n=2}^{\infty}(n+2)(n+1)c_{n+2}x^n + c_1 x + \sum_{n=2}^{\infty} n c_n x^n$$

$$+ \sum_{n=2}^{\infty} c_{n-2}x^n + 2c_0 + 2c_1 x + 2\sum_{n=2}^{\infty} c_n x^n = 0$$

$$\to (2c_0 + 2c_2) + (3c_1 + 6c_3)x$$

$$+ \sum_{n=2}^{\infty}[(n+2)(n+1)c_{n+2} + (n+2)c_n + c_{n-2}]x^n = 0$$

Example 94 (cont.)

$$(2c_0 + 2c_2) + (3c_1 + 6c_3)x$$

$$+\sum_{n=2}^{\infty}[(n+2)(n+1)c_{n+2} + (n+2)c_n + c_{n-2}]x^n = 0$$

Recall the solution method for this step:

$$K_0 + K_1(x - x_0) + K_2(x - x_0)^2 + \cdots = 0 \qquad \text{(cf. 87)}$$

Equating every power of x to zero we have:

$$c_2 = -c_0$$

$$c_3 = -\frac{1}{2}c_1$$

$$(n+2)(n+1)c_{n+2} + (n+2)c_n + c_{n-2} = 0$$

$$c_{n+2} = -\frac{(n+2)c_n + c_{n-2}}{(n+1)(n+2)}, \quad n \geq 2$$

$$c_2 = -c_0, \quad c_3 = -\frac{1}{2}c_1$$

Example 94 (cont.)

$$c_{n+2} = -\frac{(n+2)c_n + c_{n-2}}{(n+1)(n+2)}, \quad n \geq 2$$

Hence

$$c_4 = -\frac{4c_2 + c_0}{12} = \frac{1}{4}c_0$$

$$c_5 = -\frac{5c_3 + c_1}{20} = \frac{3}{40}c_1$$

The general solution is:

$$y = c_0 + c_1 x - c_0 x^2 - \frac{1}{2}c_1 x^3 + \frac{1}{4}c_0 x^4 + \frac{3}{40}c_1 x^5 + \dots$$

$$y = c_0(1 - x^2 + \frac{1}{4}x^4 + \cdots) + c_1(x - \frac{1}{2}x^3 + \frac{3}{40}x^5 + \cdots)$$

Example 95

Use power series method to solve the ivp

$$(x+1)\frac{d^2y}{dx^2} - (2-x)\frac{dy}{dx} + y = 0, \quad y(0) = 2, \ \dot{y}(0) = -1 \qquad (91)$$

$$(x+1)\left(\sum_{2}^{\infty} c_n n(n-1)x^{n-2}\right) - (2-x)\left(\sum_{1}^{\infty} c_n n x^{n-1}\right) + \sum_{0}^{\infty} c_n x^n = 0$$

$$\left(\sum_{2}^{\infty} c_n n(n-1)x^{n-1} + \sum_{2}^{\infty} c_n n(n-1)x^{n-2}\right) + \dots$$

$$\left(-2\sum_{1}^{\infty} c_n n x^{n-1} + \sum_{1}^{\infty} c_n n x^n\right) + \sum_{0}^{\infty} c_n x^n = 0$$

$$\left(\sum_{1}^{\infty} c_{k+1}(k+1)k x^k + \sum_{0}^{\infty} c_{k+2}(k+2)(k+1)x^k\right) + \dots$$

$$\left(-2\sum_{0}^{\infty} c_{k+1}(k+1)x^k + \sum_{1}^{\infty} c_k k x^k\right) + \sum_{0}^{\infty} c_k x^k = 0$$

$$\left(\sum_{1}^{\infty} c_{k+1}(k+1)kx^k + \sum_{0}^{\infty} c_{k+2}(k+2)(k+1)x^k\right) + \ldots$$

$$\left(-2\sum_{0}^{\infty} c_{k+1}(k+1)x^k + \sum_{1}^{\infty} c_k kx^k\right) + \sum_{0}^{\infty} c_k x^k = 0$$

Example 95 (cont.)

$$2c_2 - 2c_1 + c_0 +$$

$$\sum_{1}^{\infty} \left[(k+2)(k+1)c_{k+2} + (k(k+1) - 2(k+1))c_{k+1} + (1+k)c_k\right] x^k = 0$$

$$2c_2 - 2c_1 + c_0 + \sum_{1}^{\infty} \left[(k^2+3k+2)c_{k+2} + (k^2 - k - 2)c_{k+1} + (1+k)c_k\right] x^k = 0$$

$$2c_2 - 2c_1 + c_0 = 0$$

$$(k^2+3k+2)c_{k+2} + (k^2 - k - 2)c_{k+1} + (1+k)c_k = 0, \ k = 1, 2, \ldots$$

$$2c_2 - 2c_1 + c_0 = 0$$

$$(k^2 + 3k + 2)c_{k+2} + (k^2 - k - 2)c_{k+1} + (1 + k)c_k = 0, \ k = 1, 2, \dots$$

Example 95 (cont.)

$$c_2 = -\frac{1}{2}c_0 + c_1$$

for k=1

$$6c_3 - 2c_2 + 2c_1 = 0, \ \mathbf{c_3} = -\frac{1}{3}c_1 + \frac{1}{3}c_2 = -\frac{1}{3}c_1 + \frac{1}{3}(-\frac{1}{2}c_0 + c_1) = -\frac{\mathbf{1}}{\mathbf{6}}\mathbf{c_0}$$

for k=2

$$12c_4 + 3c_2 = 0, \ \mathbf{c_4} = -\frac{1}{4}c_2 = -\frac{1}{4}(-\frac{1}{2}c_0 + c_1) = \frac{\mathbf{1}}{\mathbf{8}}\mathbf{c_0} - \frac{\mathbf{1}}{\mathbf{4}}\mathbf{c_1}$$

$$c_2 = -\frac{1}{2}c_0 + c_1, \; c_3 = -\frac{1}{6}c_0, \; c_4 = \frac{1}{8}c_0 - \frac{1}{4}c_1$$

Example 95 (cont.)

Form of the solution is:

$$y(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots$$

Using the c_i's we have

$$y(x) = c_0 + c_1 x + (-\frac{1}{2}c_0 + c_1)x^2 + (-\frac{1}{6}c_0)x^3 + (\frac{1}{8}c_0 - \frac{1}{4}c_1)x^4 + \cdots$$

$$y(x) = c_0(1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{8}x^4 + \cdots) + c_1(x + x^2 - \frac{1}{4}x^4 + \cdots)$$

Example 95 (cont.)

$$y(x) = c_0(1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{8}x^4 + \cdots) + c_1(x + x^2 - \frac{1}{4}x^4 + \cdots)$$

Use the initial conditions: $y(0) = 2, \ \dot{y}(0) = -1$

$c_0(1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{8}x^4 + \cdots)_{x=0} + c_1(x + x^2 - \frac{1}{4}x^4 + \cdots)_{x=0} = 2 \rightarrow c_0 = 2$

$\frac{d}{dx}\left[c_0(1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{8}x^4 + \cdots) + c_1(x + x^2 - \frac{1}{4}x^4 + \cdots)\right]_{x=0} = -1 \rightarrow c_1 = -1$

Using $c_0 = 2$ and $c_1 = -1$ in the general solution, we obtain

$$y(x) = 2 - x - 2x^2 - \frac{1}{3}x^3 + \frac{1}{2}x^4 + \cdots$$

Solutions about singular points

Consider a second order homogeneous linear differential equation

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0 \qquad \text{(cf. 84)}$$

and assume that x_0 is a singular point of (84). We are not assured of a power series solution in positive powers of $x - x_0$. However, under certain conditions we may assume the solution of the form

$$y = |x - x_0|^r \sum_{n=0}^{\infty} c_n (x - x_0)^n \qquad (92)$$

where r is a certain (real or complex) constant.

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0 \qquad \text{(cf. 84)}$$

Let us classify the singular points. For this, normalize (84):

$$\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0 \qquad \text{(cf. 85)}$$

where $P_1(x) = \frac{a_1(x)}{a_0(x)}$ and $P_2(x) = \frac{a_2(x)}{a_0(x)}$.

Definition 27

Consider the d.e. (84) and assume at least one of the functions P_1 and P_2 in the equivalent normalized equation (85) is not analytic at x_0, so that x_0 is a singular point of (84). If the functions defined by the products

$$(x - x_0)P_1(x) \text{ and } (x - x_0)^2 P_2(x)$$

are both analytic at x_0, then x_0 is called **regular singular point** of (84). Otherwise we call it **irregular**.

If the functions defined by the products

$$(x - x_0)P_1(x) \text{ and } (x - x_0)^2 P_2(x)$$

are both analytic at x_0, then x_0 is called **regular singular point** of (84). Otherwise we call it **irregular**.

Example 96

$$2x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + (x - 5)y = 0$$

Normalized form:

$$\frac{d^2y}{dx^2} - \frac{1}{2x}\frac{dy}{dx} + \frac{x - 5}{2x^2}y = 0$$

Here $P_1(x) = -\frac{1}{2x}$ and $P_2(x) = \frac{x-5}{2x^2}$. Clearly $x_0 = 0$ is a singular point of the d.e.
The products $xP_1(x) = -\frac{1}{2}$ and $x^2P_2(x) = \frac{x-5}{2}$ are analytic at $x = 0$, so $x = 0$ is a regular singular point of the d.e.

Test: $(x - x_0)P_1(x)$ and $(x - x_0)^2 P_2(x)$

Example 97

$$x^2(x-2)^2\frac{d^2y}{dx^2} + 2(x-2)\frac{dy}{dx} + (x+1)y = 0$$

Normalized form:

$$\frac{d^2y}{dx^2} + \frac{2}{x^2(x-2)}\frac{dy}{dx} + \frac{x+1}{x^2(x-2)^2}y = 0$$

Here $P_1(x) = \frac{2}{x^2(x-2)}$ and $P_2(x) = \frac{x+1}{x^2(x-2)^2}$ have the singular points at $x = 0$ and $x = 2$.
At $x = 0$, $xP_1(x) = \frac{2}{x(x-2)}$ and $x^2P_2(x) = \frac{x+1}{(x-2)^2}$ we see that $xP_1(x)$ is not analytic at $x = 0$, so $x = 0$ is an irregular singular point of the d.e.
At $x = 2$, both $(x-2)P_1(x) = \frac{2}{x^2}$ and $(x-2)^2P_2(x) = \frac{x+1}{x^2}$ are analytic, so $x = 2$ is a regular singular point of the d.e.

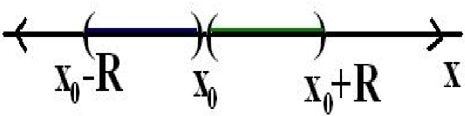
$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0 \qquad \text{(cf. 84)}$$

Theorem 25

Given that x_0 is a regular singular point of the d.e. (84), the d.e. (84) has at least one nontrivial solution of the form

$$y = |x - x_0|^r \sum_{n=0}^{\infty} c_n (x - x_0)^n \qquad \text{(cf. 92)}$$

where r is a definite (real or complex) constant which may be determined, and this solution is valid in some deleted interval $0 < |x - x_0| < R$ about x_0. For the interval $0 < x - x_0 < R$ solution becomes $y = (x - x_0)^r \sum_{n=0}^{\infty} c_n (x - x_0)^n$



Example 98

We saw in a previous example that $x = 0$ is a regular singular point of the d.e.

$$2x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + (x-5)y = 0$$

By the theorem, this equation has a nontrivial solution in the form

$$|x|^r \sum_{n=0}^{\infty} c_n x^n$$

valid in some deleted interval $0 < |x| < R$ about $x = 0$.

The Method of Frobenius

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0 \qquad \text{(cf. 84)}$$

---

1. Let x_0 be a regular singular point of the d.e. (84). We seek a solution of the form
$y = (x - x_0)^r \sum_{n=0}^{\infty} c_n(x - x_0)^n = \sum_{n=0}^{\infty} c_n(x - x_0)^{n+r}$ valid for $0 < x - x_0 < R$. Note that for $0 < x - x_0 < R$ the term $|x - x_0|^r$ becomes $(x - x_0)^r$. When $-R < x - x_0 < 0$ the following procedure may be repeated by replacing $x - x_0$ by $-(x - x_0)$.
2. Term by term differentiation:

$$y = \sum_{n=0}^{\infty} c_n(x - x_0)^{n+r} \rightarrow \frac{dy}{dx} = \sum_{n=0}^{\infty}(n + r)c_n(x - x_0)^{n+r-1}$$

$$\frac{d^2y}{dx^2} = \sum_{n=0}^{\infty}(n + r)(n + r - 1)c_n(x - x_0)^{n+r-2}$$

We substitute $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ in (84).

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0 \qquad \text{(cf. 84)}$$

---

3. Substitution results in an expression of the form

$$K_0(x - x_0)^{r+k} + K_1(x - x_0)^{r+k+1} + K_2(x - x_0)^{r+k+2} + \cdots = 0$$

4. For a solution we must set

$$K_0 = K_1 = K_2 = \cdots = 0$$

5. Equating K_0 to zero we obtain a quadratic expression in r, called indicial equation of the d.e. (84). The roots of this quadratic expression is often called the exponents of the d.e. (84). Denote the solutions r_1 and r_2 where $Re(r_1) \geq Re(r_2)$.
6. Now equate the remaining coefficients to zero. This leads to a set of conditions involving r.

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0 \qquad \text{(cf. 84)}$$

$$y = |x - x_0|^r \sum_{n=0}^{\infty} c_n(x - x_0)^n \qquad \text{(cf. 92)}$$

---

7. We substitute r_1 for r in the conditions of step 6, and choose c_n satisfying the conditions. If c_n are so chosen, the resulting series (92) with $r = r_1$ is a solution.
8. If $r_1 \neq r_2$, we may repeat the procedure of Step 7 using the root r_2. In this way we may obtain a linearly independent solution of the d.e. (84). When r_1 and r_2 are real and unequal, the second solution may or may not be linearly independent from the one obtained in Step 7. Also, when r_1 and r_2 are real and equal we do not get a new solution. These are exceptional cases and treated later.

Example 99

Solve

$$2x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + (x-5)y = 0$$

in some interval $0 < x < R$. We saw previously that $x = 0$ is a regular singular point of the d.e. We, therefore, assume the solution form

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}$$

where $c_0 \neq 0$. Then

$$\frac{dy}{dx} = \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1}$$

$$\frac{d^2y}{dx^2} = \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2}$$

$$2x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + (x-5)y = 0$$

Example 99 (cont.)

Substitute $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ in the differential equation:

$$2\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r} - \sum_{n=0}^{\infty}(n+r)c_n x^{n+r}$$

$$+\sum_{n=0}^{\infty} c_n x^{n+r+1} - 5\sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

Let us simplify this:

$$\sum_{n=0}^{\infty}[2(n+r)(n+r-1)-(n+r)-5]c_n x^{n+r} + \sum_{n=1}^{\infty} c_{n-1}x^{n+r} = 0$$

Example 99 (cont.)

$$\sum_{n=0}^{\infty}[2(n+r)(n+r-1)-(n+r)-5]c_n x^{n+r}+\sum_{n=1}^{\infty}c_{n-1}x^{n+r}=0$$

or

$$[2r(r-1)-r-5]c_0x^r$$

$$+\sum_{n=1}^{\infty}\{[2(n+r)(n+r-1)-(n+r)-5]c_n+c_{n-1}\}x^{n+r}=0$$

The lowest power of x has the factor (indicial equation)

$$2r(r-1)-r-5=0.$$

Equating this to zero yields $r_1=\frac{5}{2}$ and $r_2=-1$. These are the exponents of the the d.e. Notice that these numbers are real and unequal.

Example 99 (cont.)

The coefficients of the higher power x's are equated to zero. This gives a recurrence formula:

$$[2(n+r)(n+r-1)-(n+r)-5]c_n + c_{n-1} = 0, \quad n \geq 1$$

Letting $r = r_1 = \frac{5}{2}$ yields:

$$[2(n+\frac{5}{2})(n+\frac{3}{2})-(n+\frac{5}{2})-5]c_n + c_{n-1} = 0, \quad n \geq 1$$

This simplifies to:

$$n(2n+7)c_n + c_{n-1} = 0, \quad n \geq 1$$

or

$$c_n = -\frac{c_{n-1}}{n(2n+7)}, \quad n \geq 1$$

$$c_n = -\frac{c_{n-1}}{n(2n+7)}, \quad n \geq 1$$

Example 99 (cont.)

$$c_1 = -\frac{c_0}{9}, \quad c_2 = -\frac{c_1}{22} = \frac{c_0}{198}, \quad c_3 = -\frac{c_2}{39} = -\frac{c_0}{7722}, \dots$$

So the solution corresponding to $r = \frac{5}{2}$ is

$$\begin{aligned} y &= c_0 \left(x^{\frac{5}{2}} - \tfrac{1}{9} x^{\frac{7}{2}} + \tfrac{1}{198} x^{\frac{9}{2}} - \tfrac{1}{7722} x^{\frac{11}{2}} + \cdots \right) \\ &= c_0 x^{\frac{5}{2}} \left(1 - \tfrac{1}{9} x + \tfrac{1}{198} x^2 - \tfrac{1}{7722} x^3 + \cdots \right) \end{aligned}$$

Recall that the form of the solution is:

$$y = \sum_{n=0}^{\infty} c_n x^{n+r} = c_0 x^r + c_1 x^{1+r} + c_2 x^{2+r} + c_3 x^{3+r} + \cdots$$

Example 99 (cont.)

Now let $r = -1$ and obtain the corresponding recurrence formula

$$[2(n-1)(n-2) - (n-1) - 5]c_n + c_{n-1} = 0, \quad n \geq 1$$

This simplifies to:

$$n(2n-7)c_n + c_{n-1} = 0, \quad n \geq 1$$

or

$$c_n = -\frac{c_{n-1}}{n(2n-7)}, \quad n \geq 1$$

This yields:

$$c_1 = \frac{1}{5}c_0, \quad c_2 = \frac{1}{6}c_1 = \frac{1}{30}c_0, \quad c_3 = \frac{1}{3}c_2 = \frac{1}{90}c_0, \ldots$$

$$c_1 = \frac{1}{5}c_0, \quad c_2 = \frac{1}{6}c_1 = \frac{1}{30}c_0, \quad c_3 = \frac{1}{3}c_2 = \frac{1}{90}c_0, \ldots$$

Example 99 (cont.)

The solution

$$y = \sum_{n=0}^{\infty} c_n x^{n+r} = c_0 x^r + c_1 x^{1+r} + c_2 x^{2+r} + c_3 x^{3+r} + \cdots$$

corresponding to $r = -1$ is

$$\begin{aligned} y &= c_0 (x^{-1} + \tfrac{1}{5} + \tfrac{1}{30}x + \tfrac{1}{90}x^2 + \cdots) \\ &= c_0 x^{-1}(1 + \tfrac{1}{5}x + \tfrac{1}{30}x^2 + \tfrac{1}{90}x^3 + \cdots) \end{aligned}$$

The two solutions, corresponding to $r_1 = \frac{5}{2}$ and $r_2 = -1$, are linearly independent. Thus the general solution could be written as

$$y = C_1 x^{\frac{5}{2}}(1 - \frac{1}{9}x + \frac{1}{198}x^2 - \frac{1}{7722}x^3 + \cdots) + C_2 x^{-1}(1 + \frac{1}{5}x + \frac{1}{30}x^2 + \frac{1}{90}x^3 + \cdots)$$

It is claimed in the beginning of this section that when r_1 and r_2 are real and unequal we may or may not find a second linearly independent solution in the form of (92).

$$y = |x - x_0|^r \sum_{n=0}^{\infty} c_n (x - x_0)^n \qquad \text{(cf. 92)}$$

The following theorem states an existence condition for the linearly independent solutions.

$$y = |x - x_0|^r \sum_{n=0}^{\infty} c_n (x - x_0)^n \qquad \text{(cf. 92)}$$

Theorem 26

Let the point x_0 be a regular singular point of the d.e. (84). Let r_1 and r_2 [where $Re(r_1) \geq Re(r_2)$] be the roots of the indicial equation associated with x_0. We can conclude that:

***1.** Suppose $r_1 - r_2 \neq N$, where N is a nonnegative integer (that is, $r_1 - r_2 \neq 0, 1, 2, \ldots$). Then the d.e. (84) has two nontrivial linearly indep. solutions y_1 and y_2 of the form (92) given respectively by*

$$y_1 = |x - x_0|^{r_1} \sum_{n=0}^{\infty} c_n (x - x_0)^n \text{ and } y_2 = |x - x_0|^{r_2} \sum_{n=0}^{\infty} d_n (x - x_0)^n$$

where $c_0 \neq 0$, $d_0 \neq 0$.

Theorem 26 (cont.)

2. *Suppose $r_1 - r_2 = N$, where N is a positive integer. Then the d.e. (84) has two nontrivial linearly independent solutions y_1 and y_2 given respectively by*

$$y_1 = |x - x_0|^{r_1} \sum_{n=0}^{\infty} c_n (x - x_0)^n$$

where $c_0 \neq 0$, and

$$y_2 = |x - x_0|^{r_2} \sum_{n=0}^{\infty} d_n (x - x_0)^n + C y_1(x) \ln |x - x_0|$$

where $d_0 \neq 0$, and C is a constant which may or may not be zero.

Theorem 26 (cont.)

3. *Suppose* $r_1 - r_2 = 0$. *Then the d.e. (84) has two nontrivial linearly independent solutions* y_1 *and* y_2 *given respectively by*

$$y_1 = |x - x_0|^{r_1} \sum_{n=0}^{\infty} c_n (x - x_0)^n$$

where $c_0 \neq 0$, *and*

$$y_2 = |x - x_0|^{r_1 + 1} \sum_{n=0}^{\infty} d_n (x - x_0)^n + y_1(x) \ln |x - x_0|$$

where $d_0 \neq 0$.

Differential operators

The general linear system of two first order differential equations in two unknown functions x and y is of the form

$$\left.\begin{array}{l} a_1(t)\frac{dx}{dt} + a_2(t)\frac{dy}{dt} + a_3(t)x + a_4(t)y = F_1(t) \\ b_1(t)\frac{dx}{dt} + b_2(t)\frac{dy}{dt} + b_3(t)x + b_4(t)y = F_2(t) \end{array}\right\} \tag{93}$$

Solution of the system is an ordered pair of real functions (f, g) such that $x = f(t)$ and $y = g(t)$ simultaneously satisfy both equations in some interval $a \leq t \leq b$.

Definitions

$$Dx \triangleq \frac{dx}{dt}$$

$$D^n x \triangleq \frac{d^n x}{dt^n}$$

$$(2D+5)x = 2\frac{dx}{dt} + 5x$$

When $x = t^3 + \sin t$, this becomes

$$(2D+5)(t^3 + \sin t) = 2\frac{d(t^3 + \sin t)}{dt} + 5(t^3 + \sin t)$$

$$= 2(3t^2 + \cos t) + 5(t^3 + \sin t)$$

$$= 6t^2 + 2\cos t + 5t^3 + 5\sin t$$

A linear combination of x and its first n derivatives

$$a_0 \frac{d^n x}{dt^n} + a_1 \frac{d^{n-1} x}{dt^{n-1}} + \cdots + a_{n-1} \frac{dx}{dt} + a_n x$$

can be written in operators notation as

$$\underbrace{\left(a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n\right)}_{\text{Linear operator with constant coefficients}} \quad x$$

The operator $a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n$ is denoted by L, i.e.,

$$L \triangleq a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n$$

Assume that f_1 and f_2 are both n times differentiable functions of t, and c_1 and c_2 are constants. Then

$$L[c_1 f_1 + c_2 f_2] = c_1 L[f_1] + c_2 L[f_2]$$

Example 100

$L = 3D^2 + 5D - 2$ applies to $3t^2 + 2\sin t$, then

$$L[3t^2 + 2\sin t] = 3L[t^2] + 2L[\sin t]$$

LHS: $(3D^2 + 5D - 2)(3t^2 + 2\sin t)$

$$= (18 - 6\sin t) + (30t + 10\cos t) + (-6t^2 - 4\sin t)$$

$$= -6t^2 + 30t + 18 - 10\sin t + 10\cos t$$

RHS: $3L[t^2] + 2L[\sin t] = 3(3D^2 + 5D - 2)t^2 + 2(3D^2 + 5D - 2)\sin t$

$$= 3(3\frac{d^2}{dt^2}t^2 + 5\frac{d}{dt}t^2 - 2t^2) + 2(3\frac{d^2}{dt^2}\sin t + 5\frac{d}{dt}\sin t - 2\sin t)$$

$$= 3(6 + 10t - 2t^2) + 2(-3\sin t + 5\cos t - 2\sin t)$$

$$= -6t^2 + 30t + 18 - 10\sin t + 10\cos t$$

Suppose two linear operators L_1 and L_2 apply to f successively. If f has sufficiently many derivatives

$$L_1 L_2 f = L_2 L_1 f = Lf$$

where L is the product of L_1 and L_2 using the rules of the polynomial product.

Example 101

$$(D+1)(D+3)\sin t = (D+3)(D+1)\sin t = (D^2+4D+3)\sin t$$

Example 102

Consider

$$\left.\begin{array}{l} 2\frac{dx}{dt} - 2\frac{dy}{dt} - 3x = t \\ 2\frac{dx}{dt} + 2\frac{dy}{dt} + 3x + 8y = 2 \end{array}\right\} \tag{94}$$

In the operator notation

$$\left.\begin{array}{l} (2D-3)x - 2Dy = f_1 \\ (2D+3)x + (2D+8)y = f_2 \end{array}\right\}$$

where $f_1 \triangleq t$ and $f_2 \triangleq 2$.

Example 102 (cont.)

$$\left.\begin{array}{l}(2D-3)x-2Dy=f_1\\(2D+3)x+(2D+8)y=f_2\end{array}\right\}$$

$$\left.\begin{array}{l}L_1x+L_2y=f_1\\L_3x+L_4y=f_2\end{array}\right\}$$

$$\left.\begin{array}{l}L_1x+L_2y=f_1,\ \text{multiply by } L_4\\L_3x+L_4y=f_2,\ \text{multiply by } L_2\end{array}\right\}$$

$$\left.\begin{array}{l}L_4L_1x+L_4L_2y=L_4f_1\\L_2L_3x+L_2L_4y=L_2f_2\end{array}\right\}\ \text{subtract 2nd from the 1st}$$

$$(L_4L_1-L_2L_3)x=L_4f_1-L_2f_2$$

Example 102 (cont.)

$$(L_4L_1 - L_2L_3)x = L_4f_1 - L_2f_2$$

$$[(2D+8)(2D-3) - (-2D)(2D+3)]x = (2D+8)t - (-2D)2$$

$$[8D^2 + 16D - 24]x = 2 + 8t$$

$$[D^2 + 2D - 3]x = t + \frac{1}{4}$$

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} - 3x = t + \frac{1}{4} \tag{95}$$

$$\rightarrow x = c_1e^t + c_2e^{-3t} - \frac{1}{3}t - \frac{11}{36}$$

Example 102 (cont.)

Reconsider

$$\left.\begin{aligned} L_1 x + L_2 y &= f_1, \\ L_3 x + L_4 y &= f_2, \end{aligned}\right\}$$

$$\left.\begin{aligned} L_1 x + L_2 y &= f_1, \ \text{multiply by } L_3 \\ L_3 x + L_4 y &= f_2, \ \text{multiply by } L_1 \end{aligned}\right\}$$

$$\left.\begin{aligned} L_3 L_1 x + L_3 L_2 y &= L_3 f_1 \\ L_1 L_3 x + L_1 L_4 y &= L_1 f_2 \end{aligned}\right\} \text{ subtract the 1st from the 2nd}$$

$$(L_4 L_1 - L_2 L_3) y = L_1 f_2 - L_3 f_1$$

$$[D^2 + 2D - 3] y = -\frac{3}{8} t - 1$$

$$\frac{d^2 y}{dt^2} + 2\frac{dy}{dt} - 3y = -\frac{3}{8} t - 1 \qquad (96)$$

$$\rightarrow y = k_1 e^t + k_2 e^{-3t} + \frac{1}{8} t + \frac{5}{12}$$

Example 102 (cont.)

Solutions to (95) and (96) are:

$$\rightarrow x = c_1 e^t + c_2 e^{-3t} - \frac{1}{3}t - \frac{11}{36}$$

$$\rightarrow y = k_1 e^t + k_2 e^{-3t} + \frac{1}{8}t + \frac{5}{12}$$

In x, for arbitrarily selected constants (c_1, c_2), (95) is satisfied
In y, for arbitrarily selected constants (k_1, k_2), (96) is satisfied
Recall

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} - 3x = t + \frac{1}{4} \qquad \text{(cf. 95)}$$

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 3y = -\frac{3}{8}t - 1 \qquad \text{(cf. 96)}$$

However, arbitrarily selected constants (c_1, c_2, k_1, k_2) do not work for simultaneous solution of (94):

Example 102 (cont.)

However, arbitrarily selected constants (c_1, c_2, k_1, k_2) do not work for simultaneous solution of (94):

$$\left.\begin{array}{l} 2\frac{dx}{dt} - 2\frac{dy}{dt} - 3x = t \\ 2\frac{dx}{dt} + 2\frac{dy}{dt} + 3x + 8y = 2 \end{array}\right\} \qquad \text{(cf. 94)}$$

Let us substitute the solutions of (95) and (96) into the original equation (94) to resolve the issue of arbitrary constants. Generally substitution in one d.e.of the d. e. set is sufficient for resolving the arbitrary constants.

Example 102 (cont.)

Let us consider any of the equations in (94), for instance, the first one, to plug the solutions in it:

$$2\frac{dx}{dt} - 2\frac{dy}{dt} - 3x = t$$

$$[2c_1e^t - 6c_2e^{-3t} - \frac{2}{3}] - [2k_1e^t - 6k_2e^{-3t} + \frac{1}{4}] - [3c_1e^t + 3c_2e^{-3t} - t - \frac{11}{12}] = t$$

or

$$(-c_1 - 2k_1)e^t + (-9c_2 + 6k_2)e^{-3t} = 0$$

Thus we must have

$$k_1 = -\frac{1}{2}c_1, \quad k_2 = \frac{3}{2}c_2$$

Example 102 (cont.)

General solution of (94):

$$x = c_1 e^t + c_2 e^{-3t} - \frac{1}{3}t - \frac{11}{36}$$

$$y = -\frac{1}{2}c_1 e^t + \frac{3}{2}c_2 e^{-3t} + \frac{1}{8}t + \frac{5}{12}$$

or

$$x = -2k_1 e^t + \frac{2}{3}k_2 e^{-3t} - \frac{1}{3}t - \frac{11}{36}$$

$$y = k_1 e^t + k_2 e^{-3t} + \frac{1}{8}t + \frac{5}{12}$$

Example 103

Solve $\ddot{y} - y = 0$ using operator method.

$$(D^2 - 1)y = 0$$

$$(D-1)(D+1)y = 0$$

Let $z \triangleq (D+1)y$. The problem now can be expressed as two first order differential equations:

$$\begin{aligned} (D-1)z &= 0 \\ (D+1)y &= z \end{aligned}$$

The first one is decoupled from the the other equation; we can solve it separately:

$$(D-1)z = 0 \rightarrow \frac{dz}{dx} - z = 0 \rightarrow z(x) = ce^x$$

$$(D-1)z=0 \rightarrow \frac{dz}{dx}-z=0 \rightarrow z(x)=ce^{x}$$

Example 103 (cont.)

The second equation $(D+1)y=z$ uses the solution of the 1st equation:

$$\frac{dy}{dx}+y=ce^{x}$$

It is a 1st order linear d.e. Its integrating factor is $\mu(x)=e^{\int P(x)dx}=e^{\int 1dx}=e^{x}$. Multiply the d.e. throughout by the integrating factor:

$$e^{x}\frac{dy}{dx}+e^{x}y=ce^{2x}$$

$$\frac{d}{dx}(e^{x}y)=ce^{2x}$$

$$\frac{d}{dx}(e^x y) = ce^{2x}$$

Example 103 (cont.)

$$e^x y = \int ce^{2x} dx = \frac{c}{2} e^{2x} + b$$

$$y = \frac{c}{2} e^x + be^{-x}$$

where b and c are arbitrary constants. Compare this to the solution we had by auxiliary equations method:

$$y = c_1 e^x + c_2 e^{-x}$$

where c_1 and c_2 are arbitrary constants.

Example 104

$$\ddot{y} - 3\dot{y} + 2y = e^x$$

$$(D^2 - 3D + 2)y = e^x$$

$$(D - 1)(D - 2)y = e^x$$

Let $z \triangleq (D - 2)y$, then the other equation becomes $(D - 1)z = e^x$. The equation $(D - 1)z = e^x$ is in terms of z only. Therefore, we can solve it separately.

$$(D - 1)z = e^x \rightarrow \frac{dz}{dx} - z = e^x \rightarrow z = xe^x + Ae^x$$

We next solve the other equation $(D - 2)y = z$ using the z obtained above:

$$\dot{y} - 2y = xe^x + Ae^x$$

$$\dot{y} - 2y = xe^x + Ae^x$$

Example 104 (cont.)

This is a 1st order linear d.e. with integrating factor e^{-2x}.
Standard solution procedure yields

$$y = e^x - xe^x - Ae^x + Be^{2x}$$

Solve the above problem by the undetermined coefficients method and compare the solutions.

Example 105

Let D be the operator $\frac{d}{dt}$. Find a general solution for

$$(D+2)^2(D-1)x = 0$$

Solution

Its characteristic polynomial has the roots $-2, -2, 1$.
Corresponding general solution is

$$x(t) = c_1 e^{-2t} + c_2 t e^{-2t} + c_3 e^t$$

where c_1, c_2, c_3 are arbitrary constants.

The Laplace transform

Definition 28

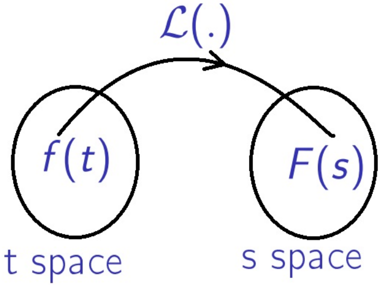
Let f be a real valued function of the real variable t, defined for $t > 0$. Let s be a variable that we shall assume to be real, and consider the function F defined by

$$F(s) = \int_0^\infty e^{-st} f(t) dt \tag{97}$$

for all values of s for which this integral exists.

The function F is called the Laplace transform of the function f.

An alternative notation for the Laplace transform F of f is $\mathcal{L}\{f\}$.



$\mathcal{L}(.)$
$f(t)$
$F(s)$
t space
s space

Two-sided Laplace transform

$$F(s) = \int_{-\infty}^{\infty} e^{-st} f(t) dt \tag{98}$$

is more general and useful for two-sided functions. It reduces to the form

$$F(s) = \int_{0}^{\infty} e^{-st} f(t) dt \qquad \text{(cf. 97)}$$

for the "causal" signals.

Example 106

$$f(t) = 1,\ t > 0 \leftrightarrow \mathcal{L}\{1\} = \int_0^\infty e^{-st} 1 dt = \lim_{R\to\infty} \int_0^R e^{-st} 1 dt$$

$$= \lim_{R\to\infty} \left[\frac{-e^{-st}}{s}\right]_0^R = \lim_{R\to\infty} \left[\frac{1}{s} - \frac{e^{-sR}}{s}\right] = \frac{1}{s}$$

for all $s > 0$.

Example 107

$$f(t) = t, \ \ t > 0$$

$$\begin{aligned}
\mathcal{L}\{t\} &= \int_0^\infty te^{-st}dt = \\
&= \lim_{R\to\infty} \int_0^R te^{-st}dt \\
&= \lim_{R\to\infty} \left\{ \left[\frac{-t}{s} e^{-st} \right]_0^R - \int_0^R \frac{-1}{s} e^{-st}dt \right\} \\
&= \lim_{R\to\infty} \left\{ \frac{-R}{s} e^{-sR} + \left[\frac{-1}{s^2} e^{-st} \right]_0^R \right\} \\
&= \lim_{R\to\infty} \left[\frac{-R}{s} e^{-sR} - \frac{-1}{s^2} e^{-sR} + \frac{1}{s^2} \right] \\
&= \frac{1}{s^2} \text{ for all } s > 0.
\end{aligned}$$

Example 108

Find Laplace transform of $f(t) = t^n,\ t > 0,\ n \in \{1, 2, \dots\}$
Integration by parts

$$\int_a^b u(t)v'(t)dt = u(t)v(t)|_a^b - \int_a^b v(t)u'(t)dt$$

with
$u = t^n,\ v'(t) = e^{-st},\ u' = nt^{n-1},\ v = -\frac{1}{s}e^{-st},\ a = 0,\ b = \infty$

$$\mathcal{L}(t^n) = \int_0^\infty t^n e^{-st} dt = t^n \left(-\frac{e^{-st}}{s} \right)\Bigg|_0^\infty \to 0 + \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}dt$$

$$= \frac{n}{s}\mathcal{L}(t^{n-1})$$

Example 108 (cont.)

$$\mathcal{L}(t^n) = \int_0^\infty t^n e^{-st} dt = t^n \left(-\frac{e^{-st}}{s} \right) \Bigg|_0^\infty \to 0 + \frac{n}{s} \int_0^\infty t^{n-1} e^{-st} dt$$

$$= \frac{n}{s} \mathcal{L}(t^{n-1})$$

Recursive use of the above formula yields

$$\mathcal{L}(t^1) = \frac{1}{s} \mathcal{L}(t^0) = \frac{1}{s} \times \frac{1}{s} = \frac{1}{s^2}$$

$$\mathcal{L}(t^2) = \frac{2}{s} \mathcal{L}(t^1) = \frac{2}{s} \times \frac{1}{s^2} = \frac{2}{s^3}$$

$$\mathcal{L}(t^3) = \frac{3}{s} \mathcal{L}(t^2) = \frac{3}{s} \times \frac{2}{s^3} = \frac{6}{s^4}$$

$$\cdots\cdots$$

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$$

Example 109

$$f(t) = e^{at},\ t > 0 \leftrightarrow \mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at} dt = \lim_{R\to\infty} \int_0^R e^{(a-s)t} dt$$

$$= \lim_{R\to\infty} \left[\frac{e^{(a-s)t}}{a-s} \right]_0^R = \lim_{R\to\infty} \left[\frac{e^{(a-s)R}}{a-s} - \frac{1}{a-s} \right] = \frac{1}{s-a}$$

for all $s > a$.

Example 110

$$f(t) = \sin bt, \ \ t > 0 \leftrightarrow \mathcal{L}\{\sin(bt)\} = \frac{b}{s^2 + b^2}, \ \ s > 0$$

Example 111

$$f(t) = \cos bt, \ \ t > 0 \leftrightarrow \mathcal{L}\{\cos(bt)\} = \frac{s}{s^2 + b^2}, \ \ s > 0$$

Existence of the Laplace Transform

Some functions, such as $f(t) = e^{t^2}$, do not have Laplace transforms. For a function to have a Laplace transform, the following integral must exist:

$$F(s) = \int_0^\infty e^{-st} f(t) dt \qquad \text{(cf. 97)}$$

When do such integrals exist? To answer this we need to define **piecewise continuity** and **being of exponential order** first.

Definition 29

A function f of t is said to be **piecewise continuous** on a finite interval $a \leq t \leq b$ if this interval can be divided into a finite number of subintervals such that

(1) f is continuous in the interior of each of these subintervals, and

(2) f approaches finite limits as t approaches either endpoint of each of the subintervals from its interior.

Piecewise continuous function

Example 112

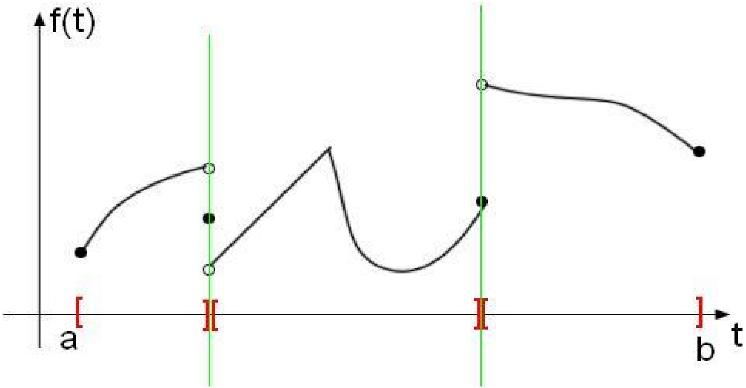


Figure: Piecewise Continuous Function on [a,b]

Example 113

$f(t) = \frac{1}{t-3}$ is discontinuous at $t = 3$. This function is not piecewise continuous on any interval containing $t = 3$, because neither $\lim_{t\to 3+}$ nor $\lim_{t\to 3-}$ exists.

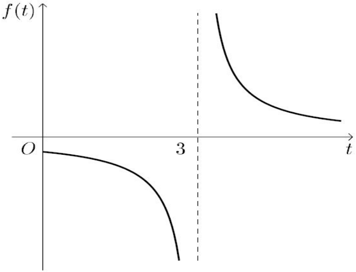


Figure: $f(t) = \frac{1}{t-3}$

Example 114

$f(t) = \begin{cases} 0 & t < 0 \\ \cos(\frac{1}{t}) & t > 0 \end{cases}$ is discontinuous at $t = 0$. This function is not piecewise continuous on any interval containing $t = 0$, because $\lim_{t \to 0+}$ does not exist.

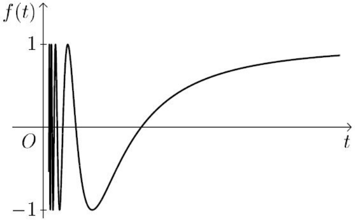


Figure: $f(t) = 0$ for $t < 0$ and $\cos\frac{1}{t}$ for $t > 0$

Definition 30

A function f of t is said to be of exponential order if there exist a constant α and positive constants t_0 and M such that

$$|f(t)| < Me^{\alpha t} \tag{99}$$

for all $t > t_0$ at which f is defined.

If f is of exponential order corresponding to some definite constant α in (99), then we say that f is of exponential order $e^{\alpha t}$.

Example 115

Every bounded function is of exponential order, for instance $\sin(bt)$
t^n is of exponential order
e^{at} is of exponential order
e^{t^2} is not of exponential order.

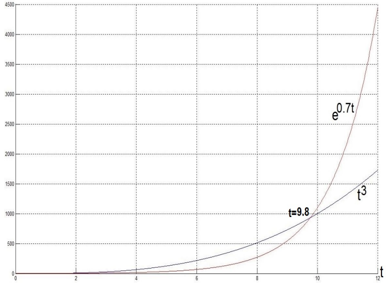


Figure: $e^{0.7t}$ and t^3 graphs show that t^3 is of exponential order.

Example 116

$f(t) = \sin(t)$ is of exponential order. Because we can find, for instance, $\alpha = 2,\ t_0 = 0.1,\ M = 5$ so that

$$|f(t)| < Me^{\alpha t}$$

is satisfied for all $t > t_0$. That is,

$$|\sin t| < 5e^{2t}, \quad \text{for all } t \geq 0.1$$

Example 117

$f(t) = t^2$ is of exponential order. Because we can find, for instance, $\alpha = 0.5,\ t_0 = 1,\ M = 5$ so that

$$|f(t)| < Me^{\alpha t}$$

is satisfied for all $t > t_0$. That is,

$$|t^2| < 5e^{0.5t}, \quad \text{for all } t \geq 1$$

Theorem 27

Let f be a real function that has the following properties:
1) f is piecewise continuous in every finite closed interval
$0 \le t \le b$, $(b > 0)$
2) f is of exponential order $e^{\alpha t}$. Then the Laplace transform

$$\int_0^\infty e^{-st} f(t)dt$$

of f exists for $s > \alpha$.

Proof Since f is of exponential order, there exist α, t_0 and M such that

$$|f(t)| < Me^{\alpha t}, \quad \text{for } t \ge t_0$$

We can write

$$\int_0^\infty e^{-st} f(t)dt = \int_0^{t_0} e^{-st} f(t)dt + \int_{t_0}^\infty e^{-st} f(t)dt$$

$$\int_0^\infty e^{-st}f(t)dt = \underbrace{\int_0^{t_0} e^{-st}f(t)dt}_{Part1} + \underbrace{\int_{t_0}^\infty e^{-st}f(t)dt}_{Part2}$$

Part 1 exists because the integral has finite limits and the function $f(t)$ is piecewise continuous.
For the second part, for $t \geq t_0$ note that

$$|f(t)| < Me^{\alpha t} \to |e^{-st}f(t)| < Me^{-(s-\alpha)t}$$

$$\to \int_{t_0}^\infty |e^{-st}f(t)|dt < M\int_{t_0}^\infty e^{-(s-\alpha)t}dt \leq M\int_0^\infty e^{-(s-\alpha)t}dt = \frac{M}{s-\alpha}$$

for $s > \alpha$.
This shows that the integral $\int_{t_0}^\infty |e^{-st}f(t)|dt$ exists. This implies that $\int_{t_0}^\infty e^{-st}f(t)dt$ exists.

$$\int_0^\infty e^{-st} f(t)dt = \underbrace{\int_0^{t_0} e^{-st} f(t)dt}_{Part1} + \underbrace{\int_{t_0}^\infty e^{-st} f(t)dt}_{Part2}$$

Integrals exist for part 1 and part 2. This shows that the integral

$$\int_0^\infty e^{-st} f(t)dt$$

exists.

Theorem 28

Let f_1 and f_2 be functions whose Laplace transforms exist, and c_1, c_2 be constants. Then $\mathcal{L}\{c_1 f_1 + c_2 f_2\} = c_1 \mathcal{L}\{f_1\} + c_2 \mathcal{L}\{f_2\}$.

Proof

$$\begin{aligned}
\mathcal{L}\{c_1 f_1 + c_2 f_2\} &= \int_0^\infty (c_1 f_1 + c_2 f_2) e^{-st} dt \\
&= \int_0^\infty (c_1 f_1 e^{-st} + c_2 f_2 e^{-st}) dt \\
&= c_1 \int_0^\infty f_1 e^{-st} dt + c_2 \int_0^\infty f_2 e^{-st} dt \\
&= = c_1 \mathcal{L}\{f_1\} + c_2 \mathcal{L}\{f_2\}
\end{aligned}$$

$c_1 f_1 + c_2 f_2$ is piecewise continuous because it is a linear combination of piecewise continuous functions. Let us show that $c_1 f_1 + c_2 f_2$ is of exponential order:

Proof (cont.)
$c_1 f_1 + c_2 f_2$ is piecewise continuous because it is a linear combination of piecewise continuous functions. Let us show that $c_1 f_1 + c_2 f_2$ is of exponential order:
$|f_1| < M_1 e^{\alpha_1 t}$ for all $t \geq t_{10}$, for some t_{10} and α_1;
$|f_2| < M_2 e^{\alpha_2 t}$ for all $t \geq t_{20}$, for some t_{20} and α_2.
Note that

$$|c_1 f_1 + c_2 f_2| \leq |c_1||f_1| + |c_2||f_2| \leq (|c_1|M + |c_2|M)e^{\alpha t}, \text{ for } t \geq t_0$$

where $\alpha = \max\{\alpha_1, \alpha_2\}, M = \max\{M_1, M_2\}, t_0 = \max\{t_{10}, t_{20}\}$
This shows that $c_1 f_1 + c_2 f_2$ is of exponential order.

□

Example 118

$$\mathcal{L}(2\sin t + 5t^3) = 2\mathcal{L}(\sin t) + 5\mathcal{L}(t^3)$$

Example 119

$$\mathcal{L}(5t^3) = 5\mathcal{L}(t^3)$$

Theorem 29

Let f be a real valued function that is continuous for $t \geq 0$ and of exponential order $e^{\alpha t}$. Let f' be piecewise continuous in every finite closed interval $0 \leq t \leq b$. Then $\mathcal{L}\{f'\}$ exists for $s > \alpha$ and $\mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0)$.

Example 120

It is known that $\mathcal{L}\{\sin bt\} = \frac{b}{s^2+b^2}$.
This implies $\mathcal{L}\{(\sin bt)'\} = s\frac{b}{s^2+b^2} - \sin(b \cdot t)_{t=0} = \frac{bs}{s^2+b^2}$
By direct computation:
$\mathcal{L}\{b\cos bt\} = \frac{bs}{s^2+b^2}$

Example 121

$\mathcal{L}\{t\} = \frac{1}{s^2} \rightarrow \mathcal{L}\{(t)'\} = s\frac{1}{s^2} - t|_{t=0} = \frac{1}{s}$
By direct computation:
$\mathcal{L}\{1\} = \frac{1}{s}$

Theorem 29

Let f be a real valued function that is continuous for $t \geq 0$ and of exponential order $e^{\alpha t}$. Let f' be piecewise continuous in every finite closed interval $0 \leq t \leq b$. Then $\mathcal{L}\{f'\}$ exists for $s > \alpha$ and $\mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0)$.

Proof of Laplace transform of a derivative

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)dt = \lim_{R\to\infty} \int_0^R e^{-st} f'(t)dt$$

In the closed interval $0 \leq t \leq R$, by hypothesis, f' has at most finite number of discontinuities, denote them by $t_1, t_2, \ldots, t_n$ with $t_1 < t_2 < \cdots < t_n$. Then we may write

$$\int_0^R e^{-st} f'(t)dt = \int_0^{t_1} e^{-st} f'(t)dt + \int_{t_1}^{t_2} e^{-st} f'(t)dt + \cdots + \int_{t_n}^{R} e^{-st} f'(t)dt$$

$$\int_0^R e^{-st} f'(t)dt = \underbrace{\int_0^{t_1} e^{-st} f'(t)dt + \int_{t_1}^{t_2} e^{-st} f'(t)dt + \cdots + \int_{t_n}^{R} e^{-st} f'(t)dt}$$

$\downarrow$

Integration by parts

$\downarrow$

$$= \left[e^{-st} f(t)\right]_0^{t_1} + s\int_0^{t_1} e^{-st} f(t)dt + \left[e^{-st} f(t)\right]_{t_1}^{t_2} + s\int_{t_1}^{t_2} e^{-st} f(t)dt$$

$$+ \left[e^{-st} f(t)\right]_{t_2}^{t_3} + s\int_{t_2}^{t_3} e^{-st} f(t)dt + \cdots + \left[e^{-st} f(t)\right]_{t_n}^{R} + s\int_{t_n}^{R} e^{-st} f(t)dt$$

$$= \left[e^{-st}f(t)\right]_0^{t_1} + s\int_0^{t_1} e^{-st}f(t)dt + \left[e^{-st}f(t)\right]_{t_1}^{t_2} + s\int_{t_1}^{t_2} e^{-st}f(t)dt$$

$$+ \left[e^{-st}f(t)\right]_{t_2}^{t_3} + s\int_{t_2}^{t_3} e^{-st}f(t)dt + \cdots + \left[e^{-st}f(t)\right]_{t_n}^{R} + s\int_{t_n}^{R} e^{-st}f(t)dt$$

$= e^{-st_1}f(t_1) - e^{-s0}f(0) + s\int_0^{t_1} e^{-st}f(t)dt$
$+e^{-st_2}f(t_2) - e^{-st_1}f(t_1) + s\int_{t_1}^{t_2} e^{-st}f(t)dt$
$+e^{-st_3}f(t_3) - e^{-st_2}f(t_2) + s\int_{t_2}^{t_3} e^{-st}f(t)dt + \cdots$
$\cdots + e^{-sR}f(R) - e^{-st_n}f(t_n) + s\int_{t_n}^{R} e^{-st}f(t)dt$
The integral thus reduces to

$$\int_0^R e^{-st}f'(t)dt = -f(0) + e^{-sR}f(R) + s\int_0^R e^{-st}f(t)dt$$

The integral thus reduces to

$$\int_0^R e^{-st} f'(t)dt = -f(0) + e^{-sR} f(R) + s\int_0^R e^{-st} f(t)dt$$

By hypothesis, f is of exponential order $e^{\alpha t}$, therefore, $|f(R)| < Me^{\alpha R}$ for all $R > R_0$ for some α, M and R_0. So we have $|e^{-sR} f(R)| < Me^{-(s-\alpha)R}$ for all $R > R_0$ for some M and R_0. Thus, if $s > \alpha$,

$$\lim_{R\to\infty} Me^{-(s-\alpha)R} = 0 \to \lim_{R\to\infty} e^{-sR} f(R) = 0$$

and

$$\lim_{R\to\infty} s\int_0^R e^{-st} f(t)dt = s\mathcal{L}\{f(t)\}$$

$$\lim_{R\to\infty} \int_0^R e^{-st} f'(t)dt = -f(0) + s\mathcal{L}\{f(t)\} \qquad \square$$

Theorem 30

Let f be a real valued function having a continuous $(n-1)st$ derivative $f^{(n-1)}$ for $t \geq 0$; and assume that $f,\ f',\ f'', \ldots, f^{(n-1)}$ are all of exponential order $e^{\alpha t}$. Suppose $f^{(n)}$ is piecewise continuous in every finite closed interval $0 \leq t \leq b$. Then $\mathcal{L}\{f^{(n)}\}$ exists for $s > \alpha$ and

$$\mathcal{L}\{f^{(n)}\} = s^n \mathcal{L}\{f\} - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \ldots - f^{(n-1)}(0)$$

Example 122

$$\mathcal{L}\ddot{f}(t) = s^2 F(s) - sf(0) - \dot{f}(0)$$
$$\mathcal{L}\dddot{f}(t) = s^3 F(s) - s^2 f(0) - s\dot{f}(0) - \ddot{f}(0)$$

Theorem 31

For a given f let $\mathcal{L}\{f\}$ exist for $s > \alpha$. Then for any constant a, $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$ for $s > \alpha + a$, where F denotes $\mathcal{L}\{f\}$.

Proof

$$\begin{aligned} \mathcal{L}\{e^{at}f(t)\} &= \int_0^\infty e^{at}f(t)e^{-st}dt \\ &= \int_0^\infty f(t)e^{-(s-a)t}dt \\ &= F(s-a) \end{aligned}$$

□

Example 123

$$t \leftrightarrow \frac{1}{s^2}$$

Then

$$e^{at} t \leftrightarrow \frac{1}{(s-a)^2}$$

$f(t) \leftrightarrow F(s)$ Then for any constant $a > 0$, $f(at) \leftrightarrow \frac{1}{a}F(\frac{s}{a})$

$$\int_0^\infty f(at)e^{-st}dt = \frac{1}{a}\int_0^\infty f(\tau)e^{-(\frac{s}{a})\tau}d\tau = \frac{1}{a}F(\frac{s}{a})$$

Example 124

$$e^t \leftrightarrow \frac{1}{s-1}$$

Then

$$e^{at} \leftrightarrow \frac{1}{a} \times \frac{1}{(\frac{s}{a})-1} = \frac{1}{s-a}$$

Theorem 32

Suppose f has Laplace transform F. Then

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}[F(s)]$$

Proof

$$F(s) = \int_0^\infty f(t)e^{-st}dt \leftrightarrow \frac{d^n F(s)}{ds^n} = \int_0^\infty f(t)\frac{d^n}{ds^n}e^{-st}dt$$

Use $\frac{d^n}{ds^n}e^{-st} = (-1)^n t^n e^{-st}$ in the above equation:

$$\frac{d^n F(s)}{ds^n} = \int_0^\infty f(t)(-1)^n t^n e^{-st}dt$$

$$(-1)^n \frac{d^n F(s)}{ds^n} = \int_0^\infty \mathbf{t^n f(t)} e^{-st}dt$$

□

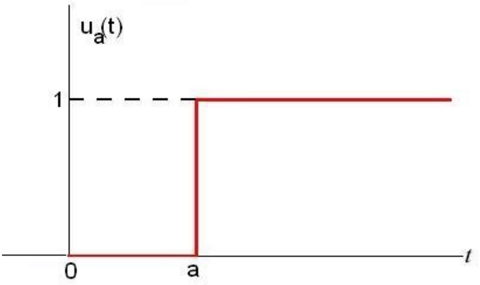
Example 125

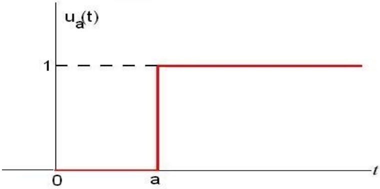
$$\mathcal{L}(t^3) = \mathcal{L}(t^2 \times t) = (-1)^2 \, \frac{d^2}{ds^2} \left[\frac{1}{s^2} \right] = \frac{6}{s^4}$$

Definition 31

For each real number $a \geq 0$, unit step function u_a is defined for nonnegative t by

$$u_a(t) = \begin{cases} 0; & t < a \\ 1, & t > a \end{cases}$$





Some Properties of u_a

$$\int_0^t u_a(t)dt = \int_a^t u_a(t)dt$$

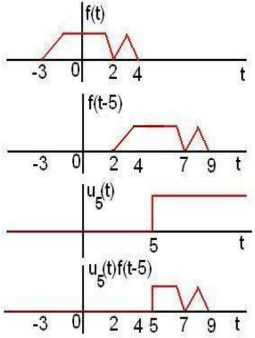
$$\mathcal{L}(u_a(t)) = \int_0^\infty u_a(t)e^{-st}dt = \int_a^\infty e^{-st}dt = \left[-\frac{e^{-st}}{s}\right]_a^\infty = \frac{e^{-as}}{s}$$

Theorem 33

Suppose f has Laplace transform F, and consider the translated function defined by

$$u_a(t)f(t-a) = \begin{cases} 0, & 0 < t < a \\ f(t-a), & t > a \end{cases}$$

Then $\mathcal{L}\{u_a(t)f(t-a)\} = e^{-as}\mathcal{L}\{f(t)\} = e^{-as}F(s)$



Proof

$$\begin{aligned}
\mathcal{L}(u_a(t)f(t-a)) &= \int_0^\infty u_a(t)f(t-a)e^{-st}dt \\
&= \int_a^\infty f(t-a)e^{-st}dt \\
&= \int_0^\infty f(u)e^{-s(u+a)}du \\
&= \int_0^\infty f(u)e^{-su}e^{-as}du \\
&= e^{-as}\int_0^\infty f(u)e^{-su}du \\
&= e^{-as}\int_0^\infty f(t)e^{-st}dt \\
&= e^{-as}F(s)
\end{aligned}$$

Example 126

$$g(t) = \begin{cases} 0, & 0 < t < 5 \\ t-3, & t > 5 \end{cases}$$

Before applying the theorem to this translated function, we must express the functional values $t-3$ for $t > 5$ in terms of $t-5$. That is we express $t-3$ as $(t-5)+2$ and write

$$g(t) = \begin{cases} 0, & 0 < t < 5 \\ (t-5)+2, & t > 5 \end{cases}$$

$$u_5(t)f(t-5) = \begin{cases} 0, & 0 < t < 5 \\ (t-5)+2, & t > 5 \end{cases}$$

where $f(t) = t+2,\ t > 0$. Hence we apply Theorem 33 with $f(t) = t+2$. $F(s) = \mathcal{L}\{t+2\} = \mathcal{L}\{t\} + 2\mathcal{L}\{1\} = \frac{1}{s^2} + \frac{2}{s}$.
Therefore,

$$\mathcal{L}\{u_5(t)f(t-5)\} = e^{-5s}F(s) = e^{-5s}(\frac{1}{s^2} + \frac{2}{s})$$

The inverse Laplace transform

Theorem 34

Let f and g be two functions that are continuous for $t \geq 0$ and that have the same Laplace transform F. Then $f(t) = g(t)$ for all $t \geq 0$.

For the continuous functions the Laplace transform is one-to-one. In other words, for continuous functions f and g, $\mathcal{L}(f) = \mathcal{L}(g)$ implies $f = g$.

Example 127

Find the inverse Laplace transform $\mathcal{L}^{-1}\{\frac{1}{s^2+6s+13}\}$.

$$\frac{1}{s^2+6s+13} = \frac{1}{(s+3)^2+2^2} = \frac{1}{2} \times \frac{2}{(s+3)^2+2^2} \leftrightarrow \frac{1}{2} e^{-3t} \sin 2t$$

Example 128

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$$

$$\frac{1}{s(s^2+1)} = \frac{A}{s}\frac{(s^2+1)}{(s^2+1)} + \frac{Bs+C}{s^2+1}\frac{s}{s}$$

$$\frac{1}{s(s^2+1)} = \frac{A(s^2+1)+(Bs+C)s}{s(s^2+1)}$$

$$1 = A(s^2+1)+(Bs+C)s$$

$$1 = (A+B)s^2 + Cs + A$$

$$\rightarrow A+B=0,\ C=0,\ A=1$$

$$\mathcal{L}^{-1}\{\frac{1}{s(s^2+1)}\} = \mathcal{L}^{-1}\{\frac{1}{s}\} - \mathcal{L}^{-1}\{\frac{s}{s^2+1}\} = 1 - \cos t$$

Example 129

$$\mathcal{L}^{-1}\{e^{-4s}(\frac{2}{s^2} + \frac{5}{s})\} = u_4(t)f(t-4)$$

with $f(t) = 2t + 5$.

Definition 32

Let f and g be two functions that are piecewise continuous on every finite closed interval $0 \leq t \leq b$ and of exponential order. The function denoted by $f * g$ and defined by

$$f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau = \int_0^t g(\tau)f(t-\tau)d\tau$$

is called the convolution of the functions f and g.

The equality of the two integrals above follows by making the change of variable $t - \tau = \xi$ in the first integral.

Properties of convolution integral:

$f * g = g * f$	commutative law
$f * (g_1 + g_2) = f * g_1 + f * g_2$	distributive law
$(f * g) * h = f * (g * h)$	associative law
$f * 0 = 0 * f = 0$	

Theorem 35

Let the functions f and g be piecewise continuous on every finite closed interval $0 \le t \le b$ and of exponential order. Then

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f\} \cdot \mathcal{L}\{g\}$$

It is not in general true that $f * 1 = f$:

$$f * 1 = \int_0^t f(t-\tau)\cdot 1 d\tau = \int_0^t f(t-\tau) d\tau$$

For instance, when $f(t) = \cos t$ we have

$$(f*1)(t) = \int_0^t f(t-\tau)d\tau = \int_0^t \cos(t-\tau)d\tau = -\sin(t-\tau)\big|_0^t$$

$$= -\sin(0) + \sin(t)$$

$$= \sin(t)$$

Similarly, it may not be true that $f * f$ is nonnegative.

Example 130

$$\frac{dy}{dt} - 2y = e^{5t},\ y(0) = 3$$

Take the Laplace transform of both sides. Let the Laplace transform of the unknown function y be Y which is also unknown meanwhile.

$$sY - y(0) - 2Y = \frac{1}{s-5} \rightarrow (s-2)Y - 3 = \frac{1}{s-5}$$

$$Y = \frac{3s-14}{(s-2)(s-5)} = \frac{A}{s-2} + \frac{B}{s-5}$$

To find A, multiply both sides by $(s-2)$ and evaluate at $s=2$:

$$\frac{3s-14}{(s-2)(s-5)} \times (s-2) = \frac{A}{s-2} \times (s-2) + \frac{B}{s-5} \times (s-2)$$

$$\left[\frac{3s-14}{(s-5)} = A + \frac{B}{s-5} \times (s-2)\right]_{s=2}$$

Example 130 (cont.)

$$\left[\frac{3s-14}{(s-5)} = A + \frac{B}{s-5} \times (s-2)\right]_{s=2}$$

$$\frac{3\times 2-14}{(2-5)} = A + \frac{B}{2-5} \times (2-2) \rightarrow A = \frac{8}{3}$$

To find B, consider the eqution

$$Y = \frac{3s-14}{(s-2)(s-5)} = \frac{A}{s-2} + \frac{B}{s-5}$$

multiply both sides by $(s-5)$ and evaluate at $s=5$. This gives B as $\frac{1}{3}$. Thus

$$Y = \frac{3s-14}{(s-2)(s-5)} = \frac{\frac{8}{3}}{s-2} + \frac{\frac{1}{3}}{s-5} \leftrightarrow \frac{8}{3}e^{2t} + \frac{1}{3}e^{5t}$$

Example 131

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} - 8y = 0,\ y(0) = 3,\ y'(0) = 6$$

$$\{s^2Y - sy(0) - y'(0)\} - 2\{sY - y(0)\} - 8Y = 0$$

$$[s^2 - 2s - 8]Y - 3s = 0$$

$$Y = \frac{3s}{(s-4)(s+2)} = \frac{A}{s-4} + \frac{B}{s+2}$$

$$A = \left[\frac{3s}{(s-4)(s+2)} \times (s-4)\right]_{s=4} = 2$$

$$B = \left[\frac{3s}{(s-4)(s+2)} \times (s+2)\right]_{s=-2} = 1$$

$$Y = \frac{3s}{(s-4)(s+2)} = \frac{2}{s-4} + \frac{1}{s+2} \leftrightarrow 2e^{4t} + e^{-2t}$$

Example 132

$$\frac{d^2y}{dt^2} + y = e^{-2t}\sin t,\ \ y(0) = 0,\ \ y'(0) = 0$$

$$\{s^2Y - sy(0) - y'(0)\} + Y = \frac{1}{[(s+2)^2+1]}$$

$$\{s^2Y - s0 - 0\} + Y = \frac{1}{[(s+2)^2+1]}$$

$$Y = \frac{1}{(s^2+1)[(s+2)^2+1]} = \frac{As+B}{s^2+1} + \frac{Cs+D}{(s+2)^2+1}$$

$$\frac{1}{(s^2+1)[(s+2)^2+1]} = \frac{As+B}{s^2+1}\frac{(s+2)^2+1}{(s+2)^2+1} + \frac{Cs+D}{(s+2)^2+1}\frac{s^2+1}{s^2+1}$$

$$\frac{1}{(s^2+1)[(s+2)^2+1]} = \frac{(As+B)((s+2)^2+1)+(Cs+D)(s^2+1)}{(s^2+1)[(s+2)^2+1]}$$

Example 132 (cont.)

$$\frac{1}{(s^2+1)[(s+2)^2+1]} = \frac{(As+B)}{s^2+1}\frac{(s+2)^2+1}{(s+2)^2+1} + \frac{(Cs+D)}{(s+2)^2+1}\frac{s^2+1}{s^2+1}$$

$$\frac{1}{(s^2+1)[(s+2)^2+1]} = \frac{(As+B)((s+2)^2+1)+(Cs+D)(s^2+1)}{(s^2+1)[(s+2)^2+1]}$$

$$1 = (As+B)(s^2+4s+5)+(Cs+D)(s^2+1)$$

$$1 = (A+C)s^3+(4A+B+D)s^2+(5A+4B+C)s+(5B+D)$$

$$\left.\begin{array}{lcl} A+C & = & 0 \\ 4A+B+D & = & 0 \\ 5A+4B+C & = & 0 \\ 5B+D & = & 1 \end{array}\right\} A = \frac{-1}{8},\ B = \frac{1}{8},\ C = \frac{1}{8},\ D = \frac{3}{8}$$

Example 132 (cont.)

$$Y = \frac{\frac{-1}{8}s + \frac{1}{8}}{s^2+1} + \frac{\frac{1}{8}s + \frac{3}{8}}{(s+2)^2+1}$$

$$= \frac{\frac{-1}{8}s}{s^2+1} + \frac{\frac{1}{8}}{s^2+1} + \frac{\frac{1}{8}s}{(s+2)^2+1}$$

$$+ \frac{\frac{2}{8}}{(s+2)^2+1} - \frac{\frac{2}{8}}{(s+2)^2+1} + \frac{\frac{3}{8}}{(s+2)^2+1}$$

$$= \frac{\frac{-1}{8}s}{s^2+1} + \frac{\frac{1}{8}}{s^2+1} + \frac{\frac{1}{8}(s+2)}{(s+2)^2+1} + \frac{\frac{1}{8}}{(s+2)^2+1}$$

$$y(t) = \frac{-1}{8}\cos t + \frac{1}{8}\sin t + \frac{1}{8}e^{-2t}\cos t + \frac{1}{8}e^{-2t}\sin t$$

Example 133

$$\frac{d^3y}{dt^3}+4\frac{d^2y}{dt^2}+5\frac{dy}{dt}+2y = 10\cos t,\ y(0)=0,\ y'(0)=0,\ y''(0)=3$$

$$\{s^3Y - s^2y(0) - sy'(0) - y''(0)\} + 4\{s^2Y - sy(0) - y'(0)\}$$

$$+5\{sY - y(0)\} + 2Y = 10\frac{s}{s^2+1}$$

$$\{s^3Y - s^2 0 - s0 - 3\} + 4\{s^2Y - s0 - 0\} + 5\{sY - 0\} + 2Y = 10\frac{s}{s^2+1}$$

$$\{s^3Y - 3\} + 4\{s^2Y\} + 5\{sY\} + 2Y = 10\frac{s}{s^2+1}$$

$$Y = \frac{3s^2+10s+3}{(s+2)(s+1)^2(s^2+1)} = \frac{A}{s+2} + \frac{B}{s+1} + \frac{C}{(s+1)^2} + \frac{Ds+E}{s^2+1}$$

$$= \frac{-1}{s+2} + \frac{2}{s+1} - \frac{2}{(s+1)^2} - \frac{s}{s^2+1} + \frac{2}{s^2+1}$$

$$y(t) = -e^{-2t} + 2e^{-t} - 2te^{-t} - \cos t + 2\sin t$$

Example 134

$$\begin{aligned} \frac{dx}{dt} - 6x + 3y &= 8e^t \\ \frac{dy}{dt} - 2x - y &= 4e^t \end{aligned}$$

$$x(0) = -1, \quad y(0) = 0$$

In Laplace domain :

$$\begin{aligned} sX + 1 - 6X + 3Y &= \frac{8}{s-1} \\ sY - 2X - Y &= \frac{4}{s-1} \end{aligned}$$

$$\begin{aligned} (s-6)X + 3Y &= \frac{-s+9}{s-1} \\ -2X + (s-1)Y &= \frac{4}{s-1} \end{aligned}$$

Example 134 (cont.)

$$\begin{array}{rcl} (s-6)X+3Y & = & \frac{-s+9}{s-1} \\ -2X+(s-1)Y & = & \frac{4}{s-1} \end{array}$$

In matrix notation:

$$\begin{bmatrix} s-6 & 3 \\ -2 & s-1 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \frac{-s+9}{s-1} \\ \frac{4}{s-1} \end{bmatrix}$$

$$X = \frac{-s+7}{(s-1)(s-4)}, \; Y = \frac{2}{(s-1)(s-4)}$$

$$\leftrightarrow x(t) = -2e^t + e^{4t}, \; y(t) = \frac{-2}{3}e^t + \frac{2}{3}e^{4t}$$

Example 135

Consider the differential equation

$$2\ddot{y} + \dot{y} + 2y = g(t), \; y(0) = 0, \; \dot{y}(0) = 0 \qquad (100)$$

where

$$g(t) = u_5(t) - u_{20}(t) = \begin{cases} 1 & 5 \leq t \leq 20 \\ 0 & t \leq 5 \text{ and } t \geq 20 \end{cases}$$

The d.e. in the Laplace domain is

$$2s^2 Y(s) - 2sy(0) - 2\dot{y}(0) + sY(s) - y(0) + 2Y(s) = \frac{e^{-5s}}{s} - \frac{e^{-20s}}{s}$$

$$2s^2Y(s) - 2sy(0) - 2\dot{y}(0) + sY(s) - y(0) + 2Y(s) = \frac{e^{-5s}}{s} - \frac{e^{-20s}}{s}$$

Example 135 (cont.)

$$2s^2Y(s) + sY(s) + 2Y(s) = \frac{e^{-5s}}{s} - \frac{e^{-20s}}{s}$$

$$Y(s) = \frac{e^{-5s} - e^{-20s}}{s(2s^2 + s + 2)}$$

It can be written as

$$Y(s) = (e^{-5s} - e^{-20s})H(s)$$

where

$$H(s) = \frac{1}{s(2s^2 + s + 2)}$$

$$h(t) = \mathcal{L}^{-1}H(s) \rightarrow y(t) = u_5(t)h(t-5) - u_{20}(t)h(t-20)$$

$$H(s) = \frac{1}{s(2s^2 + s + 2)}$$

Example 135 (cont.)

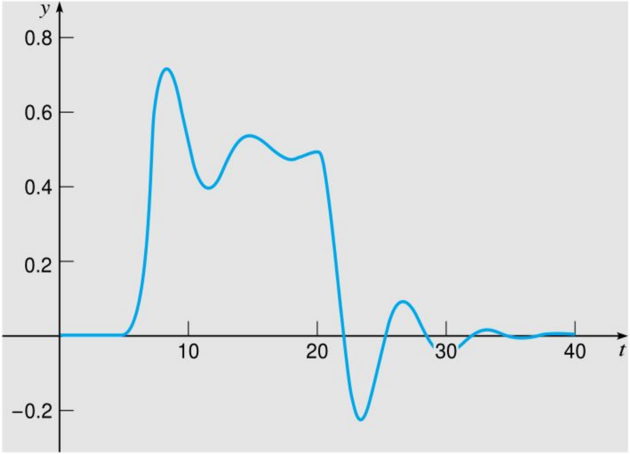
$$H(s) = \frac{a}{s} + \frac{bs + c}{2s^2 + s + 2}$$

$$H(s) = \frac{\frac{1}{2}}{s} - \frac{s + \frac{1}{2}}{2s^2 + s + 2} = \frac{\frac{1}{2}}{s} - \frac{1}{2} \frac{(s + \frac{1}{4}) + \frac{1}{4}}{(s + \frac{1}{4})^2 + \frac{15}{16}}$$

$$h(t) = \frac{1}{2} - \frac{1}{2}\left[e^{-\frac{t}{4}} \cos(\sqrt{15}t/4) + (\sqrt{15}/15)e^{-\frac{t}{4}} \sin(\sqrt{15}t/4)\right]$$

$$h(t) = \frac{1}{2} - \frac{1}{2}\left[e^{-\frac{t}{4}}\cos(\sqrt{15}t/4) + (\sqrt{15}/15)e^{-\frac{t}{4}}\sin(\sqrt{15}t/4)\right]$$
$$y(t) = u_5(t)h(t-5) - u_{20}(t)h(t-20)$$

Example 135 (cont.)



Partial Fractions Decomposition

[The Laplace Transform: Theory and Applications, Joel L. Schiff]

Consider quotient of two polynomials

$$F(s) = \frac{P(s)}{Q(s)}$$

where the degree of Q is greater than that of P, and P and Q do not have common factors. Then F can be expressed as a finite sum of partial fractions.

(i) For each factor of the form $as + b$ of Q, there corresponds a partial fraction of the form

$$\frac{A}{as + b}$$

where A is a constant.

(ii) For each repeated factor of the form $(as + b)^n$ of Q there corresponds a partial fraction of the form

$$\frac{A_1}{as + b} + \frac{A_2}{(as + b)^2} + \cdots + \frac{A_{n-1}}{(as + b)^{n-1}} + \frac{A_n}{(as + b)^n}$$

where $A_1, A_2, ..., A_n$ are constants.

(iii) For every factor of the form $as^2 + bs + c$ of Q there corresponds a partial fraction of the form

$$\frac{As + B}{as^2 + bs + c}$$

where A and B are constants.

(iv) For every repeated factor of the form $(as^2 + bs + c)^n$ of Q there corresponds a partial fraction of the form

$$\frac{A_1 s + B_1}{as^2 + bs + c} + \frac{A_2 s + B_2}{(as^2 + bs + c)^2} + \cdots + \frac{A_{n-1} s + B_{n-1}}{(as^2 + bs + c)^{n-1}} + \frac{A_n s + B_n}{(as^2 + bs + c)^n}$$

where $A_1, A_2, ..., A_n, B_1, B_2, ..., B_n$ are constants.

Example 136

$$\frac{s^2}{(s+2)(s+3)(s+7)^4} = \frac{A_1}{s+2} + \frac{A_2}{s+3} + \frac{A_3}{s+7} + \frac{A_4}{(s+7)^2} + \frac{A_5}{(s+7)^3} + \frac{A_6}{(s+7)^4}$$

Example 137

$$\frac{4s}{(s+4)(s^2+3s+7)^3} = \frac{A}{s+4} + \frac{B_1 s + B_2}{(s^2+3s+7)} + \frac{B_3 s + B_4}{(s^2+3s+7)^2} + \frac{B_5 s + B_6}{(s^2+3s+7)^3}$$

Time Domain Function	Laplace Transform
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
$\sin(bt)$	$\frac{b}{s^2+b^2}$
$\cos(bt)$	$\frac{s}{s^2+b^2}$
$t^n (n = 1, 2, \dots)$	$\frac{n!}{s^{n+1}}$
$t^n e^{at} (n = 1, 2, \dots)$	$\frac{n!}{(s-a)^{n+1}}$
$t \sin(bt)$	$\frac{2bs}{(s^2+b^2)^2}$
$t \cos(bt)$	$\frac{s^2-b^2}{(s^2+b^2)^2}$
$e^{-at} \sin(bt)$	$\frac{b}{(s+a)^2+b^2}$
$e^{-at} \cos(bt)$	$\frac{s+a}{(s+a)^2+b^2}$
$u_a(t)$	$\frac{e^{-as}}{s}$

Table: Laplace Transforms table

Summary

Laplace transforms of derivatives:

$$\mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0)$$
$$\mathcal{L}\{f''\} = s^2\mathcal{L}\{f\} - sf(0) - f'(0)$$
$$\mathcal{L}\{f^{(n)}\} = s^n\mathcal{L}\{f\} - \sum_{i=1}^{n} s^{n-i} f^{(i-1)}(0)$$

Differential equation and initial conditions:

$$\sum_{i=0}^{n} a_i f^{(i)}(t) = \phi(t), \quad f^{(i)}(0) = c_i$$

Laplace Transform:

$$\sum_{i=0}^{n} a_i \mathcal{L}\{f^{(i)}(t)\} = \mathcal{L}\{\phi(t)\}$$

$$\mathcal{L}\{f(t)\} \sum_{i=0}^{n} a_i s^i - \sum_{i=1}^{n} \sum_{j=1}^{i} a_i s^{i-j} f^{(j-1)}(0) = \mathcal{L}\{\phi(t)\}$$

$$\mathcal{L}\{f(t)\}\sum_{i=0}^{n} a_i s^i - \sum_{i=1}^{n}\sum_{j=1}^{i} a_i s^{i-j} f^{(j-1)}(0) = \mathcal{L}\{\phi(t)\}$$

$$\mathcal{L}\{f(t)\} = \frac{\mathcal{L}\{\phi(t)\} + \sum_{i=1}^{n}\sum_{j=1}^{i} a_i s^{i-j} c_{j-1}}{\sum_{i=0}^{n} a_i s^i}$$

If all the initial conditions are zero, that is,

$$f^{(i)}(0) = c_i = 0 \quad \forall i \in \{0, 1, 2, \dots n\}$$

Then the Laplace transform of the d.e. reduces to

$$f(t) = \mathcal{L}^{-1}\left\{\frac{\mathcal{L}\{\phi(t)\}}{\sum_{i=0}^{n} a_i s^i}\right\}$$

Example 138

$$2\ddot{y} + 3\dot{y} - 2y = te^{-2t}, \quad y(0) = 0, \; \dot{y}(0) = -2$$

$$2(s^2Y(s) - sy(0) - \dot{y}(0)) + 3(sY(s) - y(0)) - 2Y(s) = \frac{1}{(s+2)^2}$$

$$(2s^2 + 3s - 2)Y(s) + 4 = \frac{1}{(s+2)^2}$$

$$Y(s) = \frac{1}{(2s-1)(s+2)^3} - \frac{4}{(2s-1)(s+2)} = \frac{-4s^2 - 16s - 15}{(2s-1)(s+2)^3}$$

$$Y(s) = \frac{-4s^2 - 16s - 15}{(2s-1)(s+2)^3}$$

Example 138 (cont.)

$$Y(s) = \frac{1}{125}\left(\frac{-192}{2(s-\frac{1}{2})} + \frac{96}{s+2} - \frac{10}{(s+2)^2} - \frac{-25}{(s+2)^3}\right)$$

$$y(t) = \frac{1}{125}\left(-96e^{\frac{t}{2}} + 96e^{-2t} - 10te^{-2t} - \frac{25}{2}t^2e^{-2t}\right)$$

Example 139

Consider

$$t\ddot{y} + \dot{y} + 2y = 0, \; y(0) = 1$$

Let us transform the equation into the Laplace domain. We first do it for the first term. The properties

$$\ddot{y} \leftrightarrow s^2 Y - sy(0) - \dot{y}(0)$$

and

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}[F(s)]$$

imply

$$t\ddot{y} \leftrightarrow (-1)^1 \frac{d^1}{ds^1}[s^2 Y - sy(0) - \dot{y}(0)] \to -s^2 \dot{Y} - 2sY + 1$$

The given d.e. thus have the Laplace domain representation:

$$(-s^2 \dot{Y} - 2sY + 1) + (sY - 1) + 2Y = 0$$

Example 139 (cont.)

The given d.e. thus have the Laplace domain representation:

$$(-s^2\dot{Y} - 2sY + 1) + (sY - 1) + 2Y = 0$$

$$\dot{Y} + \left(\frac{1}{s} - \frac{2}{s^2}\right) Y = 0$$

This is a 1st order linear differential equation in independent variable s. Its integrating factor is $\mu(s) = e^{\int \left(\frac{1}{s} - \frac{2}{s^2}\right) ds} = se^{\frac{2}{s}}$
Recall that for the 1st order linear d.e.'s we have

$$[e^{\int P(x)dx} y]' = e^{\int P(x)dx} Q(x) \qquad \text{(cf. 34)}$$

Thus

$$\left[Y(s)se^{\frac{2}{s}}\right]' = 0 \rightarrow Y(s) = \frac{Ce^{-\frac{2}{s}}}{s}$$

Example 139 (cont.)

Power series expansion of the exponential term for $|s| > 0$ yields:

$$Y(s) = \frac{Ce^{-\frac{2}{s}}}{s} = \frac{C}{s}\sum_{n=0}^{\infty}\frac{(-1)^n 2^n}{n!s^n} = \frac{C}{s}\left(1 - \frac{2}{s} + \frac{2}{s^2} - \frac{4}{3s^3} + \dots\right)$$

$$Y(s) = C\sum_{n=0}^{\infty}\frac{(-1)^n 2^n}{n!s^{n+1}} = C\left(\frac{1}{s} - \frac{2}{s^2} + \frac{2}{s^3} - \frac{4}{3s^4} + \dots\right)$$

Now take the inverse Laplace transform:

$$y(t) = C\sum_{n=0}^{\infty}\frac{(-1)^n 2^n t^n}{(n!)^2} = C\left(1 - 2t + t^2 - \frac{2}{9}t^3 + \dots\right)$$

The condition $y(0) = 1$ gives $C = 1$. Thus the result is

$$y(t) = \sum_{n=0}^{\infty}\frac{(-1)^n 2^n t^n}{(n!)^2} = 1 - 2t + t^2 - \frac{2}{9}t^3 + \dots$$

Bessel's Equation

Consider the second order linear differential equation

$$t^2 \frac{d^2 y}{dt^2} + t\frac{dy}{dt} + (t^2 - v^2)y = 0 \qquad (101)$$

where v is a constant. Equation (101) is known as Bessel's equation. For each constant v, Bessel's equation has two linearly independent solutions denoted by $J_v(t)$ and $Y_v(t)$, called, respectively, Bessel function of order v of the 1st and 2nd kind. Consider the zero-th order case, i.e., $v = 0$:

$$t\frac{d^2 y}{dt^2} + \frac{dy}{dt} + ty = 0$$

with initial conditions $y(0) = 3$, $\dot{y}(0) = 0$. Let us transform the above equation into the Laplace domain.

$$-\frac{d}{ds}[s^2 Y - sy(0) - \dot{y}(0)] + [sY - y(0)] - \frac{d}{ds}Y = 0$$

$$-\frac{d}{ds}[s^2Y - sy(0) - \dot{y}(0)] + [sY - y(0)] - \frac{d}{ds}Y = 0$$

$$-\frac{d}{ds}[s^2Y - 3s] + [sY - 3] - \frac{d}{ds}Y = 0$$

$$-s^2\frac{dY}{ds} - 2sY + 3 + sY - 3 - \frac{d}{ds}Y = 0$$

$$(s^2+1)\frac{dY}{ds} + sY = 0$$

$$\frac{dY}{Y} + \frac{sds}{s^2+1} = 0$$

$$\ln Y + \frac{1}{2}\ln(s^2+1) = c_1$$

$$\ln Y(s^2+1)^{\frac{1}{2}} = c_1, \quad Y(s^2+1)^{\frac{1}{2}} = e^{c_1}, \; Y = \frac{c}{\sqrt{s^2+1}}$$

$$y(t) = cJ_0(t)$$

$$y(t) = cJ_0(t)$$

Noting that $y(0) = 3$ and $J_0(0) = 1$ we write

$$y(0) = cJ_0(0) \rightarrow 3 = c \cdot 1 \rightarrow c = 3$$

Solution is, therefore,

$$y(t) = 3J_0(t)$$

Next we obtain an explicit representation of J_0 in terms of known elementary functions. This will show that $J_0(0) = 1$ is indeed true.

We used the fact

$$\mathcal{L}(J_0) = \frac{1}{\sqrt{s^2+1}}, \ s > 0$$

$$\mathcal{L}(J_0) = \frac{1}{\sqrt{s^2(1+\frac{1}{s^2})}} = \frac{1}{s}(1+\frac{1}{s^2})^{-\frac{1}{2}}, \ s > 0$$

Binomial theorem "$(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \cdots$" implies

$$\mathcal{L}(J_0) = \frac{1}{s} - \frac{1}{2}\frac{1}{s^3} + \frac{3}{8}\frac{1}{s^5} - \frac{5}{16}\frac{1}{s^7} + \cdots, \quad s > 0$$

Now we take the inverse Laplace transform:

$$J_0(t) = 1 - \frac{t^2}{4} + \frac{t^4}{64} - \frac{t^6}{2304} + \cdots$$

$$J_0(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{2^{2n}(n!)^2}$$

is the power series representation of the 0th order Bessel function of the 1st kind.

The matrix method

Consider the linear system of the form

$$
\begin{aligned}
\frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\
\frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\
&\vdots \\
\frac{dx_n}{dt} &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n
\end{aligned}
\tag{102}
$$

Define

$$
A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}; \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}
$$

Now (102) can be written as

$$
\frac{dX}{dt} = AX \tag{103}
$$

Example 140

Consider the linear system of the form

$$\begin{aligned} \frac{dx_1}{dt} &= 4x_1 + 3x_2 - 3x_3 \\ \frac{dx_2}{dt} &= 6x_2 + 2x_3 \\ \frac{dx_3}{dt} &= x_1 + 5x_3 \end{aligned}$$

Define

$$A \triangleq \begin{bmatrix} 4 & 3 & -3 \\ 0 & 6 & 2 \\ 1 & 0 & 5 \end{bmatrix}; \quad X \triangleq \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

A compact notation

$$\frac{dX}{dt} = AX$$

Definition 33

By a solution of the system (102), that is, of the vector differential equation (103), we mean an $n \times 1$ column vector function

$$\Phi = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_n \end{bmatrix}$$

whose components $\phi_1, \phi_2, \ldots, \phi_n$ have a continuous derivative on the real interval $a \leq t \leq b$, and

$$\begin{aligned}
\frac{d\phi_1}{dt} &= a_{11}\phi_1 + a_{12}\phi_2 + \cdots + a_{1n}\phi_n \\
\frac{d\phi_2}{dt} &= a_{21}\phi_1 + a_{22}\phi_2 + \cdots + a_{2n}\phi_n \\
&\vdots \\
\frac{d\phi_n}{dt} &= a_{n1}\phi_1 + a_{n2}\phi_2 + \cdots + a_{nn}\phi_n
\end{aligned}$$

holds for all t on $a \leq t \leq b$.

Example 141

Consider the linear system of the form

$$\begin{aligned}\frac{dx_1}{dt} &= x_1 - 3x_2 \\ \frac{dx_2}{dt} &= -2x_1 + 2x_2\end{aligned}$$

Following vector function is a solution:

$$\Phi = \begin{bmatrix} e^{4t} \\ -e^{4t} \end{bmatrix}$$

Above function has continuous derivatives and satisfies the linear system in $(-\infty, \infty)$:

$$\begin{aligned}\frac{d(e^{4t})}{dt} &= e^{4t} - 3(-e^{4t}) \\ \frac{d(-e^{4t})}{dt} &= -2e^{4t} + 2(-e^{4t})\end{aligned}$$

$$\begin{aligned}
\frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\
\frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\
&\vdots \\
\frac{dx_n}{dt} &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n
\end{aligned} \qquad \text{(cf. 102)}$$

$$\frac{dX}{dt} = AX \qquad \text{(cf. 103)}$$

Theorem 36

Any linear combination of the solutions of the homogeneous linear system (102) is itself a solution of the system (102).

Example 142

Consider the linear system of the form

$$\begin{aligned} \frac{dx_1}{dt} &= x_1 - 3x_2 \\ \frac{dx_2}{dt} &= -2x_1 + 2x_2 \end{aligned}$$

and the functions

$$\Phi_1 = \begin{bmatrix} e^{4t} \\ -e^{4t} \end{bmatrix}, \quad \Phi_2 = \begin{bmatrix} 3e^{-t} \\ 2e^{-t} \end{bmatrix}$$

Φ_1 and Φ_2 are solutions to the given linear system in $(-\infty, \infty)$, so is their linear combination

$$7\begin{bmatrix} e^{4t} \\ -e^{4t} \end{bmatrix} + 9\begin{bmatrix} 3e^{-t} \\ 2e^{-t} \end{bmatrix}$$

$$
\begin{aligned}
\frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\
\frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\
&\vdots \\
\frac{dx_n}{dt} &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n
\end{aligned} \qquad \text{(cf. 102)}
$$

$$\frac{dX}{dt} = AX \qquad \text{(cf. 103)}$$

Theorem 37

There exist n linearly independent solutions of the homogeneous linear system (102). Every solution of system (102) can be written as a linear combination of any n linearly independent solutions of (102).

Definition 34

Let

$$\Phi_1 = \begin{bmatrix} \phi_{11} \\ \phi_{21} \\ \vdots \\ \phi_{n1} \end{bmatrix}; \quad \Phi_2 = \begin{bmatrix} \phi_{12} \\ \phi_{22} \\ \vdots \\ \phi_{n2} \end{bmatrix}; \cdots ; \Phi_n = \begin{bmatrix} \phi_{1n} \\ \phi_{2n} \\ \vdots \\ \phi_{nn} \end{bmatrix}$$

be n linearly independent solutions of the homogeneous linear system (102). Let $c_1, c_2, \ldots, c_n$ be n arbitrary constants. Then the solution

$$X = c_1\Phi_1 + c_2\Phi_2 + \cdots + c_n\Phi_n$$

is called a general solution of the system (102).

Example 143

Consider the linear system of the form

$$\begin{aligned} \frac{dx_1}{dt} &= x_1 - 3x_2 \\ \frac{dx_2}{dt} &= -2x_1 + 2x_2 \end{aligned}$$

The functions Φ_1 and Φ_2

$$\Phi_1 = \begin{bmatrix} e^{4t} \\ -e^{4t} \end{bmatrix}, \quad \Phi_2 = \begin{bmatrix} 3e^{-t} \\ 2e^{-t} \end{bmatrix}$$

are two linearly independent solutions to the given linear system in $(-\infty, \infty)$. General solution of the linear system is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} e^{4t} \\ -e^{4t} \end{bmatrix} + c_2 \begin{bmatrix} 3e^{-t} \\ 2e^{-t} \end{bmatrix}$$

where c_1 and c_2 are arbitrary constants.

Definition 35

Consider the n vector functions $\Phi_1, \Phi_2, \ldots, \Phi_n$ defined respectively, by

$$\Phi_1 = \begin{bmatrix} \phi_{11} \\ \phi_{21} \\ \vdots \\ \phi_{n1} \end{bmatrix}; \quad \Phi_2 = \begin{bmatrix} \phi_{12} \\ \phi_{22} \\ \vdots \\ \phi_{n2} \end{bmatrix}; \cdots ; \Phi_n = \begin{bmatrix} \phi_{1n} \\ \phi_{2n} \\ \vdots \\ \phi_{nn} \end{bmatrix}$$

The $n \times n$ determinant

$$\begin{vmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & & & \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{vmatrix}$$

is called Wronskian of the n vector functions $\Phi_1, \Phi_2, \ldots, \Phi_n$. We will denote its value at t by $W(\Phi_1(t), \Phi_2(t), \ldots, \Phi_n(t))$.

$$\begin{aligned} \frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ \frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ &\vdots \\ \frac{dx_n}{dt} &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{aligned} \qquad \text{(cf. 102)}$$

Theorem 38

n solutions $\Phi_1, \Phi_2, \ldots, \Phi_n$ of the homogeneous linear system (102) are linearly independent on an interval $a \leq t \leq b$ if and only if $W(\Phi_1(t), \Phi_2(t), \ldots, \Phi_n(t)) \neq 0$ for all $t \in [a, b]$.

Theorem 39

Let $\Phi_1, \Phi_2, \ldots, \Phi_n$ *be n solutions of the homogeneous linear differential equation (102) on an interval* $a \le t \le b$*. Then either* $W(\Phi_1(t), \Phi_2(t), \ldots, \Phi_n(t)) = 0$ *for all* $t \in [a, b]$ *or* $W(\Phi_1(t), \Phi_2(t), \ldots, \Phi_n(t)) = 0$ *for no* $t \in [a, b]$*.*

Example 144

Consider the linear system of the form

$$\begin{aligned} \frac{dx_1}{dt} &= x_1 - 3x_2 \\ \frac{dx_2}{dt} &= -2x_1 + 2x_2 \end{aligned}$$

and their solutions

$$\Phi_1 = \begin{bmatrix} e^{4t} \\ -e^{4t} \end{bmatrix}, \quad \Phi_2 = \begin{bmatrix} 3e^{-t} \\ 2e^{-t} \end{bmatrix}$$

Their Wronskian is

$$W(\Phi_1(t), \Phi_2(t)) = \begin{vmatrix} e^{4t} & 3e^{-t} \\ -e^{4t} & 2e^{-t} \end{vmatrix} = e^{4t} \cdot 2e^{-t} - 3e^{-t} \cdot (-e^{4t}) = 5e^{3t}$$

Since $5e^{3t} \neq 0$, Φ_1 and Φ_2 are linearly independent.

Solutions have the form $X(t) = Ve^{\lambda t}$ where $V = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ and λ is a scalar. Recalling

$$\frac{dX(t)}{dt} = AX(t) \qquad \text{(cf. 103)}$$

substitute $X(t) = Ve^{\lambda t}$ into (103) to obtain

$$\begin{aligned} \lambda Ve^{\lambda t} &= AVe^{\lambda t} \\ \rightarrow AV &= \lambda V \\ \rightarrow AV &= \lambda IV \\ \rightarrow AV - \lambda IV &= 0 \\ (A - \lambda I)V &= 0 \end{aligned}$$

$$(A - \lambda I)V = 0 \qquad (104)$$

$$\left(\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 1 \end{bmatrix}\right) \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

This equation holds only for certain λ and $\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ pairs.

This equation set has a non trivial solution if and only if

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0,$$

or in matrix notation $|A - \lambda I| = 0$. This is called **characteristic equation** for system (103). The λ values satisfying the characteristic equation are called **characteristic values** of (103). Solutions of (104) corresponding to characteristic values are called **characteristic vectors** of (103). Recall that

$$\frac{dX}{dt} = AX \qquad \text{(cf. 103)}$$

$$(A - \lambda I)V = 0 \qquad \text{(cf. 104)}$$

$$\frac{dX}{dt} = AX \qquad \text{(cf. 103)}$$

General solution form of (103) depends on characteristic values and characteristic vectors of A. There are 2 cases:

Case 1 A has n distinct characteristic values.

Case 2 A has less than n distinct characteristic values.
In case 2 at least one of the characteristic values is repeated. Denote its multiplicity by m, and its number of linearly independent characteristic vectors by p (where $p \leq m$)

Subcase 2a $p = m$

Subcase 2b $p < m$

Case 1: n distinct characteristic values

Suppose that each of the n characteristic values $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the $n \times n$ square coefficient matrix A of the differential equation is distinct and let $V^{(1)}, V^{(2)}, \ldots, V^{(n)}$ be a set of n respective corresponding characteristic vectors of A. Then the n distinct vector functions $X_1, X_2, \ldots, X_n$ defined respectively by

$$X_1(t) = V^{(1)} e^{\lambda_1 t} = \begin{bmatrix} v_{11} \\ v_{21} \\ \vdots \\ v_{n1} \end{bmatrix} e^{\lambda_1 t}, \; X_2(t) = V^{(2)} e^{\lambda_2 t} = \begin{bmatrix} v_{12} \\ v_{22} \\ \vdots \\ v_{n2} \end{bmatrix} e^{\lambda_2 t}$$

$$\ldots, X_n(t) = V^{(n)} e^{\lambda_n t} = \begin{bmatrix} v_{1n} \\ v_{2n} \\ \vdots \\ v_{nn} \end{bmatrix} e^{\lambda_n t}$$

are solutions of the vector differential equation (103) on every real interval $[a, b]$. This can be verified by direct substitution.

Now consider the Wronskian of the n solutions $X_1, X_2, \ldots, X_n$:

$$\begin{vmatrix} v_{11}e^{\lambda_1 t} & v_{12}e^{\lambda_2 t} & \cdots & v_{1n}e^{\lambda_n t} \\ v_{21}e^{\lambda_1 t} & v_{22}e^{\lambda_2 t} & \cdots & v_{2n}e^{\lambda_n t} \\ \vdots & & & \vdots \\ v_{n1}e^{\lambda_1 t} & v_{n2}e^{\lambda_2 t} & \cdots & v_{nn}e^{\lambda_n t} \end{vmatrix}$$

$$= e^{(\lambda_1+\lambda_2+\cdots+\lambda_n)t} \begin{vmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & & & \vdots \\ v_{n1} & v_{n2} & \cdots & v_{nn} \end{vmatrix} \neq 0$$

Since exponential functions never result in zero, and from linear algebra eigenvectors corresponding to distinct eigenvalues are linearly independent which makes the determinant above nonzero. The n solutions $X_1, X_2, \ldots, X_n$ are linearly independent.

Theorem 40

Consider the homogeneous linear system

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \ldots & \ldots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

That is, the vector differential equation

$$\frac{dX}{dt} = AX$$

with obvious definitions. Suppose each of the n characteristic values $\lambda_1, \lambda_2, \ldots, \lambda_n$ of A is distinct; and let $V^{(1)}, V^{(2)}, \ldots, V^{(n)}$ be a set of respective corresponding characteristic vectors of A.

$$
\begin{aligned}
\frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\
\frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\
&\vdots \\
\frac{dx_n}{dt} &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n
\end{aligned}
\qquad \text{(cf. 102)}
$$

Theorem 40 (cont.)

Then on every real interval, the n vector functions defined by

$$V^{(1)}e^{\lambda_1 t}, V^{(2)}e^{\lambda_2 t}, \ldots, V^{(n)}e^{\lambda_n t}$$

form a linearly independent set of solutions of (102), and

$$X(t) = c_1 V^{(1)}e^{\lambda_1 t} + c_2 V^{(2)}e^{\lambda_2 t} + \ldots + c_n V^{(n)}e^{\lambda_n t},$$

where $c_1, c_2, \ldots, c_n$ *are n arbitrary constants, is a general solution of (102).*

Example 145

Consider

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{bmatrix} = \begin{bmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

or in vector-matrix notation

$$\frac{dX}{dt} = AX$$

Example 145 (cont.)

$$\rightarrow |A - \lambda I| = \begin{vmatrix} 7-\lambda & -1 & 6 \\ -10 & 4-\lambda & -12 \\ -2 & 1 & -1-\lambda \end{vmatrix} = \lambda^3 - 10\lambda^2 + 31\lambda - 30$$

Characteristic values are obtained by equating characteristic polynomial above to zero:

$$\lambda_1 = 2,\ \lambda_2 = 3,\ \lambda_3 = 5$$

Let us find characteristic vectors for each characteristic value. To find a characteristic vector for $\lambda_1 = 2$, we need to solve

$$(A - \lambda_1 I)V = 0 \text{ or } \begin{bmatrix} 7-\lambda_1 & -1 & 6 \\ -10 & 4-\lambda_1 & -12 \\ -2 & 1 & -1-\lambda_1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Example 145 (cont.)

$$\begin{bmatrix} 7-\lambda_1 & -1 & 6 \\ -10 & 4-\lambda_1 & -12 \\ -2 & 1 & -1-\lambda_1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 7-2 & -1 & 6 \\ -10 & 4-2 & -12 \\ -2 & 1 & -1-2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -1 & 6 \\ -10 & 2 & -12 \\ -2 & 1 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \underbrace{\rightarrow}_{GaussianElim.} V^{(1)} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

Example 145 (cont.)

Next find a characteristic vector for $\lambda_2 = 3$

$$\begin{bmatrix} 7-\lambda_2 & -1 & 6 \\ -10 & 4-\lambda_2 & -12 \\ -2 & 1 & -1-\lambda_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 7-3 & -1 & 6 \\ -10 & 4-3 & -12 \\ -2 & 1 & -1-3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -1 & 6 \\ -10 & 1 & -12 \\ -2 & 1 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \underbrace{\rightarrow}_{GaussianElim.} \quad V^{(2)} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$$

Example 145 (cont.)

Next find a characteristic vector for $\lambda_3 = 5$

$$\begin{bmatrix} 7-\lambda_3 & -1 & 6 \\ -10 & 4-\lambda_3 & -12 \\ -2 & 1 & -1-\lambda_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 7-5 & -1 & 6 \\ -10 & 4-5 & -12 \\ -2 & 1 & -1-5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 6 \\ -10 & -1 & -12 \\ -2 & 1 & -6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \underbrace{\rightarrow}_{GaussianElim.} V^{(3)} = \begin{bmatrix} 3 \\ -6 \\ -2 \end{bmatrix}$$

Example 145 (cont.)

For $\lambda = \lambda_1 = 2 \rightarrow V^{(1)} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

For $\lambda = \lambda_2 = 3 \rightarrow V^{(2)} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$

For $\lambda = \lambda_3 = 5 \rightarrow V^{(3)} = \begin{bmatrix} 3 \\ -6 \\ -2 \end{bmatrix}$ We have distinct characteristic values and corresponding characteristic vectors. For a general solution, we use them in the solution formula:

$$X(t) = c_1 \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} e^{3t} + c_3 \begin{bmatrix} 3 \\ -6 \\ -2 \end{bmatrix} e^{5t}$$

Example 146

Consider

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

It has the characteristic values $2, 3$ and 5, which are real and distinct.

Example 146 (cont.)

Characteristic values and corresponding characteristic vectors are

$$\lambda_1 = 2 \rightarrow K \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 3 \rightarrow L \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

$$\lambda_3 = 5 \rightarrow M \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

for arbitrary nonzero constants K, L, and M.

Example 146 (cont.)

For $\lambda_1 = 2 \rightarrow V^{(1)} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

For $\lambda_2 = 3 \rightarrow V^{(2)} = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$

For $\lambda_3 = 5 \rightarrow V^{(3)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

We have distinct characteristic values and corresponding characteristic vectors. A general solution:

$$X(t) = c_1 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} e^{3t} + c_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{5t}$$

Case 2: Repeated characteristic values

We again consider the vector differential equation

$$\frac{dX}{dt} = AX$$

where A is an $n \times n$ real constant matrix. We suppose that A has a real characteristic value λ_1 of multiplicity m, where $1 < m \leq n$, and that all the other characteristic values $\lambda_{m+1}, \lambda_{m+2}, \ldots, \lambda_n$ (if there are any) are distinct.

Example 147

Let 6×6 matrix A have the characteristic equation $(\lambda - 7)^4(\lambda - 2)(\lambda - 5) = 0$. Here $\lambda_1 = 7$ repeated 4 times; $\lambda_5 = 2$ and $\lambda_6 = 5$ are distinct. Linear algebra says we obtain 4 or less linearly independent characteristic vectors for $\lambda_1 = 7$, depending on the matrix A.

We know that the repeated characteristic value λ_1 of multiplicity m has p linearly independent characteristic vectors, where $1 \leq p \leq m$. Now we consider two subcases (1) $p = m$ and (2) $p < m$.

Case 1 If $p = m$ then we will have totally n linearly independent characteristic vectors for the matrix A. In this case the general solution has the form that is the same as the one for all distinct characteristic values. The next example illustrates this:

Example 148

Consider

$$\frac{dX}{dt} = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 3 & 3 & -1 \end{bmatrix} X$$

or in vector-matrix notation

$$\frac{dX}{dt} = AX$$

$$\rightarrow |A - \lambda I| = \begin{vmatrix} 3-\lambda & 1 & -1 \\ 1 & 3-\lambda & -1 \\ 3 & 3 & -1-\lambda \end{vmatrix} = \lambda^3 - 5\lambda^2 + 8\lambda - 4$$

Characteristic values are obtained by equating characteristic polynomial to zero:

$$\underbrace{\lambda_1 = 1}_{\text{distinct}},\ \underbrace{\lambda_2 = 2,\ \lambda_3 = 2}_{\text{repeated}}$$

Example 148 (cont.)

Evaluate $(A - \lambda I)V = 0$ at the characteristic values:
At $\lambda = 1$

$$V^{(1)} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

is a characteristic vector.
At $\lambda = 2$

$$V^{(2)} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad V^{(3)} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

are characteristic vectors. $V^{(2)}$ and $V^{(3)}$ are linearly independent.
General solution is

$$X(t) = c_1 \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t}$$

Case 2 $p < m$: In this case there are less than m linearly independent characteristic vectors corresponding to the characteristic value λ_1 of multiplicity m.
Hence there are less than m linearly independent solutions of system (102) of the form $Ve^{\lambda_1 t}$ corresponding to λ_1. Thus there is not a full set of n linearly independent solutions of (102) of basic exponential form $Ve^{\lambda t}$.
Clearly we must seek linearly independent solutions of another form.

Let λ be a characteristic value of multiplicity $m = 2$. Suppose $p = 1 < m$, so that there is only one type of characteristic vector V and hence only one type of solution of the basic exponential form $Ve^{\lambda t}$ corresponding to λ. We need two linearly independent solutions in order to write the general solution. The second solution is of the form

$$(Vt + W)e^{\lambda t}$$

together with $Ve^{\lambda t}$ form a linearly independent set of two solutions. Substituting this in the differential equation

$$\frac{dX}{dt} = AX$$

yields

$$(Vt + W)\lambda e^{\lambda t} + Ve^{\lambda t} = A(Vt + W)e^{\lambda t}$$

$$(Vt + W)\lambda e^{\lambda t} + Ve^{\lambda t} = A(Vt + W)e^{\lambda t}$$

Dividing throughout by $e^{\lambda t}$ and rearranging, this can be written as

$$(\lambda V - AV)t + (\lambda W + V - AW) = 0$$

This implies

$$(A - \lambda I)V = \underline{0}$$

$$\lambda W + V - AW = 0$$

We already know the V satisfying the first equation. From the second equation we want to find W:

$$(A - \lambda I)W = V$$

Upon finding W, the general solution will be

$$X(t) = c_1 Ve^{\lambda t} + c_2(Vt + W)e^{\lambda t}$$

Now let λ be a characteristic value of multiplicity $m = 3$, and suppose $p < m$. Here there are two possibilities: $p = 1$ and $p = 2$. If $p = 1$, there is only one type of characteristic vector V and hence only one type of solution of the form

$$Ve^{\lambda t}$$

corresponding to λ. Then a second solution corresponding to λ is of the form

$$(Vt + W)e^{\lambda t}$$

Substituting this in the d.e. $\frac{dX}{dt} = AX$ results

$$(Vt + W)\lambda e^{\lambda t} + Ve^{\lambda t} = A(Vt + W)e^{\lambda t}$$

$$(Vt + W)\lambda e^{\lambda t} + Ve^{\lambda t} = A(Vt + W)e^{\lambda t}$$

Dividing throughout by $e^{\lambda t}$ and rearranging, this can be written as

$$(\lambda V - AV)t + (\lambda W + V - AW) = 0$$

This implies

$$(A - \lambda I)V = \underline{0}$$

$$\lambda W + V - AW = 0$$

We already know the V satisfying the first equation. From the second equation we want to find W:

$$(A - \lambda I)W = V$$

Upon finding W, an already found part of the general solution will be

$$X(t) = c_1 Ve^{\lambda t} + c_2(Vt + W)e^{\lambda t}$$

In this case the third solution corresponding to λ is of the form

$$(V\frac{t^2}{2} + Wt + Z)e^{\lambda t}$$

Upon substituting this in the d.e. $\frac{dX}{dt} = AX$ we observe that Z satisfies

$$(A - \lambda I)Z = W$$

Z obtained from this is used in the third solution. These three solutions obtained are linearly independent. The general solution will be

$$X(t) = c_1 Ve^{\lambda t} + c_2(Vt + W)e^{\lambda t} + c_3(V\frac{t^2}{2} + Wt + Z)e^{\lambda t}$$

If $p = 2$, there are two linearly independent characteristic vectors $V^{(1)}$ and $V^{(2)}$ corresponding to λ and hence there are two linearly independent solutions of the form

$$V^{(1)}e^{\lambda t} \text{ and } V^{(2)}e^{\lambda t}$$

Then a third solution corresponding to λ is of the form

$$(Vt + W)e^{\lambda t}$$

where V satisfies

$$(A - \lambda I)V = \underline{0} \tag{105}$$

and W satisfies

$$(A - \lambda I)W = V \tag{106}$$

The V in (105) is defined by $k_1 V^{(1)} + k_2 V^{(2)}$. We need to determine k_1 and k_2 which satisfy (106).

Example 149

Consider

$$\frac{dX}{dt} = \begin{bmatrix} 4 & 3 & 1 \\ -4 & -4 & -2 \\ 8 & 12 & 6 \end{bmatrix} X$$

or in vector-matrix notation

$$\frac{dX}{dt} = AX$$

$$\rightarrow |A - \lambda I| = \begin{vmatrix} 4-\lambda & 3 & 1 \\ -4 & -4-\lambda & -2 \\ 8 & 12 & 6-\lambda \end{vmatrix} = \lambda^3 - 6\lambda^2 + 12\lambda - 8$$

Characteristic values are obtained by equating characteristic polynomial to zero:

$$\lambda_1 = \lambda_2 = \lambda_3 = 2$$

Example 149 (cont.)

Evaluate $(A - \lambda I)V = 0$ at the characteristic value. We obtain two linearly independent characteristic vectors:

$$\rightarrow V^{(1)} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \quad V^{(2)} = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$$

For the third solution we solve

$$(A - \lambda I)W = V$$

with

$$V = k_1 V^{(1)} + k_2 V^{(2)} = \begin{bmatrix} k_1 \\ k_2 \\ -2k_1 - 3k_2 \end{bmatrix}$$

Example 149 (cont.)

$$(A-2I)W = V \to \begin{bmatrix} 2 & 3 & 1 \\ -4 & -6 & -2 \\ 8 & 12 & 4 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ -2k_1 - 3k_2 \end{bmatrix}$$

Notice that rows on the lefthand side of the equality are proportional. For consistency we must have $k_2 = -2k_1$. Select $k_1 = 1,\ k_2 = -2 \to V = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$ Solving for W we obtain $W = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Thus the general solution is

$$X(t) = c_1 \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} e^{2t} + c_3 \left(\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} t + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) e^{2t}$$

Example 150

Consider

$$\frac{dX}{dt} = \begin{bmatrix} 3 & 0 & 1 \\ 2 & 1 & 1 \\ -4 & 0 & -1 \end{bmatrix} X, \quad X(0) = \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix}$$

or in vector-matrix notation

$$\frac{dX}{dt} = AX$$

$$\rightarrow |A - \lambda I| = \begin{vmatrix} 3-\lambda & 0 & 1 \\ 2 & 1-\lambda & 1 \\ -4 & 0 & -1-\lambda \end{vmatrix} = (\lambda - 1)^3$$

Characteristic values are $\lambda_1 = \lambda_2 = \lambda_3 = 1$

Example 150 (cont.)

Evaluate $(A - \lambda I)_{\lambda=1} V = 0$. We obtain two linearly independent characteristic vectors:

$$\to V^{(1)} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \quad V^{(2)} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

For the third solution we solve

$$(A - \lambda I)W = V$$

with

$$V = k_1 V^{(1)} + k_2 V^{(2)} = \begin{bmatrix} k_1 + k_2 \\ k_2 \\ -2k_1 - 2k_2 \end{bmatrix}$$

$$V = k_1 V^{(1)} + k_2 V^{(2)} = \begin{bmatrix} k_1 + k_2 \\ k_2 \\ -2k_1 - 2k_2 \end{bmatrix}$$

Example 150 (cont.)

$$(A - 1 \cdot I)W = V$$

$$\rightarrow \begin{bmatrix} 2 & 0 & 1 \\ 2 & 0 & 1 \\ -4 & 0 & -2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} k_1 + k_2 \\ k_2 \\ -2k_1 - 2k_2 \end{bmatrix}$$

For consistency we must have $k_1 = 0$. Arbitrarily assign 1 to k_2:

$$\begin{bmatrix} 2 & 0 & 1 \\ 2 & 0 & 1 \\ -4 & 0 & -2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 1 \\ 2 & 0 & 1 \\ -4 & 0 & -2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

Example 150 (cont.)

A solution is $W = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Thus the general solution is

$$X(t) = c_1 \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} e^t + c_3 \left(\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right) e^t$$

Note that for different W solutions, one gets another equivalent form of the general solution.

$$X(t) = c_1 \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} e^t + c_3 \left(\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right) e^t$$

Example 150 (cont.)

Let us use the initial conditions:

$$\begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \cdot 1 + c_2 \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \cdot 1 + c_3 \left(\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \cdot 0 + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right) \cdot 1$$

$$\rightarrow \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$$

Thus the solution is

$$X(t) = - \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} e^t + 3 \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} e^t$$

Example 151

Consider

$$\frac{dX}{dt} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 2 & -2 & 3 \end{bmatrix} X$$

or in vector-matrix notation

$$\frac{dX}{dt} = AX$$

$$\rightarrow |A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 & 0 \\ 1 & 4-\lambda & 1 \\ 2 & -2 & 3-\lambda \end{vmatrix} = (\lambda - 3)^3$$

Characteristic values are $\lambda_1 = \lambda_2 = \lambda_3 = 3$

Example 151 (cont.)

Evaluate $(A - \lambda I)_{\lambda=3} V = 0$. We obtain one linearly independent characteristic vector:

$$\rightarrow V = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

The general solution has the form

$$X(t) = c_1 V e^{\lambda t} + c_2 (Vt + W) e^{\lambda t} + c_3 (V \frac{t^2}{2} + Wt + Z) e^{\lambda t}$$

For the second term above we need to solve

$$(A - \lambda I) W = V$$

that is,

$$\rightarrow \begin{bmatrix} -1 & 1 & 0 \\ 1 & 1 & 1 \\ 2 & -2 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

$$X(t) = c_1 V e^{\lambda t} + c_2(Vt + W)e^{\lambda t} + c_3(V\frac{t^2}{2} + Wt + Z)e^{\lambda t}$$

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & 1 & 1 \\ 2 & -2 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

Example 151 (cont.)

A solution is $W = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

For the third term in the general solution, we solve

$$\rightarrow \begin{bmatrix} -1 & 1 & 0 \\ 1 & 1 & 1 \\ 2 & -2 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$X(t) = c_1 Ve^{\lambda t} + c_2(Vt + W)e^{\lambda t} + c_3(V\frac{t^2}{2} + Wt + Z)e^{\lambda t}$$

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & 1 & 1 \\ 2 & -2 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Example 151 (cont.)

A solution is $Z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Thus the general solution is

$$X(t) = c_1 \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} e^{3t} + c_2(\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix})e^{3t}$$

$$+c_3(\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \frac{t^2}{2} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix})e^{3t}$$

Example 152

Find a general solution for the differential equation set

$$\dot{x} = \begin{bmatrix} 12 & -8 \\ 16 & -4 \end{bmatrix} x$$

Finding the characteristic values and vectors:

$$\left| \lambda I - \begin{bmatrix} 12 & -8 \\ 16 & -4 \end{bmatrix} \right| = \begin{vmatrix} \lambda - 12 & 8 \\ -16 & \lambda + 4 \end{vmatrix} = \lambda^2 - 8\lambda + 80$$

Characteristic values are $4 + 8i$ and $4 - 8i$.
The following solution procedure for this problem is similar to that of Example 76.

Example 152 (cont.)

Characteristic vector for $4+8i$:

$$(A-\lambda I)V|_{\lambda=4+8i}=0 \rightarrow \begin{bmatrix} 4+8i-12 & 8 \\ -16 & 4+8i+4 \end{bmatrix} V = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -8+8i & 8 \\ -16 & 8+8i \end{bmatrix} V = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\begin{bmatrix} 1+i \\ 2 \end{bmatrix}$ is a characteristic vector.

Characteristic vector for $4-8i$:

$$(A-\lambda I)V|_{\lambda=4-8i}=0 \rightarrow \begin{bmatrix} 4-8i-12 & 8 \\ -16 & 4-8i+4 \end{bmatrix} V = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -8-8i & 8 \\ -16 & 8-8i \end{bmatrix} V = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\begin{bmatrix} 1-i \\ 2 \end{bmatrix}$ is a characteristic vector.

Example 152 (cont.)

$$x(t) = c_1 e^{(4+8i)t} \begin{bmatrix} 1+i \\ 2 \end{bmatrix} + c_2 e^{(4-8i)t} \begin{bmatrix} 1-i \\ 2 \end{bmatrix} \qquad (107)$$

$x(t)$ satisfies the d.e, however, it is not a solution; it is not real. Consider the first term in Eq. (107):

$$x(t) = c_1 e^{(4+8i)t} \begin{bmatrix} 1+i \\ 2 \end{bmatrix} = c_1 e^{4t}(\cos 8t + i \sin 8t)(\begin{bmatrix} 1 \\ 2 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix})$$

Decompose it into real and imaginary components:

$$x(t) = c_1 e^{4t} \left(\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cos 8t - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin 8t \right\} + i \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos 8t + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \sin 8t \right\} \right.$$

Example 152 (cont.)

$$x(t) = c_1 e^{4t}\left(\left\{\begin{bmatrix}1\\2\end{bmatrix}\cos 8t - \begin{bmatrix}1\\0\end{bmatrix}\sin 8t\right\} + i\left\{\begin{bmatrix}1\\0\end{bmatrix}\cos 8t + \begin{bmatrix}1\\2\end{bmatrix}\sin 8t\right\}\right.$$

Each of the real and imaginary components satisfies the d.e. Each one is real, and they are together linearly independent.

$$e^{4t}\left(\begin{bmatrix}1\\2\end{bmatrix}\cos 8t - \begin{bmatrix}1\\0\end{bmatrix}\sin 8t\right), e^{4t}\left(\begin{bmatrix}1\\0\end{bmatrix}\cos 8t + \begin{bmatrix}1\\2\end{bmatrix}\sin 8t\right)$$

A general solution is, therefore,

$$x(t) = c_1 e^{4t}\left(\begin{bmatrix}1\\2\end{bmatrix}\cos 8t - \begin{bmatrix}1\\0\end{bmatrix}\sin 8t\right) + c_2 e^{4t}\left(\begin{bmatrix}1\\0\end{bmatrix}\cos 8t + \begin{bmatrix}1\\2\end{bmatrix}\sin 8t\right)$$

In Eq. (107), we used the 1st term of the summation to obtain two linearly independent solutions. Alternatively, we could have started with the 2nd term to obtain a general solution.

Example 153

Solve the following

$$\dot{X} = \begin{bmatrix} -1 & \frac{3}{2} \\ \frac{-1}{6} & -2 \end{bmatrix} X, \; X(2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} -1 - \lambda & \frac{3}{2} \\ \frac{-1}{6} & -2 - \lambda \end{vmatrix} = \lambda^2 + 3\lambda + \frac{9}{4} = (\lambda + \frac{3}{2})^2$$

Repeated characteristic values at $\frac{-3}{2}$.

Example 153 (cont.)

$$(A - \lambda I)V = 0 \rightarrow \begin{bmatrix} -1-\lambda & \frac{3}{2} \\ \frac{-1}{6} & -2-\lambda \end{bmatrix} V = 0$$

At $\lambda = \frac{-3}{2}$:

$$\begin{bmatrix} \frac{1}{2} & \frac{3}{2} \\ \frac{-1}{6} & -\frac{1}{2} \end{bmatrix} V = 0$$

This has only one linearly independent solution:

$$V = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

Other linearly independent solution has the form $(Vt + W)e^{\frac{-3}{2}t}$.

Example 153 (cont.)

Substitute $(Vt + W)e^{\frac{-3}{2}t}$ in the d.e. we obtain

$$(A - \lambda I)_{\lambda = \frac{-3}{2}} W = V$$

$$\begin{bmatrix} -1-\lambda & \frac{3}{2} \\ \frac{-1}{6} & -2-\lambda \end{bmatrix}_{\lambda=\frac{-3}{2}} W = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2} & \frac{3}{2} \\ \frac{-1}{6} & -\frac{1}{2} \end{bmatrix} W = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

This leads to

$$W = \begin{bmatrix} -6-3\rho \\ \rho \end{bmatrix}$$

Arbitrarily choosing $\rho = 0$ we have

$$W = \begin{bmatrix} -6 \\ 0 \end{bmatrix}$$

Example 153 (cont.)

Other linearly independent solution is $\left(\begin{bmatrix} -3 \\ 1 \end{bmatrix} t + \begin{bmatrix} -6 \\ 0 \end{bmatrix}\right) e^{\frac{-3}{2}t}$.

The general solution is

$$X(t) = c_1 \begin{bmatrix} -3 \\ 1 \end{bmatrix} e^{\frac{-3}{2}t} + c_2\left(\begin{bmatrix} -3 \\ 1 \end{bmatrix} t + \begin{bmatrix} -6 \\ 0 \end{bmatrix}\right) e^{\frac{-3}{2}t}$$

Apply the initial conditions:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} -3 \\ 1 \end{bmatrix} e^{-3} + c_2\left(\begin{bmatrix} -3 \\ 1 \end{bmatrix} 2 + \begin{bmatrix} -6 \\ 0 \end{bmatrix}\right) e^{-3}$$

Need to solve

$$\begin{aligned} -3e^{-3}c_1 - 12e^{-3}c_2 &= 1 \\ e^{-3}c_1 + 2e^{-3}c_2 &= 0 \end{aligned}$$

$$-3e^{-3}c_1 - 12e^{-3}c_2 = 1$$
$$e^{-3}c_1 + 2e^{-3}c_2 = 0$$

Example 153 (cont.)

$$c_1 = \frac{e^3}{3},\ c_2 = -\frac{e^3}{6}$$

The solution is

$$X(t) = \frac{e^3}{3}\begin{bmatrix} -3 \\ 1 \end{bmatrix} e^{\frac{-3}{2}t} - \frac{e^3}{6}\left(\begin{bmatrix} -3 \\ 1 \end{bmatrix} t + \begin{bmatrix} -6 \\ 0 \end{bmatrix}\right)e^{\frac{-3}{2}t}$$

Sturm-Liouville Boundary Value Problems

Definition 36

Sturm-Liouville BVP is boundary value problem which consists of
(a) A second order homogeneous linear d.e. of the form

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + [q(x) + \lambda r(x)]\, y = 0 \tag{108}$$

where p, q, and r are real functions such that p has a continuous derivative, q and r are continuous, and $p(x) > 0$ and $r(x) > 0$ for all x on a real interval $a \leq x \leq b$; and λ is a parameter independent of x; and
(b) two supplementary conditions

$$\begin{aligned} A_1 y(a) + A_2 y'(a) &= 0 \\ B_1 y(b) + B_2 y'(b) &= 0 \end{aligned} \tag{109}$$

where A_1, A_2, B_1 and B_2 are real constants such that A_1 and A_2 are not both zero, and B_1 and B_2 are not both zero.
This type of problem is called Sturm-Liouville problem.

Remarks: Sturm-Liouville d.e. is a linear differential equation. Continuity of q, r, and derivative of p makes coefficients of y,$\dot{y}$, and $\ddot{y}$ continuous, so that a sufficient condition for existence of a solution is satisfied. Conditions $p(x) > 0$ and $r(x) > 0$ make the problem "regular"; a consequence is that the problem has real "characteristic values" only.

Example 154

The boundary value problem

$$\frac{d}{dx}\left[x\frac{dy}{dx}\right] + \left[2x^2 + \lambda x^3\right] y = 0$$

$$3y(1) + 4y'(1) = 0$$
$$5y(2) - 3y'(2) = 0$$

is a Sturm-Liouville problem.

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + [q(x) + \lambda r(x)]\, y = 0 \qquad \text{(cf. 108)}$$

Example 155

$$\frac{d^2y}{dx^2} + \lambda y = 0$$
$$y(0) = 0, \quad y(\pi) = 0$$

is a Sturm Liouville problem. The differential equation may be written as

$$\frac{d}{dx}\left[\underbrace{1}_{p(x)} \cdot \frac{dy}{dx}\right] + \left[\underbrace{0}_{q(x)} + \lambda \cdot \underbrace{1}_{r(x)}\right] y = 0$$

$$\underbrace{A_1}_{1} y(\underbrace{0}_{a}) + \underbrace{A_2}_{0} y'(\underbrace{0}_{a}) = 0$$

$$\underbrace{B_1}_{1} y(\underbrace{\pi}_{b}) + \underbrace{B_2}_{0} y'(\underbrace{\pi}_{b}) = 0$$

This verifies the claim.

Example 156

Find nontrivial solutions of the Sturm-Liouville problem

$$\frac{d^2y}{dx^2} + \lambda y = 0$$

$$y(0) = 0, \quad y(\pi) = 0$$

Solution

We consider three cases $\lambda = 0$, $\lambda < 0$, and $\lambda > 0$.

Case 1: $\lambda = 0$ reduces the the problem to

$$\frac{d^2y}{dx^2} = 0$$

The general solution is

$$y(x) = c_1 + c_2 x$$

The first condition $y(0) = 0$ yields $c_1 = 0$. The second condition $y(\pi) = c_1 + c_2\pi = 0$ yields $c_2 = 0$.

Example 156 (cont.)

Thus, when $\lambda = 0$ we have $c_1 = c_2 = 0$ and solution is $y(x) = 0 + 0 \cdot x$, the trivial solution.

Case 2: For the d.e

$$\frac{d^2y}{dx^2} + \lambda y = 0, \; y(0) = 0, \quad y(\pi) = 0$$

when $\lambda < 0$, the characteristic equation is

$$m^2 + \lambda = 0$$

Its roots $\pm\sqrt{-\lambda}$ are real and unequal. The corresponding general solution is

$$y(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$$

where $\alpha = \sqrt{-\lambda}$. Apply the conditions $y(0) = 0$ and $y(\pi) = 0$:

$$c_1 + c_2 = 0, \quad c_1 e^{\alpha\pi} + c_2 e^{-\alpha\pi} = 0$$

Example 156 (cont.)

Solve the equations arising from applying the condition:

$$\begin{aligned} c_1 + c_2 &= 0 \\ c_1 e^{\alpha\pi} + c_2 e^{-\alpha\pi} &= 0 \end{aligned}$$

The only solution is $c_1 = c_2 = 0$
$\therefore$ When $\lambda < 0$ we have $c_1 = c_2 = 0$ and

$$y(x) = 0 \cdot e^{\alpha x} + 0 \cdot e^{-\alpha x}$$

the trivial solution.

Example 156 (cont.)

Case 3: $\lambda > 0$ implies that the characteristic equation has the roots $\pm i\sqrt{\lambda}$.This leads to the general solution

$$y(x) = c_1 \sin\sqrt{\lambda}x + c_2 \cos\sqrt{\lambda}x$$

Now apply the condition $y(0) = 0$:

$$c_1 \sin 0 + c_2 \cos 0 = 0$$

This results in $c_2 = 0$. The other condition $y(\pi) = 0$ yields:

$$c_1 \sin\sqrt{\lambda}\pi + c_2 \cos\sqrt{\lambda}\pi = 0$$

Because $c_2 = 0$, this reduces to

$$c_1 \sin\sqrt{\lambda}\pi = 0$$

If we let $c_1 = 0$, then we get a trivial solution. This is not desired. Therefore we make $\sin\sqrt{\lambda}\pi = 0$

The general solution corresponding to $\lambda > 0$ from the previous slide:

$$y(x) = c_1 \sin \sqrt{\lambda}x + \cancel{c_2 \cos \sqrt{\lambda}x} \rightarrow 0$$

Example 156 (cont.)

$\sin \sqrt{\lambda}\pi = 0$ is satisfied if $\sqrt{\lambda} = n$, or equivalently, $\lambda = n^2$. In other words, λ must be a member of the infinite sequence

$$1, 4, 9, 16, \ldots$$

$\therefore$ For $\lambda = n^2$ $(n = 1, 2, 3, \ldots)$ we have nontrivial solutions

$$y(x) = c_n \sin nx,$$

Or, more explicitly

$$c_1 \sin x, c_2 \sin 2x, c_3 \sin 3x, \ldots$$

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + [q(x) + \lambda r(x)]\, y = 0 \qquad \text{(cf. 108)}$$

$$\begin{aligned} A_1 y(a) + A_2 y'(a) &= 0 \\ B_1 y(b) + B_2 y'(b) &= 0 \end{aligned} \qquad \text{(cf. 109)}$$

Definition 37

Consider the Sturm-Liouville equation (108) and the supplementary conditions (109). The values of the parameter λ in (108) for which there exists nontrivial solutions of the problem are called the **characteristic values** of the problem. The corresponding nontrivial solutions themselves are called the **characteristic functions** of the problem.

Example 157

Find the characteristic values and characteristic functions of

$$\frac{d}{dx}\left[x\frac{dy}{dx}\right]+\frac{\lambda}{x}y=0$$

$$y'(1)=0,\quad y'(e^{2\pi})=0$$

where we assume that the parameter λ is nonnegative.

Solution: We consider separately the cases $\lambda = 0$ and $\lambda > 0$.

Case 1: $\lambda = 0$ reduces the problem to

$$\frac{d}{dx}\left[x\frac{dy}{dx}\right]=0$$

Integrate twice for the general solution:

$$x\frac{dy}{dx}=C\rightarrow\frac{dy}{dx}=\frac{C}{x}\rightarrow y(x)=C\ln|x|+C_0$$

where C and C_0 are arbitrary constants.

$$y(x) = C \ln|x| + C_0 \to y'(x) = \frac{C}{x}$$

$$y'(1) = 0, \quad y'(e^{2\pi}) = 0$$

$$\to y(x) = C \ln|x| + C_0$$

Example 157 (cont.)

Apply the supplementary conditions to the general solution:

$$y'(1) = \frac{C}{1} = 0 \to C = 0, \ \& \ y'(e^{2\pi}) = \frac{C}{e^{2\pi}} = 0 \to C = 0$$

Thus C becomes 0. There is no condition imposed on C_0. Solution becomes

$$y(x) = C_0$$

Thus $\lambda = 0$ is a characteristic value and the corresponding characteristic functions are $y(x) = C_0$, where C_0 is an arbitrary nonzero constant.

Example 157 (cont.)

Case 2: $\lambda > 0$:

$$\frac{d}{dx}\left[x\frac{dy}{dx}\right] + \frac{\lambda}{x}y = 0$$

$$1 \cdot \frac{dy}{dx} + x\frac{d^2y}{dx^2} + \frac{\lambda}{x}y = 0$$

$$x \cdot \frac{dy}{dx} + x^2\frac{d^2y}{dx^2} + \lambda y = 0$$

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \lambda y = 0$$

For $x \neq 0$, this is a Cauchy Euler equation. Letting $x = e^t$, the solution is found to be

$$y(t) = c_1 \sin\sqrt{\lambda}t + c_2 \cos\sqrt{\lambda}t.$$

Example 157 (cont.)

$$y(t) = c_1 \sin \sqrt{\lambda} t + c_2 \cos \sqrt{\lambda} t.$$

Back to the x gives

$$y(x) = c_1 \sin(\sqrt{\lambda} \ln x) + c_2 \cos(\sqrt{\lambda} \ln x).$$

Apply the supplementary conditions $y'(1) = 0, \;\; y'(e^{2\pi}) = 0$ to the general solution. Let us apply the first condition first:

$$\frac{dy}{dx} = \frac{c_1 \sqrt{\lambda}}{x} \cos(\sqrt{\lambda} \ln x) - \frac{c_2 \sqrt{\lambda}}{x} \sin(\sqrt{\lambda} \ln x)$$

$$y'(1) = 0 \rightarrow \frac{c_1 \sqrt{\lambda}}{1} \cos(\sqrt{\lambda} \ln 1) - \frac{c_2 \sqrt{\lambda}}{1} \sin(\sqrt{\lambda} \ln 1) = 0$$

$$\rightarrow c_1 \sqrt{\lambda} = 0 \rightarrow c_1 = 0$$

Example 157 (cont.)

Now apply the second supplementary conditions $y'(e^{2\pi}) = 0$ to the general solution. This leads to

$$c_2\sqrt{\lambda}e^{-2\pi}\sin(2\pi\sqrt{\lambda}) = 0$$

Nontrivial solutions will require $\lambda = \frac{n^2}{4}$, $(n = 1, 2, 3, \ldots)$ Thus, corresponding to the characteristic values $\lambda = \frac{n^2}{4}$, $(n = 1, 2, 3, \ldots)$, with $x > 0$, the characteristic functions are

$$y(x) = C_n \cos\left(\frac{n \ln x}{2}\right)$$

Recall that the solution before using the supplementary conditions is

$$y(x) = c_1 \sin(\sqrt{\lambda}\ln x) + c_2 \cos(\sqrt{\lambda}\ln x)$$

Theorem 41

Hypothesis *Consider the Sturm Liouville problem consisting of*
1. the differential equation

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + [q(x) + \lambda r(x)]\, y = 0$$

where $p, q,$ *and* r *are real functions such that* p *has continuous derivative,* q *and* r *are continuous,* $p(x) > 0$ *and* $r(x) > 0$ *for all* x *on a real interval* $a \leq x \leq b$*, and* λ *is a parameter independent of* x*; and*
2. the conditions

$$\begin{aligned} A_1 y(a) + A_2 y'(a) &= 0 \\ B_1 y(b) + B_2 y'(b) &= 0 \end{aligned}$$

where $A_1, A_2, B_1,$ *and* B_2 *are real constants such that* A_1 *and* A_2 *are not both zero, and* B_1 *and* B_2 *are not both zero.*

Theorem 41 (cont.)

Conclusions:

1. There exists an infinite number of characteristic values λ_n of the given problem. These characteristic values can be arranged in a monotonic increasing sequence

$$\lambda_1 < \lambda_2 < \lambda_3 < \dots$$

and such that $\lambda_n \to +\infty$ as $n \to +\infty$.

2.Corresponding to each characteristic value λ_n there exists a one parameter family of characteristic functions ϕ_n. Each of these characteristic functions is defined on $a \leq x \leq b$, and any two characteristic functions corresponding to the same characteristic value are nonzero constant multiples of each other.

3. Each characteristic function ϕ_n corresponding to the characteristic value λ_n $(n = 1, 2, 3, \dots)$ has exactly $(n - 1)$ zeros in the open interval $a < x < b$.

Example 158

Consider the Sturm Liouville problem

$$\frac{d^2y}{dx^2} + \lambda y = 0$$

$$y(0) = 0, \quad y(\pi) = 0$$

It has already been solved and was found that it has infinitely many characteristic values, and these values are growing towards infinity, therefore, the 1st conclusion is verified. Let us verify the 2nd conclusion for any two characteristic functions. For instance, for $\lambda = 9$, corresponding functions are $c \sin 3x$ where c is arbitrary. Take any two of them, for instance, $5 \sin 3x$ and $-2.2 \sin 3x$, and observe that the conclusion holds. That is, for the same characteristic value, corresponding characteristic functions are multiple of each other.

Example 158 (cont.)

Conclusion 3 suggests the characteristic function $c_n \sin nx$ corresponding to $\lambda = n^2$ has exactly $n-1$ zeros in the open interval $0 < x < \pi$. We know that $\sin nx = 0$ if and only if $nx = k\pi$, where k is an integer. Thus the zeros of $c_n \sin nx$ are given by

$$x = \frac{k\pi}{n}, \quad (k = 0, \pm 1, \pm 2, \ldots) \tag{110}$$

The zeros of (110) which lie in the open interval $0 < x < \pi$ are the ones corresponding to $k = 1, 2, \ldots, n-1$. Totally, there are $n-1$ zeros in the interval.

Orthogonality of Characteristic Functions

Definition 38

Two functions f and g are called **orthogonal** with respect to the **weight function** r on the interval $a \leq x \leq b$ if and only if

$$\int_a^b f(x)g(x)r(x)dx = 0$$

Example 159

The functions $\sin x$ and $\sin 2x$ are orthogonal with respect to the weight function having the constant value 1 on the interval $0 \leq x \leq \pi$:

$$\int_0^\pi (\sin x)(\sin 2x)(1)dx = \left.\frac{2\sin^3 x}{3}\right|_0^\pi = 0$$

Definition 39

Let $\{\phi_n\}$, $n = 1, 2, 3, \ldots$, be an infinite set of functions defined on the interval $a \le x \le b$. The **set** $\{\phi_n\}$ is called **orthogonal system** with respect to the weight function r on $a \le x \le b$ if every two distinct functions of the set are orthogonal with respect to r on $a \le x \le b$. That is, the set $\{\phi_n\}$ is orthogonal with respect to r on $a \le x \le b$ if

$$\int_a^b \phi_m(x)\phi_n(x)r(x)dx = 0, \quad \text{for } m \neq n$$

Example 160

Consider the infinite set of functions $\{\sin x, \sin 2x, \sin 3x, \ldots\}$ on the interval $0 \le x \le \pi$. Let the weight function be 1. Then this set is orthogonal wrt this weight function:

$$\int_0^\pi (\sin mx)(\sin nx)(1)dx = \left[\frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n}\right]_0^\pi = 0$$

Theorem 42

Hypothesis *Consider the Sturm Liouville problem consisting of*

1. the differential equation

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + [q(x) + \lambda r(x)]\, y = 0$$

where $p, q,$ and r are real functions such that p has continuous derivative, q and r are continuous, $p(x) > 0$ and $r(x) > 0$ for all x on a real interval $a \leq x \leq b$, and λ is a parameter independent of x; and

2. the conditions

$$\begin{aligned} A_1 y(a) + A_2 y'(a) &= 0 \\ B_1 y(b) + B_2 y'(b) &= 0 \end{aligned}$$

where $A_1, A_2, B_1,$ and B_2 are real constants such that A_1 and A_2 are not both zero, and B_1 and B_2 are not both zero.

Theorem 42 (cont.)

Let λ_m and λ_n be two distinct characteristic values of this problem. Let ϕ_m be a characteristic function for λ_m and ϕ_n be a characteristic function for λ_n.

Conclusion *The characteristic functions ϕ_m and ϕ_n are orthogonal with respect to the weight function r on the interval $a \leq x \leq b$.*

Example 161

Consider the Sturm Liouville problem

$\frac{d^2y}{dx^2} + \lambda y = 0, \quad y(0) = 0, \quad y(\pi) = 0$

where $r = 1$. Corresponding to each characteristic value $\lambda = n^2$ $(n = 1, 2, \ldots)$ we have characteristic functions $c_n \sin nx$ $(n = 1, 2, \ldots)$. Define $\phi_n(x) = \sin nx, \; n = 1, 2, \ldots$ The set $\{\phi_n\}, \; n = 1, 2, \ldots$, is an orthogonal system because

$$\int_0^{\pi} (\sin mx)(\sin nx)(1)dx = 0, \text{ for } m = 1, 2, \ldots, \; n = 1, 2, \ldots, \; m \neq n$$

Picard's iterations

We have previously seen the following conditions on existence and uniqueness of solutions of a 1st order d.e.

Theorem 43

If both $f(x, y)$ and $f_y(x, y)$ are continuous on the rectangle $R = \{(x, y) : \ a \leq x \leq b, c \leq y \leq d\}$ and $(x_0, y_0) \in R$, then there exists a unique solution to the initial value problem

$$\frac{dy}{dx} = f(x, y), \ y(x_0) = y_0 \tag{111}$$

for all values of x in some (smaller) interval $[x_0 - \delta, x_0 + \delta]$ contained in $a \leq x \leq b$.

$$\frac{dy}{dx} = f(x, y), \; y(x_0) = y_0 \qquad \text{(cf. 111)}$$

Integrate both sides of (111) from x_0 to x:

$$y(x) - y_0 = \int_{x_0}^{x} f(x, y)dx \qquad (112)$$

Theorem 44 (Picard's Iterations)

For the initial value problem satisfying conditions of Theorem 43 define the following iterations:

$$Y_0(x) = y_0$$

$$Y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, Y_n(t))dt$$

Then the solution $y(x)$ to (111) is the limit

$$y(x) = \lim_{n \to \infty} Y_n(x)$$

$$Y_0(x) = y_0$$

$$Y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, Y_n(t))dt$$

Example 162

$$\frac{dy}{dx} = y - x, \;\; y(0) = 2 \tag{113}$$

$Y_0(x) = 2$
$Y_1(x) = 2 + \int_0^x f(t, 2)dt = 2 + \int_0^x (2 - t)dt = 2 + 2x - \frac{x^2}{2}$
$Y_2(x) = 2 + \int_0^x f(t, 2 + 2t - \frac{t^2}{2})dt = 2 + \int_0^x (2 + 2t - \frac{t^2}{2} - t)dt$
$= 2 + 2x + \frac{x^2}{2} - \frac{x^3}{6}$
$Y_3(x) = 2 + 2x + \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{24}$
$Y_4(x) = 2 + 2x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120}$
$Y_5(x) = 2 + 2x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} - \frac{x^6}{720}$

Example 162 (cont.)

$$Y_5(x) = 2 + 2x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} - \frac{x^6}{720}$$

For a better approximation more iterations can be used. However, one may notice that the pattern above belongs to the known function

$$1 + x + e^x$$

Example 163

$$\frac{dy}{dx} = 1 + y^2, \; y(0) = 0$$

Let us use Picard's iterations to find the solution:

$$\begin{aligned}
Y_0(x) &= 0 \\
Y_1(x) &= x \\
Y_2(x) &= x + \tfrac{x^3}{3} \\
Y_3(x) &= x + \tfrac{x^3}{3} + \tfrac{2x^5}{15} + \tfrac{x^7}{63} \\
Y_4(x) &= x + \tfrac{x^3}{3} + \tfrac{2x^5}{15} + \tfrac{17x^7}{315} + \tfrac{38x^9}{2835} + \tfrac{134x^{11}}{51975} + \tfrac{4x^{13}}{12285} + \tfrac{x^{15}}{59535}
\end{aligned}$$

Approximation may be improved by using more iterations. The term $Y_4(x)$ above has pattern matching with that of $\tan x$. Indeed $y(x) = \tan x$ solves the given d.e.

How does it work? Notice that the following equations have the same solution:

$$\frac{dy}{dx} = f(x, y),\ y(x_0) = y_0 \qquad \text{(cf. 111)}$$

$$y(x) - y_0 = \int_{x_0}^{x} f(x, y)dx \qquad \text{(cf. 112)}$$

It can be shown that the iterations

$$Y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, Y_n(t))dt,\ Y_0(x) = y_0$$

have a limit, call it $y(x)$.

As $n \to \infty$ we have $Y_{n+1}(x) = Y_n(x) \triangleq y(x)$.
Substituting $y(x)$ for $Y_{n+1}(x)$ and $Y_n(x)$ in the above equation will result in Equation (112). The $y(x)$ satisfying (112) satisfies the original differential equation (111).

$$Y_0(x) = y_0$$
$$Y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, Y_n(t))dt$$

Example 164

$$\frac{dy}{dx} = 2(y+1), \; y(0) = 0 \tag{114}$$

$Y_0(x) = 0$
$Y_1(x) = \int_0^x f(t,0)dt = \int_0^x 2dt = 2x$
$Y_2(x) = \int_0^x f(t,2t)dt = \int_0^x 2(2t+1)dt = 2x^2 + 2x$
$Y_3(x) = \frac{(2x)^3}{3!} + \frac{(2x)^2}{2!} + 2x$
...
...
The pattern above is

$$Y_n(x) = \sum_{k=1}^{n} \frac{(2x)^k}{k!}$$

$$\frac{dy}{dx} = 2(y+1),\ y(0) = 0 \qquad \text{(cf. 114)}$$

$$Y_n(x) = \sum_{k=1}^{n} \frac{(2x)^k}{k!}$$

Example 164 (cont.)

This pattern corresponds to

$$y(x) = e^{2x} - 1$$

Euler Equation

$$L[y] := x^2\ddot{y} + \alpha x\dot{y} + \beta y = 0 \tag{115}$$

where $L \triangleq x^2D^2 + \alpha xD + \beta$ with α, β are real constants, is called Euler equation. $x = 0$ is a regular singular point. Let $y = x^r$ be a solution candidate for $x > 0$.

$$L[x^r] = x^2(x^r)'' + \alpha x(x^r)' + \beta x^r = x^2r(r-1)x^{r-2} + \alpha xrx^{r-1} + \beta x^r$$

$$x^r\left[r(r-1) + \alpha r + \beta\right] = x^r\left[r^2 + (\alpha-1)r + \beta\right] := x^rF(r)$$

If r satisfies $F(r) = 0$ then x^r is a solution. F is a quadratic polynomial; it has two solutions:

$$r_{1,2} = \frac{-(\alpha-1) \pm \sqrt{(\alpha-1)^2 - 4\beta}}{2}$$

We may write

$$F(r) = (r - r_1)(r - r_2)$$

If r_1 and r_2 are real and distinct then the general solution is

$$y(x) = c_1 x^{r_1} + c_2 x^{r_2} \tag{116}$$

Example 165

$$2x^2 \ddot{y} + 3x\dot{y} - y = 0, \; x > 0$$

Substituting x^r above results

$$x^r[2r(r-1) + 3r - 1] = x^r(r - \frac{1}{2})(r+1)$$

Equation above is satisfied if $r = \frac{1}{2}$ or $r = -1$. The two roots are real and distinct. Therefore, the general solution is

$$y(x) = c_1 x^{\frac{1}{2}} + c_2 x^{-1}$$

Real and equal roots case ($r_1 = r_2$)

$$L[x^r] = x^r \overbrace{[(r(r-1) + \alpha r + \beta]}^{F(r)} = x^r (r - r_1)^2$$

$y = x^{r_1}$ is a solution. For a general solution we need to find another linearly independent solution. Noting that $\frac{d}{dr} x^r = x^r \ln x$

$$\frac{\partial}{\partial r} L[x^r] = \frac{\partial}{\partial r}[x^r (r - r_1)^2] = (r - r_1)^2 x^r \ln x + 2(r - r_1) x^r$$

On the left change order of differentiation, keep the rhs as it is:

$$L \frac{\partial}{\partial r} x^r = L[x^r \ln x] = (r - r_1)^2 x^r \ln x + 2(r - r_1) x^r$$

At $r = r_1$ we have $L[x^r \ln x] = 0$, therefore, $x^{r_1} \ln x$ is a solution. It is linearly independent. Thus the general solution is

$$y(x) = c_1 x^{r_1} + c_2 x^{r_1} \ln x$$

Example 166

$$x^2\ddot{y} + 5x\dot{y} + 4y = 0, \; x > 0$$

Substituting x^r above results

$$x^r[r^2 + 4r + 4] = 0$$

Equation above is satisfied if $r_1 = r_2 = -2$. The two roots are real and equal. Therefore, the general solution is

$$y(x) = c_1 x^{-2} + c_2 x^{-2} \ln x$$

$r = \lambda \pm i\mu$ case: $x^{\lambda+i\mu}$ satisfies the differential equation. Note that

$$x^{\lambda+i\mu} = e^{\ln x^{\lambda+i\mu}} = e^{(\lambda+i\mu)\ln x} = e^{\lambda \ln x} e^{i\mu \ln x} =$$
$$= e^{\lambda \ln x}[\cos(\mu \ln x) + i \sin(\mu \ln x)]$$

The real and imaginary parts are linearly independent. The general solution is

$$y(x) = c_1 e^{\lambda \ln x} \cos(\mu \ln x) + c_2 e^{\lambda \ln x} \sin(\mu \ln x) \tag{117}$$

Example 167

$$x^2 \ddot{y} + x\dot{y} + y = 0, \; x > 0$$

Substituting x^r above results

$$x^r[r^2 + 1] = 0$$

Equation above is satisfied if $r = \pm i$. The roots are complex. Plug $\lambda = 0$ and $\mu = 1$ in (117) to obtain the general solution as

$$y(x) = c_1 \cos(\ln x) + c_2 \sin(\ln x)$$

Example 168

$$2x^2\ddot{y} + 3x\dot{y} - 15y = 0, \;\; y(1) = 0, \; \dot{y}(1) = 1, \; x > 0$$

Substituting x^r in the above equation results

$$2x^2(x^r)'' + 3x(x^r)' - 15x^r = 0$$

$$(2r(r-1) + 3r - 15)x^r = 0$$

$$(2r^2 + r - 15)x^r = (r+3)(r - \frac{5}{2})x^r = 0$$

Euler d.e. with real and distinct roots! The general solution, therefore, has the form (see Eq. (116))

$$y(x) = c_1 x^{r_1} + c_2 x^{r_2} \rightarrow y(x) = c_1 x^{\frac{5}{2}} + c_2 x^{-3}$$

Using initial conditions we obtain

$$y(x) = \frac{2}{11} x^{\frac{5}{2}} - \frac{2}{11} x^{-3}$$

Examples for Regular Singular Points

Example 1: Bessel's differential equation of v-th order has a singular point at $x = 0$

$$x^2\ddot{y} + x\dot{y} + (x^2 - v^2)y = 0$$

In standard form:

$$\ddot{y} + \frac{1}{x}\dot{y} + \frac{(x^2 - v^2)}{x^2}y = 0$$

Example 2: Legendre equation has a singular point at $x = 1$ and $x = -1$

$$(1 - x^2)\ddot{y} - 2x\dot{y} + n(n+1)y = 0$$

In standard form:

$$\ddot{y} - \frac{2x}{(1 - x^2)}\dot{y} + \frac{n(n+1)}{(1 - x^2)}y = 0$$

Examples for Regular Singular Points

$$\ddot{y} + \frac{1}{x}\dot{y} + \frac{(x^2 - v^2)}{x^2}y = 0$$

$x \times \frac{1}{x} = 1; \quad x^2 \times \frac{(x^2 - v^2)}{x^2} = (x^2 - v^2)$

$\therefore$ Bessel's singular point at 0 is regular singular point.

$$\ddot{y} - \frac{2x}{(1 - x^2)}\dot{y} + \frac{n(n+1)}{(1 - x^2)}y = 0$$

$(x-1) \times \frac{-2x}{(1-x^2)} = \frac{2x}{(1+x)}; \quad (x-1)^2 \times \frac{n(n+1)}{(1-x^2)} = \frac{(1-x)n(n+1)}{(1+x)}$

$\therefore$ Legendre's singular point at $x = 1$ is regular.

It can be shown that the singular point at $x = -1$ is also regular.

Solving Bessel's DE of 0-th Order using Power Series

Bessel's differential of 0-th order is as follows:

$$x^2\ddot{y} + x\dot{y} + x^2 y = 0$$

Solution $y(x)$ has the form

$$y(x) = |x^r| \sum_{n=0}^{\infty} a_n x^n$$

For the solution domain $x > 0$ we have

$$y(x) = \sum_{n=0}^{\infty} a_n x^{r+n}; \quad \dot{y}(x) = \sum_{n=0}^{\infty} (r+n) a_n x^{r+n-1};$$

$$\ddot{y}(x) = \sum_{n=0}^{\infty} (r+n)(r+n-1) a_n x^{r+n-2}$$

$$x^2\ddot{y} + x\dot{y} + x^2 y = 0$$

Solving Bessel's DE of 0-th Order using Power Series

Substitute in the d.e.

$$x^2 \sum_{n=0}^{\infty} (r+n)(r+n-1)a_n x^{r+n-2} + x \sum_{n=0}^{\infty} (r+n)a_n x^{r+n-1}$$

$$+x^2 \sum_{n=0}^{\infty} a_n x^{r+n} = 0$$

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n x^{r+n} + \sum_{n=0}^{\infty}(r+n)a_n x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+2} = 0$$

Use transformation $m = n + 2$

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n x^{r+n} + \sum_{n=0}^{\infty}(r+n)a_n x^{r+n} + \sum_{m=2}^{\infty} a_{m-2} x^{r+m} = 0$$

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n x^{r+n} + \sum_{n=0}^{\infty}(r+n)a_n x^{r+n} + \sum_{n=2}^{\infty} a_{n-2}x^{r+n} = 0$$

Solving Bessel's DE of 0-th Order using Power Series

Let all sums start at $n = 2$:

$$a_0[(r(r-1)+r]x^r + a_1[(r+1)r+(r+1)]x^{r+1}$$

$$+\sum_{n=2}^{\infty}(a_n[(r+n)(r+n-1)+(r+n)] + a_{n-2})\,x^{r+n} = 0$$

$$a_0 r^2 x^r + a_1(r+1)^2 x^{r+1} + \sum_{n=2}^{\infty}[a_n(r+n)^2 + a_{n-2}]x^{r+n} = 0$$

Indicial equation

$$r^2 = 0 \rightarrow r_1 = 0,\ \ r_2 = 0$$

$$a_0 r^2 x^r + a_1(r+1)^2 x^{r+1} + \sum_{n=2}^{\infty} [a_n(r+n)^2 + a_{n-2}] x^{r+n} = 0$$

Solving Bessel's DE of 0-th Order using Power Series

Set $r = 0$, then equating the factor of x^{r+1} to zero we get $a_1 = 0$. Equating factors of x^{r+n} for $n \geq 2$ we get a recurrence formula

$$a_n = -\frac{a_{n-2}}{(r+n)^2}, \quad n \geq 2$$

Since $a_1 = 0$ we conclude that $a_3 = a_5 = a_7 = \cdots = 0$. And we conclude

$$a_2 = -\frac{a_0}{2^2}, \; a_4 = -\frac{a_2}{4^2} = \frac{a_0}{4^2 2^2} = \frac{a_0}{2^4 2^2}, \; a_6 = -\frac{a_0}{2^6 (3 \cdot 2 \cdot 1)^2}$$

Noticing the pattern we can write
$a_{2n} = \frac{(-1)^n a_0}{2^{2n}(n!)^2}, \quad n = 1, 2, 3, \ldots.$

Solving Bessel's DE of 0-th Order using Power Series

We have obtained a solution, 0-th order Bessel's equation of first kind:

$$y(x) = a_0 \underbrace{\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}\right]}_{J_0(x)}$$

where a_0 is arbitrary constant. □

Bessel's differential equation under consideration

$$x^2\ddot{y} + x\dot{y} + (x^2 - v^2)y = 0$$

is a second order homogeneous linear differential equation. Therefore, it has two linearly independent solutions. These two linearly independent solutions serve as a basis for the general solution. The form of the 2nd linearly independent solution depends on the order of the Bessel's d.e. Next theorem states this relation.

Linearly independent solutions of Bessel's DE

Theorem Given the Bessel's equation of order $v \geq 0$

$$x^2\ddot{y} + x\dot{y} + (x^2 - v^2)y = 0$$

then we have the following:
(1) If $v \notin \{0, 1, 2, \ldots\}$ the equation has two linearly independent solutions $J_v(x)$ and $J_{-v}(x)$ and the general solution is

$$y(x) = c_1 J_v(x) + c_2 J_{-v}(x)$$

where c_1 and c_2 are arbitrary constants. Note that the indicial equation is $r^2 - v^2 = 0$. Its roots are $\pm v$. Setting r to $\pm v$ we get two linearly independen solutions.
(2) If $v \in \{0, 1, 2, \ldots\}$ the equation has only one Bessel's function of the first kind $J_v(x)$, another linearly independent solution is the Bessel's function of the second kind Y_v. The general solution is $y(x) = c_1 J_v(x) + c_2 Y_v(x)$ where c_1 and c_2 are arbitrary constants.

Linearly independent solutions of Bessel's DE

Case (1) is straightforward: Whatever done for v is repeated for $-v$, and form the general solution as indicated.

The case v = 0 is a particular of Case (2), we have found solution J_0, we need to find the second linearly independent solution in the form of Bessel's function of the second kind. There are two popular methods for obtaining the second independent solution $Y_0(x)$. Let us start with one of them.

Let $0 < \epsilon < 1$. Because ϵ is not an integer, $J_\epsilon(x)$ and $J_{-\epsilon}(x)$ are two linearly independent solutions corresponding the Bessel's d.e. of order ϵ. Define $Y_\epsilon(x)$ by

$$Y_\epsilon(x) = \frac{J_\epsilon(x) - J_{-\epsilon}(x)}{\epsilon}$$

Because $Y_\epsilon(x)$ is a linear combination of solutions, it is also a solution to Bessel's d.e. of order ϵ.

$$Y_\epsilon(x) = \frac{J_\epsilon(x) - J_{-\epsilon}(x)}{\epsilon}$$

Linearly independent solutions of Bessel's DE

Define $Y_0(x)$ by

$$Y_0(x) = \lim_{\epsilon \to 0} Y_\epsilon(x) = \lim_{\epsilon \to 0} \frac{J_\epsilon(x) - J_{-\epsilon}(x)}{\epsilon}$$

It can be proven that the limit $Y_0(x)$ is a solution to Bessel's d.e. of order 0; and $J_0(x)$ and $Y_0(x)$ are linearly independent. We have seen a method for obtaining the second solution from that of the first one.

More on Bessel Equation

Bessel function of order zero, denoted by J_0, is defined by

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots \tag{118}$$

If u_r denotes r-th term of the series above, then we have

$$\frac{u_{r+1}}{u_r} = -\frac{x^2}{(2r)^2}$$

This shows that, regardless of x value, the above ratio tends to 0 as $r \to \infty$. $J_0(x)$ converges for all x values.

A digression

Ratio test (Also known as d'Alembert's criterion)

Suppose that there exists C such that

$$\lim_{r\to\infty} \left| \frac{a_{r+1}}{a_r} \right| = C.$$

If $C < 1$, then the series is absolutely convergent. If $C > 1$, then the series diverges. If $C = 1$, the ratio test is inconclusive, and the series may converge or diverge. **EOD**

Bessel function of order n, where n is a positive integer, is defined by

$$J_n(x) = \frac{x^n}{2^n n!}(1 - \frac{x^2}{2 \cdot 2n+2} + \frac{x^4}{2 \cdot 4 \cdot 2n+2 \cdot 2n+4} - \cdots) \quad (119)$$

$J_n(x)$ also converges for every x.

$$J_n(x) = \frac{x^n}{2^n n!}(1 - \frac{x^2}{2 \cdot 2n+2} + \frac{x^4}{2 \cdot 4 \cdot 2n+2 \cdot 2n+4} - \cdots) \quad \text{(cf. 119)}$$

For $n = 1$, Bessel function becomes

$$J_1(x) = \frac{x}{2} - \frac{x^3}{2^2 \cdot 4} + \frac{x^5}{2^2 \cdot 4^2 \cdot 6} - \frac{x^7}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8} \cdots \tag{120}$$

For $n = 2$, it becomes

$$J_2(x) = \frac{x^2}{2 \cdot 4} - \frac{x^4}{2^2 \cdot 4^2 \cdot 6} + \frac{x^6}{2^2 \cdot 4^2 \cdot 6 \cdot 8} - \frac{x^8}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8 \cdot 10} \cdots \tag{121}$$

Note that $J_n(x)$ is even function of x when n is even, odd when n is odd. Also note that $J_0(0) = 1$ and $J_1(0) = 0$.

Verifying that J_0 solves the Bessel's DE

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots \quad \text{(cf. 118)}$$

$$J_1(x) = \frac{x}{2} - \frac{x^3}{2^2 \cdot 4} + \frac{x^5}{2^2 \cdot 4^2 \cdot 6} - \frac{x^7}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8} \cdots \quad \text{(cf. 120)}$$

Note the following relationships between J_0 and J_1.

$$\frac{dJ_0(x)}{dx} = -J_1(x) \tag{122}$$

$$\frac{d}{dx}\left[xJ_1(x)\right] = xJ_0(x) \tag{123}$$

Verifying that J_0 solves the Bessel's DE

$$\frac{dJ_0(x)}{dx} = -J_1(x) \qquad \text{(cf. 122)}$$

$$\frac{d}{dx}\left[xJ_1(x)\right] = xJ_0(x) \qquad \text{(cf. 123)}$$

Use (122) in (123):

$$\frac{d}{dx}\left[x\frac{dJ_0(x)}{dx}\right] + xJ_0(x) = 0 \qquad (124)$$

$$x\frac{d^2J_0(x)}{dx^2} + \frac{dJ_0(x)}{dx} + xJ_0(x) = 0$$

This shows that $J_0(x)$ is a solution to the 0-th order Bessel differential equation

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0 \qquad (125)$$

The 2nd linearly independent solution of the Bessel's DE

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0 \qquad \text{(cf. 125)}$$

It has been shown that J_0 is a solution to (125). Since (125) is a 2nd order homogeneous linear differential equation it must have two linearly independent solutions. Let the linearly independent 2nd solution that we seek be u. And let $v \triangleq J_0$. Equation (125) leads to

$$xu'' + u' + xu = 0$$

$$xv'' + v' + xv = 0$$

Multiplying the first of these equations by v and the second by u and subtracting we have

$$x(u''v - uv'') + u'v - uv' = 0$$

The 2nd linearly independent solution of the Bessel's DE

$$xu'' + u' + xu = 0$$

$$xv'' + v' + xv = 0$$

Multiplying the first of these equations by v and the second by u and subtracting we have

$$x(u''v - uv'') + u'v - uv' = 0$$

Because

$$u''v - uv'' = \frac{d}{dx}(u'v - uv')$$

we can write

$$\frac{d}{dx}\left[x(u'v - uv')\right] = 0$$

The 2nd linearly independent solution of the Bessel's DE

$$\frac{d}{dx}\left[x(u'v - uv')\right] = 0$$

implies

$$x(u'v - uv') = B$$

where B is a constant. Dividing by xv^2

$$\frac{u'v - uv'}{v^2} = \frac{B}{xv^2}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{B}{xv^2}$$

This leads to

$$\frac{u}{v} = A + B\int \frac{dx}{xv^2}$$

and the second solution is, therefore

$$u = AJ_0(x) + BJ_0(x)\int \frac{dx}{xJ_0^2(x)}$$

The 2nd linearly independent solution of the Bessel's DE

$$u = AJ_0(x) + BJ_0(x) \int \frac{dx}{xJ_0^2}$$

We next evaluate the integral above. Note that

$$\frac{1}{xJ_0^2(x)} = \frac{1}{x} + \frac{x}{2} + \frac{5x^3}{32} + \cdots$$

The integral above becomes

$$\begin{aligned}
&J_0(x) \int \tfrac{dx}{xJ_0^2} \\
&= J_0(x) \left[\ln x + \tfrac{x^2}{4} + \tfrac{5x^4}{128} + \cdots \right] \\
&= J_0(x) \ln x + (1 - \tfrac{x^2}{2^2} + \cdots)(\tfrac{x^2}{4} + \tfrac{5x^4}{128} + \cdots) \\
&= J_0(x) \ln x + \tfrac{x^2}{4} + \tfrac{3x^4}{128} + \cdots
\end{aligned}$$

The 2nd linearly independent solution of the Bessel's DE

$$
\begin{aligned}
&J_0(x) \int \frac{dx}{xJ_0^2} \\
&= J_0(x) \left[\ln x + \frac{x^2}{4} + \frac{5x^4}{128} + \cdots \right] \\
&= J_0(x) \ln x + (1 - \frac{x^2}{2^2} + \cdots)(\frac{x^2}{4} + \frac{5x^4}{128} + \cdots) \\
&= J_0(x) \ln x + \frac{x^2}{4} + \frac{3x^4}{128} + \cdots
\end{aligned}
$$

If we define

$$
Y_0(x) \triangleq J_0(x) \ln x + \frac{x^2}{4} + \frac{3x^4}{128} + \cdots
$$

then Y_0 is a linearly independent 2nd solution of the Bessel equation of the 0-th order.

Alternative way of finding $Y_0(x)$

Regarding Part 3 of Theorem 26 (*The case of a regular singular point x_0 of the d.e. (84). The roots r_1 and r_2 of the indicial equation associated with x_0 satisfy $r_1 - r_2 = 0$*), let a solution of Bessel's equation of order zero have the form

$$y(x) = J_0(x) \ln x + \sum_{n=1}^{\infty} b_n x^n$$

Its derivatives are

$$\dot{y}(x) = \dot{J}_0(x) \ln x + \frac{J_0(x)}{x} + \sum_{n=1}^{\infty} n b_n x^{n-1}$$

$$\ddot{y}(x) = \ddot{J}_0(x) \ln x + 2\frac{\dot{J}_0(x)}{x} - \frac{J_0(x)}{x^2} + \sum_{n=2}^{\infty} n(n-1) b_n x^{n-2}$$

Substitute $y, \dot{y}$, and $\ddot{y}$ in the Bessel's equation of order zero $x^2\ddot{y} + x\dot{y} + x^2 y = 0$

Alternative way of finding $Y_0(x)$

Substitute $y, \dot{y}$, and $\ddot{y}$ in the DE: $x^2\ddot{y} + x\dot{y} + x^2 y = 0$

$$x^2\left[\ddot{J}_0(x)\ln x + 2\frac{\dot{J}_0(x)}{x} - \frac{J_0(x)}{x^2} + \sum_{n=2}^{\infty} n(n-1)b_n x^{n-2}\right] + x\left[\dot{J}_0(x)\ln x\right.$$

$$\left. + \frac{J_0(x)}{x} + \sum_{n=1}^{\infty} nb_n x^{n-1}\right] + x^2\left[J_0(x)\ln x + \sum_{n=1}^{\infty} b_n x^n\right] = 0$$

$$\ln x\left[x^2\ddot{J}_0(x) + x\dot{J}_0(x) + x^2 J_0(x)\right] + x\frac{J_0(x)}{x} - x^2\frac{J_0(x)}{x^2} +$$

$$+2x\dot{J}_0(x) + x^2\sum_{n=2}^{\infty} n(n-1)b_n x^{n-2} + x\sum_{n=1}^{\infty} nb_n x^{n-1} + x^2\sum_{n=1}^{\infty} b_n x^n = 0$$

Coefficients of $\ln x$ add up to zero and blue colored terms cancel out. Thus

$$2x\dot{J}_0(x) + \sum_{n=2}^{\infty} n(n-1)b_n x^n + \sum_{n=1}^{\infty} nb_n x^n + \sum_{n=1}^{\infty} b_n x^{n+2} = 0 \quad (126)$$

Alternative way of finding $Y_0(x)$

Substitute $y, \dot{y}$, and $\ddot{y}$ in the Bessel's equation of order zero $x^2\ddot{y} + x\dot{y} + x^2 y = 0$. Some terms cancel out and we obtain

$$\underbrace{2x\dot{J}_0(x)}_{1st Term} + \underbrace{\sum_{n=2}^{\infty} n(n-1)b_n x^n}_{2nd Term} + \underbrace{\sum_{n=1}^{\infty} nb_n x^n}_{3rd Term} + \underbrace{\sum_{n=1}^{\infty} b_n x^{n+2}}_{4th Term} = 0$$

(cf. 126)

Further simplify the terms.

1st term:

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2} x^{2n} \to 2x\dot{J}_0(x) = 2\sum_{n=1}^{\infty} \frac{(-1)^n 2n}{2^{2n}(n!)^2} x^{2n}$$

2nd term:

$$\sum_{n=2}^{\infty} n(n-1)b_n x^n = 2b_2 x^2 + \sum_{n=3}^{\infty} n(n-1)b_n x^n$$

Alternative way of finding $Y_0(x)$

3rd term:

$$\sum_{n=1}^{\infty} nb_n x^n = b_1 x + 2b_2 x^2 + \sum_{n=3}^{\infty} nb_n x^n$$

4th term: Change of index: m=n+2

$$\sum_{n=1}^{\infty} b_n x^{n+2} = \sum_{m-2=1}^{\infty} b_{m-2} x^m = \sum_{n=3}^{\infty} b_{n-2} x^n$$

The expression (126) now can be written as

$$b_1 x + 4b_2 x^2 + \sum_{n=3}^{\infty} [n^2 b_n + b_{n-2}] x^n = -2 \sum_{n=1}^{\infty} \frac{(-1)^n 2n}{2^{2n}(n!)^2} x^{2n}$$

We have even powers on the righthand side, therefore, for the equality to hold there cannot be odd powers on the left. Thus $b_1 = 0$

Alternative way of finding $Y_0(x)$

$$b_1 x + 4b_2 x^2 + \sum_{n=3}^{\infty}[n^2 b_n + b_{n-2}]x^n = -2\sum_{n=1}^{\infty}\frac{(-1)^n 2n}{2^{2n}(n!)^2}x^{2n}$$

We have even powers on the righthand side, therefore, for the equality to hold there cannot be odd powers on the left. Thus $b_1 = 0$

$$4b_2 x^2 + \sum_{n=3}^{\infty}[n^2 b_n + b_{n-2}]x^n = -2\sum_{n=1}^{\infty}\frac{(-1)^n 2n}{2^{2n}(n!)^2}x^{2n}$$

The expression $\sum_{n=3}^{\infty}[n^2 b_n + b_{n-2}]x^n$ has to result in zero for odd powers of x. So. $n^2 b_n + b_{n-2} = 0$, $n = 3, 5, \ldots$ imply $b_3 = b_5 = \cdots = 0$. Equating like powers of x for even n we obtain $b_2 = 1/4$.

Alternative way of finding $Y_0(x)$

$$4b_2x^2 + \sum_{n=3}^{\infty}[n^2 b_n + b_{n-2}]x^n = -2\sum_{n=1}^{\infty}\frac{(-1)^n 2n}{2^{2n}(n!)^2}x^{2n}$$

For $n \geq 2$, coefficients of x^{2n} on the left are $(2n)^2 b_{2n} + b_{2(n-1)}$; which are $-\frac{(-1)^n n}{2^{2(n-1)}(n!)^2}$ on the right. Thus

$$(2n)^2 b_{2n} + b_{2(n-1)} = -\frac{(-1)^n n}{2^{2(n-1)}(n!)^2}$$

This results in

$$b_{2n} = \frac{1}{(2n)^2}\left[-b_{2(n-1)} - \frac{(-1)^n n}{2^{2(n-1)}(n!)^2}\right]$$

Alternative way of finding $Y_0(x)$

$$b_{2n} = \frac{1}{(2n)^2}\left[-b_{2(n-1)} - \sum_{n=1}^{\infty} \frac{(-1)^n n}{2^{2(n-1)}(n!)^2}\right]$$

Using this

$$b_4 = \frac{1}{2^2 2^2}\left(-\frac{1}{4} - \frac{2}{2^2 2^2}\right) = -\frac{1}{2^2 4^2}\left(1 + \frac{1}{2}\right)$$

$$b_6 = \frac{1}{2^2 3^2}\left[\frac{1}{2^2 4^2}\left(1 + \frac{1}{2}\right) + \frac{3}{2^4 (3!)^2}\right] = \frac{1}{2^2 4^2 6^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right)$$

$$b_{2n} = \frac{(-1)^{n+1}}{2^{2n}(n!)^2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)$$

Noting $H_n := 1 + \frac{1}{2} + \cdots + \frac{1}{n}$, called Harmonic number, yields

$$y(x) = J_0(x) \ln x + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{H_n}{2^{2n}(n!)^2} x^{2n}$$

Alternative way of finding $Y_0(x)$

$$y(x) = J_0(x)\ln x + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{H_n}{2^{2n}(n!)^2}x^{2n}$$

A normalization of the above solution is the 2nd kind Bessel function of order zero:

$$Y_0(x) = \frac{2}{\pi}[y(x) + (\gamma - \ln 2)J_0(x)]$$

where γ is the Euler–Mascheroni constant which roughly equals 0.5772.

A Digression: Definition

$$\gamma \triangleq \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln n\right) \approx 0.577216...$$

World record (2009), 14.9 billion digits have been computed. **EOD**

Dynamics of Disease Spreading

Let the population be subdivided into a set of distinct classes: **Susceptible**, **Infectious** and **Recovered**.

This is termed the SIR model.

Individuals are born into the **susceptible** class. Susceptible individuals have never come into contact with the disease and are able to catch the disease, after which they move into the infectious class.

Infectious individuals spread the disease to susceptibles, and remain in the infectious class for a given period of time (the infectious period) before moving into the recovered class.

Individuals in the **recovered** class are assumed to be immune for life.

R_0, **basic reproductive ratio**, is defined by epidemiologists as "the average number of secondary cases caused by an infectious individual in a totally susceptible population".
When R_0 is greater than 1, the disease can enter a totally susceptible population and the number of cases will increase, whereas when R_0 is less than 1, the disease will always fail to spread.

Disease	R_0
AIDS	2 to 5
Smallpox	3 to 5
Measles	16 to 18
Malaria	> 100

Table: The value of R_0 for some well-known diseases

Consider the situation **when a new strain of influenza enters a totally susceptible population**. Simple intuition tells us that the disease will spread rapidly through the population, infecting a large proportion of the population in a very short time. It is therefore plausible to ignore births and deaths in the population and only concentrate on the disease dynamics. However, if we wish to model a disease that is endemic, that is, persists indefinitely in the population, our SIR model must also include births to correct the number of susceptibles.

The differential equation model

$$\frac{dS}{dt} = B - \beta SI - dS;$$

$$\frac{dI}{dt} = \beta SI - gI - dI;$$

$$\frac{dR}{dt} = gI - dR;$$

$$R_0 = \frac{\beta}{g}.$$

Here B is the birth rate, d is the death rate, $1/g$ is the infectious period, and β is the contact rate.

Population growth model

The rate of change of population should be proportional to the population itself, that is,

$$\frac{dP}{dt} = rP$$

where P is the population at time t, and r is some real constant. This is a first order linear differential equation. For an initial population $P(0) = P_0$ its solution is

$$P(t) = P_0 e^{rt}$$

This is an overly simplified model assuming the resources (food, space, etc.) are unlimited.

When the resources are limited, the concept of a maximum sustainable population (a "carrying capacity", called K), must be considered. That is,

$$\frac{dP}{dt} = 0 \text{ when } P = K$$

$$\frac{dP}{dt} < 0 \text{ if } P > K$$

$$\frac{dP}{dt} = rP \text{ when } P \ll K$$

This is called **logistic** growth model.

Carrying capacity effect may be viewed in 50-year change of country populations.

Country	1970	2020
Italy	53.3M	59.5M
France	50.5M	64.5M
UK	55.5M	67.5M
Greece	8.8M	10.5M
Türkiye	35.7M	84 M

Table: Population data for some countries

The logistic equation contains the carrying capacity feature:

$$\frac{dP}{dt} = rP(1 - \frac{P}{K})$$

Notice that when $P = K$ it reduces to

$$\frac{dP}{dt} = rP(1 - 1) = 0$$

Also notice the cases $P > K$ and $P < K$.

Let us solve

$$\frac{dP}{dt} = rP(1 - \frac{P}{K})$$

It is separable

$$\frac{dP}{P(1-\frac{P}{K})} = rdt$$

$$\left(\frac{1}{P} + \frac{1}{K-P}\right) dP = rdt$$

$$\ln P - \ln |K - P| = rt + c_1$$

$$\ln \left|\frac{P}{K-P}\right| = rt + c_1$$

$$\frac{P}{K-P} = c_2 e^{rt}$$

$$P(t) = \frac{Kc_2 e^{rt}}{1 + c_2 e^{rt}}$$

$$P(t) = \frac{Kc_2 e^{rt}}{1 + c_2 e^{rt}}$$

$$P(t) = \frac{K}{1 + c_3 e^{-rt}}$$

Apply the initial condition $P(0) = P_0$ to obtain $c_3 = \frac{K-P_0}{P_0}$.

$$\therefore P(t) = \frac{K}{1 + \frac{K-P_0}{P_0} e^{-rt}}$$

Units

Consider $\frac{dP}{dt} = rP$.

$\frac{dP}{dt}$ has unit "population / time", therefore, rP must have unit "population / time". This implies that r has to have unit $(\text{time})^{-1}$.

An application: Carbon dating

Living matter is constantly taking up carbon from the air. The result is that within such material the ratio of the number of isotopes of radioactive carbon 14 ($^{14}\mathrm{C}$) to the number of isotopes of stable carbon 12 ($^{12}\mathrm{C}$) is essentially constant. Once the specimen is dead (for example, a tree is cut down for its wood, or cotton is harvested for weaving), the radioactive $^{14}\mathrm{C}$ atoms begin to decay according to the model

$$\frac{d}{dt}N(t) = -0.0001216N(t) \tag{127}$$

The unit of t is years. This equation implies initial $^{14}\mathrm{C}$ level N(0) reduces to its half value in 5700 years.

$$\frac{d}{dt}N(t) = -0.0001216N(t) \qquad \text{(cf. 127)}$$

By examining the ratio of the number of isotopes of carbon 12 to carbon 14 in a sample of the material that we want to date, it is possible to work out the proportion remaining of the $^{14}\mathrm{C}$ atoms that were initially present. Suppose that the sample stopped taking up carbon from the air when at time t_0, and that the number of $^{14}\mathrm{C}$ atoms present then was N_{t_0}. If we know that the sample now (at time t) contains only a fraction p of the initial level of $^{14}\mathrm{C}$, then $N(t) = pN_{t_0}$. Using our explicit solution to (127)

$$N(t) = N_{t_0} e^{-0.0001216(t-t_0)}$$

we should have

$$pN_{t_0} = N(t) = N_{t_0} e^{-0.0001216(t-t_0)}$$

.

Cancelling the factor of N_{t_0} in the two outside terms yields the equation

$$p = e^{-0.0001216(t-t_0)}$$

and so the year t_0 is

$$t_0 = t + \frac{\ln p}{0.0001216}$$

Example 169

In 1988, a historical linen cloth, was analyzed in Arizona. p was found to be 0.92. This led to

$$t_0 = 1988 + \frac{\ln 0.92}{0.0001216} \approx \mathbf{1302}$$

So, the shroud belongs to the middle Age.

Predator-Prey Equation(Lotka–Volterra model)

Example 170

Let x be the number of prey (for example, rabbits); y be the number of some predator (for example, foxes); consequently, $\frac{dx}{dt}$ and $\frac{dy}{dt}$ represent the instantaneous growth rates of the two populations; t represent time; $\alpha, \beta, \gamma, \delta$ be positive real parameters describing the interaction of the two species. The populations change through time according to

$$\frac{dx}{dt} = \alpha x - \beta xy,$$
$$\frac{dy}{dt} = \delta xy - \gamma y,$$

Example 170 (cont.)

$$\frac{dx}{dt} = \alpha x - \beta xy,$$
$$\frac{dy}{dt} = \delta xy - \gamma y,$$

The above equations are based on the following assumptions:

1. The prey population finds ample food at all times.
2. The food supply of the predator population depends entirely on the size of the prey population.
3. The rate of change of population is proportional to its size.
4. During the process, the environment does not change in favour of one species, and genetic adaptation is inconsequential.
5. Predators have limitless appetite.

Example 170 (cont.)

$$\frac{dx}{dt} = \alpha x - \beta xy,$$
$$\frac{dy}{dt} = \delta xy - \gamma y,$$

The prey are assumed to have an unlimited food supply and to reproduce exponentially, unless subject to predation; this exponential growth is represented in the equation above by the term αx. The rate of predation upon the prey is assumed to be proportional to the rate at which the predators and the prey meet, this is represented above by βxy. If either x or y is zero, then there can be no predation.

In the 2nd equation, δxy represents the growth of the predator population. The term γy represents the loss rate of the predators due to either natural death or emigration, it leads to an exponential decay in the absence of prey.

Definitions

Recall that **partial differential equation** is a differential equation which involves partial derivatives of one or more dependent variables wrt one or more independent variables.

Solution of a partial differential equation is an explicit or implicit relation between the variables which does not contain derivatives and which satisfies the differential equations.

Example 171

$$\frac{\partial u}{\partial x} = x^2 + y^2$$

$$\rightarrow u = \int (x^2 + y^2)\partial x + \phi(y) = \frac{x^3}{3} + xy^2 + \phi(y)$$

where ϕ is an arbitrary function of y

Example 172

$$\frac{\partial^2 u}{\partial y \partial x} = x^3 - y$$

can be written in the form

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = x^3 - y$$

$$\rightarrow \frac{\partial u}{\partial x} = x^3 y - \frac{y^2}{2} + \phi(x)$$

where ϕ is an arbitrary function of x

$$u = \frac{x^4 y}{4} - \frac{xy^2}{2} + f(x) + g(y)$$

where $f(x) \triangleq \int \phi(x)dx$ and g is an arbitrary function of y

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = 0 \qquad (128)$$

Theorem 45

Let $f_1, f_2, \ldots, f_n$ be n solutions of Equation (128) in a region R of the xy plane. The linear combination $c_1 f_1 + c_2 f_2 + \ldots + c_n f_n$ where $c_1, c_2, \ldots, c_n$ are arbitrary constants, is also a solution of Equation (128) in the region R.

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = 0 \quad \text{(cf. 128)}$$

Theorem 46

Let $f_1, f_2, \ldots,$ be infinite set of solutions of Equation (128) in a region R of the xy plane. Suppose the infinite series $\sum_{n=1}^{\infty} f_n = f_1 + f_2 + \ldots$ converges to f in R. Suppose this series may be differentiated term by term in R to obtain the various derivatives (of f) which appear in Equation (128). Then the function f is also a solution of Equation (128) in R.

General Solutions

In general, we cannot find "general solutions" (i.e., relatively simple formulas describing all possible solutions) to second-order partial differential equations. An exception is with the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

It can be shown that it has a general solution

$$u(x, t) = f(x - ct) + g(x + ct)$$

where f and g are arbitrary sufficiently differentiable functions of their respective arguments.

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

$$u(x,t) = f(x-ct) + g(x+ct)$$

Noting that

$$u_t(x,t) = -c\dot{f}(x-ct) + c\dot{g}(x+ct), \quad u_{tt}(x,t) = c^2\ddot{f}(x-ct) + c^2\ddot{g}(x+ct)$$

$$u_x(x,t) = \dot{f}(x-ct) + \dot{g}(x+ct), \quad u_{xx}(x,t) = \ddot{f}(x-ct) + \ddot{g}(x+ct)$$

and substituting in the d.e. we verify that u is a solution:

$$\left[c^2\ddot{f}(x-ct) + c^2\ddot{g}(x+ct)\right] - c^2\left[\ddot{f}(x-ct) + \ddot{g}(x+ct)\right] = 0$$

Consider the partial differential equation

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} = 0 \tag{129}$$

where a, b, c are constant scalars. Consider the solution of the form $u = f(y + mx)$. Then

$$\frac{\partial^2 u}{\partial x^2} = m^2 \ddot{f}(y + mx)$$

$$\frac{\partial^2 u}{\partial x \partial y} = m\ddot{f}(y + mx)$$

$$\frac{\partial^2 u}{\partial y^2} = \ddot{f}(y + mx)$$

Substitute in the pde:

$$am^2\ddot{f}(y + mx) + bm\ddot{f}(y + mx) + c\ddot{f}(y + mx) = 0$$

$$am^2\ddot{f}(y+mx)+bm\ddot{f}(y+mx)+c\ddot{f}(y+mx)=0$$
$$\ddot{f}(y+mx)[am^2+bm+c]=0$$

Thus $f(y+mx)$ is a solution of

$$a\frac{\partial^2 u}{\partial x^2}+b\frac{\partial^2 u}{\partial x\partial y}+c\frac{\partial^2 u}{\partial y^2}=0 \qquad \text{(cf. 129)}$$

if m satisfies

$$am^2+bm+c=0 \qquad (130)$$

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{(cf. 129)}$$

$$am^2 + bm + c = 0 \qquad \text{(cf. 130)}$$

If Equation (130) has distinct roots m_1 and m_2 then $f(y + m_1 x) + g(y + m_2 x)$ is a solution.

If Equation (130) has repeated root m then $f(y + mx) + xg(y + mx)$ is a solution.

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{(cf. 129)}$$

$$am^2 + bm + c = 0 \qquad \text{(cf. 130)}$$

If in the Equation (130), $a = 0, \ b \neq 0$ then the quadratic equation (130) reduces to $bm + c = 0$ and hence has only one root. Denoting this root by m_1 the partial differential equation has the solution $f(y + m_1x)$. Noting that an arbitrary $g(x)$ satisfies the pde, we have $f(y + m_1x) + g(x)$ as a solution.

If in the Equation (130), $a = 0, b = 0 \ c \neq 0$ then $f(x) + yg(x)$ is a solution.

In the above solutions f and g are arbitrary functions of their respective arguments.

Digression

Definition The rate of change of $f(x, y)$ in the direction of the unit vector $u = (a, b)$ is called the **directional derivative**. It is defined by

$$\lim_{h \to 0} \frac{f(x + ah, y + bh) - f(x, y)}{h}$$

In more familiar terms, directional derivative in the direction of the unit vector $u = (a, b)$ can be written as

$$f_x(x, y)a + f_y(x, y)b$$

Example 173

Solve

$$4u_x - 3u_y = 0$$

together with the auxiliary condition $u(0, y) = y^3$. Equation says that the directional derivative towards the vector $V \triangleq (4, -3)$ must be zero. So, $u(x, y)$ in the direction V must be constant. This implies that u changes in the orthogonal direction $(3, 4)$ only. Thus $u(x, y) = f(3x + 4y)$, where f is any function of one variable. This is the general solution of the PDE.

Example 173 (cont.)

$$4u_x - 3u_y = 0$$
$$u(x, y) = f(3x + 4y)$$
$$u(0, y) = y^3$$

Let us use the auxiliary condition by setting $x = 0$: $f(4y) = y^3$, Let $w \triangleq 4y$, then $f(w) = \frac{w^3}{64}$. Use argument of f as $3x + 4y$ to obtain the solution

$$u(x, y) = \frac{(3x + 4y)^3}{64}$$

$$4u_x - 3u_y = 0$$
$$u(x, y) = f(3x + 4y)$$
$$u(0, y) = y^3$$

Example 173 (cont.)

Let us verify that the solution $u(x, y) = \frac{(3x+4y)^3}{64}$ satisfies the PDE $4u_x - 3u_y = 0$. Note that

$$u_x = \frac{9(3x + 4y)^2}{64}, \quad u_y = \frac{12(3x + 4y)^2}{64}$$

These partial derivatives satisfy the d.e. $4u_x - 3u_y = 0$.
Furthermore, $u(0, y) = \frac{(3 \cdot 0+4y)^3}{64} = y^3$ shows that the auxiliary condition is also satisfied.

Example 174

Consider the pde

$$au_x + bu_y + cu = f(x, y) \tag{131}$$

Use the transformation

$$w = bx - ay, \quad z = y$$

which is invertible:

$$x = \frac{1}{b}(w + az), \quad y = z$$

$$\begin{bmatrix} b & -a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} w \\ z \end{bmatrix}$$

$$au_x + bu_y + cu = f(x, y) \qquad \text{(cf. 131)}$$

$$w = bx - ay, \quad z = y$$

$$x = \frac{1}{b}(w + az), \quad y = z$$

Example 174 (cont.)

Define the solution in terms of the new variables w and z as $v(w, z)$ Let us express $au_x + bu_y$ in terms of the new function v:

$$au_x + bu_y = a\frac{\partial}{\partial x}v(w, z) + b\frac{\partial}{\partial y}v(w, z)$$

$$\begin{aligned} au_x + bu_y &= a\left[\frac{\partial v}{\partial w}\frac{\partial w}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right] + b\left[\frac{\partial v}{\partial w}\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial y}\right] \\ &= a\left[v_w \cdot b + v_z \cdot 0\right] + b\left[v_w(-a) + v_z \cdot 1\right] \\ &= bv_z \end{aligned} \qquad (132)$$

$$au_x + bu_y + cu = f(x, y) \qquad \text{(cf. 131)}$$

$$w = bx - ay, \quad z = y$$

$$x = \frac{1}{b}(w + az), \quad y = z$$

Example 174 (cont.)

$$au_x + bu_y = a\frac{\partial}{\partial x}v(w, z) + b\frac{\partial}{\partial y}v(w, z)$$

$$\begin{aligned} au_x + bu_y &= a\left[\frac{\partial v}{\partial w}\frac{\partial w}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right] + b\left[\frac{\partial v}{\partial w}\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial y}\right] \\ &= a\left[v_w \cdot b + v_z \cdot 0\right] + b\left[v_w(-a) + v_z \cdot 1\right] \\ &= bv_z \end{aligned}$$

(cf. 132)

Using the result above we can write (131) as

$$bv_z + cv = f\left(\frac{1}{b}(w + az), z\right) \qquad (133)$$

Example 175

Using the solution procedure in the previous example solve

$$3u_x - 2u_y + u = x$$

Use the transformation

$$w = -2x - 3y, \quad z = y$$

and define the solution in terms of w and z as v. Using the path in the previous example we obtain

$$3u_x - 2u_y = -2v_z$$

The pde in the transformed variables becomes

$$-2v_z + v = -\frac{1}{2}(w + 3z)$$

$$\to \frac{\partial v}{\partial z} - \frac{1}{2}v = \frac{1}{4}(w + 3z)$$

$$au_x + bu_y + cu = f(x, y) \qquad \text{(cf. 131)}$$

$$w = bx - ay, \quad z = y$$

$$x = \frac{1}{b}(w + az), \quad y = z$$

$$bv_z + cv = f(\frac{1}{b}(w + az), z)$$

DE: $3u_x - 2u_y + u = x$

Transform: $w = -2x - 3y, \quad z = y \to 3u_x - 2u_y = -2v_z$

Transformed PDE: $-2v_z + v = -\frac{1}{2}(w + 3z)$

Example 175 (cont.)

$$\rightarrow \frac{\partial v}{\partial z} - \frac{1}{2}v = \frac{1}{4}(w + 3z)$$

Integrating factor μ of the above d.e. equals $e^{-\frac{z}{2}}$. Thus

$$\frac{\partial}{\partial z}(e^{-\frac{z}{2}} \cdot v) = \frac{1}{4}(w + 3z)e^{-\frac{z}{2}}$$

$$\rightarrow e^{-\frac{z}{2}} v(w, z) = \frac{1}{4}\int^{z} (w + 3\zeta)e^{-\frac{\zeta}{2}} d\zeta$$

$$\rightarrow e^{-\frac{z}{2}} v(w, z) = -\frac{1}{2}(w + 6 + 3z)e^{-\frac{z}{2}} + c(w)$$

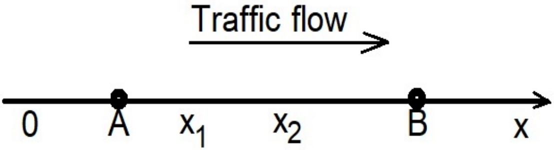
$$\rightarrow v(w, z) = -\frac{1}{2}(w + 6 + 3z) + c(w)e^{\frac{z}{2}}$$

In terms of the original variables x and y:

$$u(x, y) = x - 3 + c(-2x - 3y)e^{\frac{y}{2}}$$

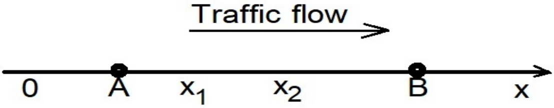
Example 176

Consider one-way flow of traffic along a long straight road between a point A at one end and a point B at the other, on the assumption that vehicles move to the right and can neither enter nor leave the road between A and B. See the figure below where the x axis lies along the road, with the sense of increasing x taken in the direction of the traffic flow.



Let us now consider the flow of vehicles past any two fixed points x_l and x_2 located between A and B, with $x_1 < x_2$.

Example 176 (cont.)



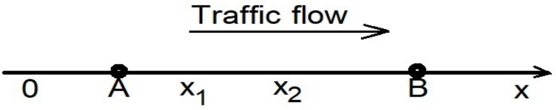
As vehicles can neither enter nor leave the road between A and B, it follows that the increase in the number of vehicles in this interval in a unit of time must be the difference between the number of vehicles entering at x_1 and leaving at x_2. In terms of the vehicle density ρ, at time t total number of cars in the interval is

$$\int_{x_1}^{x_2} \rho(x,t)dx$$

so the rate of change of this quantity is

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho(x,t)dx$$

Example 176 (cont.)



In terms of the flux $q(x, t)$ of vehicles at time t, which we assume to be continuous and differentiable, the difference between the number of vehicles entering at x_1 and leaving at x_2 in a unit of time is $q(x_1, t) - q(x_2, t)$, so as this difference must equal:

$$\frac{d}{dt} \int_{x_1}^{x_2} \rho(x, t) dx = q(x_1, t) - q(x_2, t)$$

Use the integral formula

$$\int_{x1}^{x2} \frac{\partial}{\partial x} q(x, t) dx = q(x_2, t) - q(x_1, t)$$

Example 176 (cont.)

$$\frac{d}{dt}\int_{x_1}^{x_2}\rho(x,t)dx = q(x_1,t) - q(x_2,t), \quad \int_{x1}^{x2}\frac{\partial}{\partial x}q(x,t)dx = q(x_2,t) - q(x_1,t)$$

$$\frac{d}{dt}\int_{x_1}^{x_2}\rho(x,t)dx = -\int_{x1}^{x2}\frac{\partial}{\partial x}q(x,t)dx$$

$$\int_{x_1}^{x_2}\left[\rho_t(x,t) + q_x(x,t)\right]dx = 0$$

Since the above equation holds for arbitrary x_1 and x_2 we must have

$$\rho_t(x,t) + q_x(x,t) = 0$$

In the simplest case vehicle flux q and the traffic density ρ is related by $q = c\rho$, where c denotes constant speed of the vehicles, consequently, $q_x = c\rho_x$. Using this, the pde becomes

$$\rho_t(x,t) + c\rho_x(x,t) = 0$$

Example 176 (cont.)

$$\rho_t(x,t) + c\rho_x(x,t) = 0$$

This pde has zero directional derivative in the $(x,t) = (c,1)$ direction. Solution is constant, therefore, does not change in this direction. Solution is function of the orthogonal direction $(x,t) = (1,-c)$. Thus, we have solution of the form

$$\rho(x,t) = f(x - tc) \tag{134}$$

where f is arbitrary, differentiable function of its argument $x - tc$. This can be verified by substituting in the pde.

$$\rho(x,t) = f(x - tc) \qquad \text{(cf. 134)}$$

Example 176 (cont.)

To discover the connection between the arbitrary function f and the traffic flow in the solution described by (134), we must consider how the traffic flow started. Suppose at time $t = 0$ the traffic density distribution was $\rho(x,0) = F(x)$, where $F(x)$ is a known function of x found by observation at time $t = 0$. Then the condition $\rho(x,0) = F(x)$ imposed on the traffic density at $t = 0$. Setting $t = 0$ in (134) shows $f(x) = F(x)$. This leads to the solution:

$$\rho(x,t) = F(x - ct)$$

Fourier Series

Consider

$$\frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L}) \qquad (135)$$

where a_i, b_i, and L are constants On the set of points where the series (135) converges, it defines a function f, whose value at each point is the sum of the series for that value of x. In this case the series (135) is said to be the **Fourier series** for f.

The first term in the series (135) is written as $\frac{a_0}{2}$ rather than simply as a_0 to simplify a formula for the Fourier series coefficients.

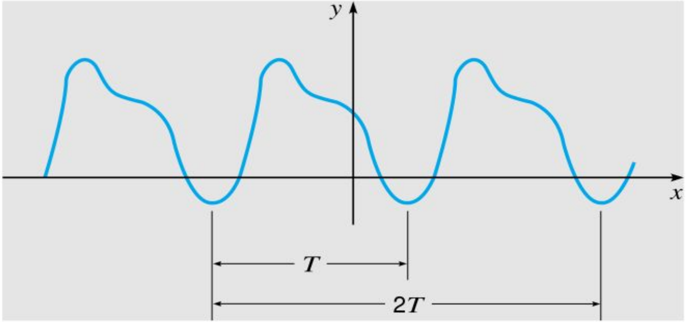
A function f is said to be **periodic** with period $T > 0$ if the domain of f contains $x + T$ whenever it contains x, and if

$$f(x + T) = f(x) \qquad (136)$$

for every value of x.

$$f(x + T) = f(x) \qquad \text{(cf. 136)}$$

The smallest value of T for which Eq. (136) holds is called the fundamental period of f.



If f and g are any two periodic functions with common period T, then their product fg and any linear combination $c_1 f + c_2 g$ are also periodic with period T. To prove the latter statement, let $F(x) = c_1 f(x) + c_2 g(x)$; then for any x

$$F(x+T) = c_1 f(x+T) + c_2 g(x+T) = c_1 f(x) + c_2 g(x) = F(x).$$

Orthogonality of the Sine and Cosine Functions

The standard inner product (u, v) of two real-valued functions u and v on the interval $\alpha \le x \le \beta$ is defined by

$$(u, v) = \int_{\alpha}^{\beta} u(x)v(x)dx.$$

The functions u and v are said to be orthogonal on $\alpha \le x \le \beta$ if their inner product is zero, that is, if

$$\int_{\alpha}^{\beta} u(x)v(x)dx = 0.$$

A set of functions is said to be **mutually orthogonal** if each distinct pair of functions in the set is orthogonal.
The functions $\sin(m\pi x/L)$ and $\cos(m\pi x/L)$, $m = 1, 2, \ldots,$ form a mutually orthogonal set of functions on the interval $-L \leq x \leq L$.
Let m, n be positive integers. then

$$\int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}dx = \begin{cases} 0 & m \neq n \\ L & m = n \end{cases}$$

$$\int_{-L}^{L} \cos\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx = 0 \quad \text{all } m, n$$

$$\int_{-L}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx = \begin{cases} 0 & m \neq n \\ L & m = n \end{cases}$$

$$\frac{a_0}{2} + \sum_{m=1}^{\infty}(a_m \cos\frac{m\pi x}{L} + b_m \sin\frac{m\pi x}{L}) \qquad \text{(cf. 135)}$$

Let (135) converges and let us call it $f(x)$:

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty}(a_m \cos\frac{m\pi x}{L} + b_m \sin\frac{m\pi x}{L}) \qquad (137)$$

First multiply Eq. (137) by $\cos(n\pi x/L)$, where n is a fixed positive integer, and integrate with respect to x from $-L$ to L;

$$\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx = \frac{a_0}{2}\int_{-L}^{L}\cos\frac{n\pi x}{L}dx +$$

$$\sum_{m=1}^{\infty}(a_m\int_{-L}^{L}\cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}dx + b_m\int_{-L}^{L}\sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}dx)$$

$$\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx = \frac{a_0}{2}\int_{-L}^{L}\cos\frac{n\pi x}{L}dx +$$

$$\sum_{m=1}^{\infty}\left(a_m\int_{-L}^{L}\cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}dx + b_m\int_{-L}^{L}\sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}dx\right)$$

When $m = n$, the red colored summation above equals $a_n L$:

$$\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx = a_n L$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx$$

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty}(a_m \cos\frac{m\pi x}{L} + b_m \sin\frac{m\pi x}{L}) \qquad \text{(cf. 137)}$$

To determine a_0 we can integrate Eq. (137) from $-L$ to L, obtaining

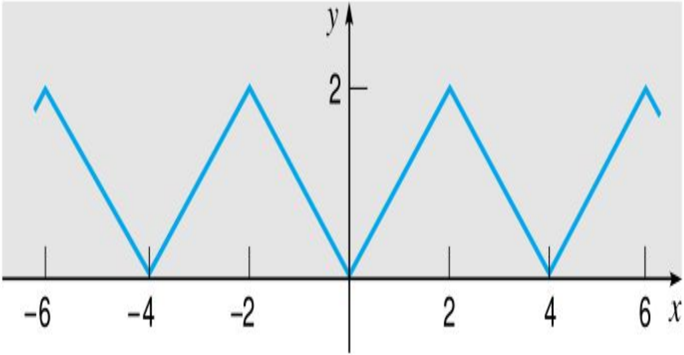
$$\int_{-L}^{L} f(x)dx = La_0$$

$$\rightarrow a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx$$

Likewise

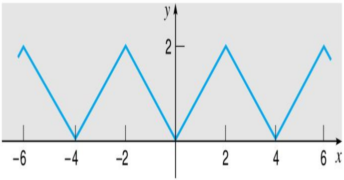
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx$$

Example 177



Consider the triangular wave above. It is periodic with $T = 4$, so $L = 2$.

Example 177 (cont.)



$$f(x) = \begin{cases} -x, & -2 \leq x \leq 0 \\ x, & 0 \leq x \leq 2 \end{cases}$$

Using the formulas derived, we obtain

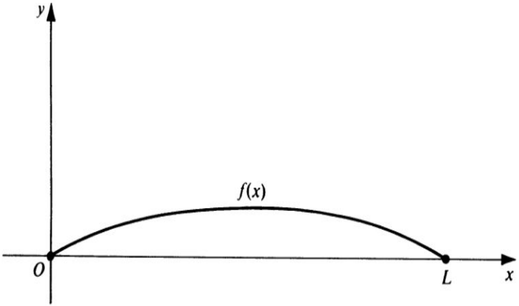
$$a_0 = 2,\ a_m = \begin{cases} -\frac{8}{(m\pi)^2}, & m \text{ odd} \\ 0, & m \text{ even} \end{cases},\quad b_m = 0, m = 1, 2, \ldots$$

$$\rightarrow f(x) = 1 - \frac{8}{\pi^2}\left(\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + \cdots\right)$$

The Vibrating String Problem

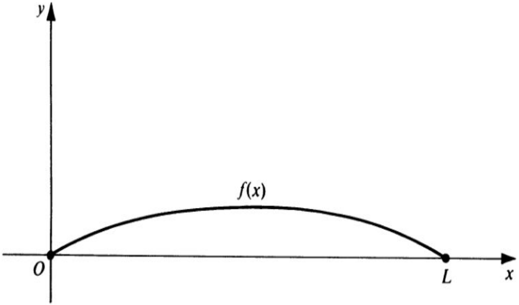
Example 178

The Physical Problem Consider a tightly stretched elastic string the ends of which are fixed on the x axis at $x = 0$ and $x = L$. Suppose that for each x in the interval $0 < x < L$ the string is displaced into the xy plane and that for each such x the displacement from the x axis is given by $f(x)$, where f is a known function of x.



Example 178 (cont.)

Suppose that at $t = 0$ the string is released from the initial position defined by $f(x)$, with an initial velocity given at each point of the interval $0 < x < L$ by $g(x)$, where g is a known function of x. Obviously the string will vibrate, and its displacement in the y direction at any point x at any time t will be a function of both x and t. We seek to find this displacement as a function of x and t; we denote it by y or $y(x, t)$.



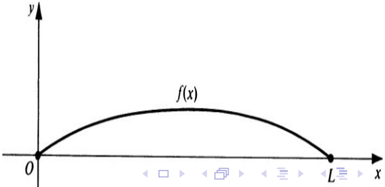
Example 178 (cont.)

Mathematical Problem The string satisfies

$$\alpha^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2} \tag{138}$$

where $\alpha^2 \triangleq \frac{\tau}{\rho}$, with τ and ρ denote string tension and density respectively. Boundary conditions of the string are:

$$\begin{aligned} y(0,t) &= 0, && \text{for } 0 < t < \infty \\ y(L,t) &= 0, && \text{for } 0 < t < \infty \\ y(x,0) &= f(x), && \text{for } 0 \leq x \leq L \\ \tfrac{\partial y(x,0)}{\partial t} &= g(x), && \text{for } 0 \leq x \leq L \end{aligned} \tag{139}$$



$$\alpha^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2} \qquad \text{(cf. 138)}$$

Example 178 (cont.)

Let the solution have the form

$$y(x, t) = X(x)T(t) \tag{140}$$

$$\rightarrow \frac{\partial^2 y}{\partial x^2} = T\frac{d^2 X}{dx^2}, \quad \frac{\partial^2 y}{\partial t^2} = X\frac{d^2 T}{dt^2}$$

Substituting in the pde:

$$\alpha^2 T \frac{d^2 X}{dx^2} = X \frac{d^2 T}{dt^2}$$

Dividing througout XT:

$$\alpha^2 \frac{\frac{d^2 X}{dx^2}}{X} = \frac{\frac{d^2 T}{dt^2}}{T} \tag{141}$$

Example 178 (cont.)

$$\alpha^2 \frac{\frac{d^2X}{dx^2}}{X} = \frac{\frac{d^2T}{dt^2}}{T} \qquad \text{(cf. 141)}$$

Since X is a function of x only, the left member of (141) is a function of x only and is independent of t. Further, since T is a function of t only, the right member of (141) is a function of t only and hence is independent of x. Since one of the two equal expressions in (141) is independent of t and the other one is independent of x, both of them must be equal to a constant k.

$$\alpha^2 \frac{\frac{d^2X}{dx^2}}{X} = k, \quad \frac{\frac{d^2T}{dt^2}}{T} = k$$

From this we obtain the two ordinary differential equations:

Example 178 (cont.)

From this we obtain the two ordinary differential equations:

$$\frac{d^2X}{dx^2} - \frac{k}{\alpha^2}X = 0 \qquad (142) \qquad\qquad \frac{d^2T}{dt^2} - kT = 0 \qquad (143)$$

For the ode (142) the boundary conditions in (139) become

$$\begin{aligned} y(0,t) &= X(0)T(t) = 0, && \text{for } 0 < t < \infty \\ y(L,t) &= X(L)T(t) = 0, && \text{for } 0 < t < \infty \end{aligned} \qquad (144)$$

Satisfying the above equations by assuming $T(t) = 0$ for $0 < t < \infty$ leads to trivial solution $y(x,t) = X(x)T(t) = 0$ for $0 < t < \infty$; therefore we do not assume that. This leads to:

$$X(0) = 0, \ X(L) = 0$$

Example 178 (cont.)

Thus, one of the equation to be solved is

$$\frac{d^2X}{dx^2} - \frac{k}{\alpha^2}X = 0 \qquad \text{(cf. 142)}$$

with $X(0) = 0$, $X(L) = 0$. This is a Sturm-Liouville equation (See the class notes). $k = 0$ leads to the trivial solution. $k > 0$, likewise, leads to the trivial solution. $k < 0$ leads to

$$k = -\frac{n^2\pi^2\alpha^2}{L^2}, \; n = 1, 2, \ldots$$

and the solutions (characteristic functions)

$$X_n = c_n \sin\frac{n\pi x}{L}, \; n = 1, 2, \ldots \qquad (145)$$

Example 178 (cont.)

Now consider the other de:

$$\frac{d^2 T}{dt^2} - kT = 0 \qquad \text{(cf. 143)}$$

Using the k values obtained for the 1st de, the equation above becomes

$$\frac{d^2 T}{dt^2} + \frac{n^2 \pi^2 \alpha^2}{L^2} T = 0, \;\; n = 1, 2, \ldots \tag{146}$$

For each value of n, this differential equation has solutions of the form

$$T_n = c_{n,1} \sin \frac{n\pi\alpha t}{L} + c_{n,2} \cos \frac{n\pi\alpha t}{L}, \quad n = 1, 2, \ldots \tag{147}$$

Example 178 (cont.)

We obtained expressions for X_n and T_n, so that

$$X_n T_n = \left[c_n \sin \frac{n\pi x}{L}\right] \left[c_{n,1} \sin \frac{n\pi\alpha t}{L} + c_{n,2} \cos \frac{n\pi\alpha t}{L}\right]$$

which leads to the solution

$$y_n(x,t) = \left[\sin \frac{n\pi x}{L}\right] \left[a_n \sin \frac{n\pi\alpha t}{L} + b_n \cos \frac{n\pi\alpha t}{L}\right], \; n = 1, 2, \ldots$$

with $a_n := c_n c_{n,1}$, $b_n := c_n c_{n,2}$

Example 178 (cont.)

$$y_n(x,t) = \left[\sin \frac{n\pi x}{L}\right] \left[a_n \sin \frac{n\pi\alpha t}{L} + b_n \cos \frac{n\pi\alpha t}{L}\right], \; n = 1, 2, \ldots$$

The last two initial conditions in (139) shown below are to be satisfied:

$$\begin{array}{ll} y(x,0) = f(x), & \text{for } 0 \leq x \leq L \\ \frac{\partial y(x,0)}{\partial t} = g(x), & \text{for } 0 \leq x \leq L \end{array} \qquad (148)$$

The 1st condition above results in

$$b_n \sin \frac{n\pi x}{L} = f(x), \; 0 \leq x \leq L$$

which cannot be satisfied for an arbitrary $f(x)$
(unless $f(x) = A \sin \frac{n\pi x}{L}$).

Example 178 (cont.)

Knowing that linear combinations of the solutions are also solutions, and infinite series of solutions, with convergence criterion satisfied, are also solutions, we express the solution as

$$\sum_{n=1}^{\infty} y_n(x,t) = \sum_{n=1}^{\infty} \left[\sin \frac{n\pi x}{L}\right] \left[a_n \sin \frac{n\pi\alpha t}{L} + b_n \cos \frac{n\pi\alpha t}{L}\right]$$

Denoting the sum on the left by $y(x,t)$ we can write

$$y(x,t) = \sum_{n=1}^{\infty} \left[\sin \frac{n\pi x}{L}\right] \left[a_n \sin \frac{n\pi\alpha t}{L} + b_n \cos \frac{n\pi\alpha t}{L}\right] \tag{149}$$

Digression Let m, n be positive integers. then

$$\int_0^L \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}dx = \begin{cases} 0 & m \neq n \\ \frac{L}{2} & m = n \end{cases}$$

$$\int_0^L \cos\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx = 0 \ \text{ all } m, n$$

$$\int_0^L \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx = \begin{cases} 0 & m \neq n \\ \frac{L}{2} & m = n \end{cases}$$

Example 178 (cont.)

Next we apply the boundary condition

$$\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} = f(x), \quad 0 \leq x \leq L$$

The coefficients b_n can be found by

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx, \quad (n = 1, 2, \dots)$$

$$y(x,t)=\sum_{n=1}^{\infty}\left[\sin\frac{n\pi x}{L}\right]\left[a_n\sin\frac{n\pi\alpha t}{L}+b_n\cos\frac{n\pi\alpha t}{L}\right]\quad \text{(cf. 149)}$$

Example 178 (cont.)

The only condition which remains to be satisfied is

$$\frac{\partial y(x,0)}{\partial t}=g(x),\ \text{for } 0\leq x\leq L$$

Taking the derivative of (149) we obtain

$$\frac{\partial y(x,t)}{\partial t}=\sum_{n=1}^{\infty}\left[\frac{n\pi\alpha}{L}\right]\left[\sin\frac{n\pi x}{L}\right]\left[a_n\cos\frac{n\pi\alpha t}{L}-b_n\sin\frac{n\pi\alpha t}{L}\right]$$

Using the 2nd i.c.:

$$\sum_{n=1}^{\infty}\frac{a_n n\pi\alpha}{L}\sin\frac{n\pi x}{L}=g(x),\quad 0\leq x\leq L$$

Example 178 (cont.)

$$\frac{\partial y(x,0)}{\partial t} = g(x), \text{ for } 0 \leq x \leq L$$

$$\frac{\partial y(x,t)}{\partial t} = \sum_{n=1}^{\infty} \left[\frac{n\pi\alpha}{L}\right] \left[\sin \frac{n\pi x}{L}\right] \left[a_n \cos \frac{n\pi\alpha t}{L} - b_n \sin \frac{n\pi\alpha t}{L}\right]$$

$$\sum_{n=1}^{\infty} \frac{a_n n\pi\alpha}{L} \sin \frac{n\pi x}{L} = g(x), \quad 0 \leq x \leq L$$

This may compactly be written as

$$\sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} = g(x), \quad 0 \leq x \leq L$$

where $A_n := \frac{a_n n\pi\alpha}{L}$.

Example 178 (cont.)

$$\sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} = g(x), \quad 0 \leq x \leq L$$

The coefficients A_n can be obtained by

$$A_n = \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi x}{L} dx, \quad (n = 1, 2, \dots)$$

Since $A_n := \frac{a_n n\pi\alpha}{L}$ the above formula becomes

$$a_n = \frac{2}{n\pi\alpha} \int_0^L g(x) \sin \frac{n\pi x}{L} dx, \quad (n = 1, 2, \dots)$$

Example 178 (cont.)

Solution recap

$$y(x,t) = \sum_{n=1}^{\infty} \left[\sin \frac{n\pi x}{L}\right] \left[a_n \sin \frac{n\pi\alpha t}{L} + b_n \cos \frac{n\pi\alpha t}{L}\right]$$

$$a_n = \frac{2}{n\pi\alpha} \int_0^L g(x) \sin \frac{n\pi x}{L} dx, \quad (n = 1, 2, \ldots)$$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx, \quad (n = 1, 2, \ldots)$$

Approximate Methods of Solving First-Order Equations

Consider

$$\frac{dy}{dx} = f(x, y) \tag{150}$$

where f is a real function of x and y. The explicit solutions of (150) are certain real functions, and the graphs of these solution functions are curves in the xy plane called the integral curves of (150).

At each point (x, y) at which $f(x, y)$ is defined, the differential equation (150) defines the slope f(x, y) at the point (x, y) of the integral curve of (150) that passes through this point.

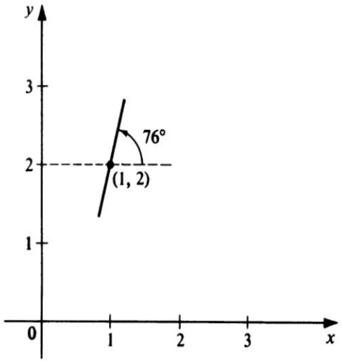
Thus we may construct the tangent to an integral curve of (150) at a given point (x, y) without actually knowing the solution function.

Through the point (x, y) we draw a short segment of the tangent to the integral curve of (150) that passes through this point. That is, through (x, y) we construct a short segment the slope of which is $f(x, y)$, as given by the differential equation (150). Such a segment is called **a line element** of the differential equation (150). For example, let us consider the differential equation

$$\frac{dy}{dx} = 2x + y. \tag{151}$$

Here $f(x, y) = 2x + y$, and the slope of the integral curve of (151) that passes through the point (1,2) has at this point the value $f(1, 2) = 4$. Thus through the point (1, 2) we construct a short segment of slope 4 or, in other words, of angle of inclination approximately 76^o. This short segment is the line element of the differential equation (151) at the point (1,2). It is tangent to the integral curve of (151) which passes through this point.

$f(x, y) = 2x + y$, and the slope of the integral curve that passes through the point (1,2) has at this point the value $f(1, 2) = 4$.



Consider

$$\frac{dy}{dx} = f(x, y) \qquad \text{(cf. 150)}$$

Let us now return to the general equation (150). A line element of (150) can be constructed at every point (x, y) at which $f(x, y)$ in (150) is defined. Doing so for a selection of different points (x, y) leads to a configuration of selected line elements that indicates the directions of the integral curves at the various selected points. We shall refer to such a configuration as a **line element configuration**.

For each point (x, y) at which $f(x, y)$ is defined, the differential equation (150) thus defines a line segment with slope $f(x, y)$, or, in other words, **a direction**. Each such point, taken together with the corresponding direction so defined, constitutes the so-called **direction field** of the differential equation (150). We say that the differential equation (150) defines this direction field, and this direction field is represented graphically by a line element configuration.

Example 179

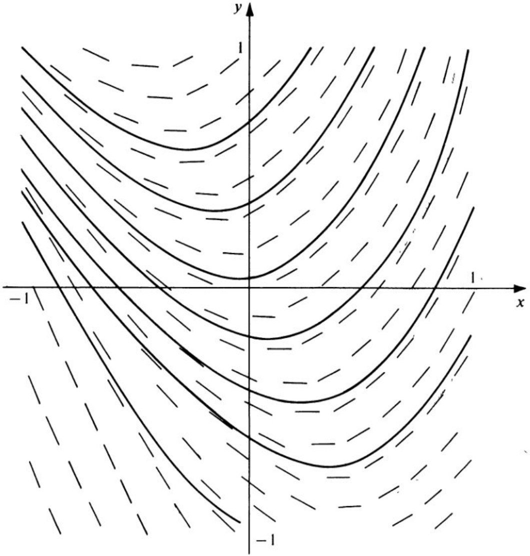
Construct a line element configuration for the differential equation

$$\frac{dy}{dx} = 2x + y. \qquad \text{(cf. 151)}$$

and use this configuration to sketch the approximate integral curves.

(x,y)	slope	inclination
$(0.5, -0.5)$	0.5	27^0
$(0.5, 0)$	1	45^0
$(0.5, 0.5)$	1.5	56^0
$(0.5, 1)$	2	63^0
$(1, -0.5)$	1.5	56^0
$(1, 0)$	2	63^0
$(1, 0.5)$	2.5	68^0
$(1, 1)$	3	72^0

Example 179 (cont.)



The Method of Isoclines

Consider

$$\frac{dy}{dx} = f(x, y) \qquad \text{(cf. 150)}$$

A curve along which the slope $f(x, y)$ has a constant value c is called an **isocline** of the differential equation (150). That is, the isoclines of (150) are the curves $f(x, y) = c$, for different values of the parameter c.

Caution. Note carefully that the isoclines of the differential equation (150) are not in general integral curves of (150). An isocline is merely a curve along which all of the line elements have a single, fixed inclination. This is why isoclines are useful. Since the line elements along a given isocline all have the same inclination, a great number of line elements can be constructed with ease and speed.

Example 180

Employ the method of isoclines to sketch the approximate integral curves of

$$\frac{dy}{dx} = 2x + y. \qquad \text{(cf. 151)}$$

Solution We have already noted that the isoclines of the differential equation (151) are the straight lines $2x + y = c$ or

$$y = -2x + c. \qquad (152)$$

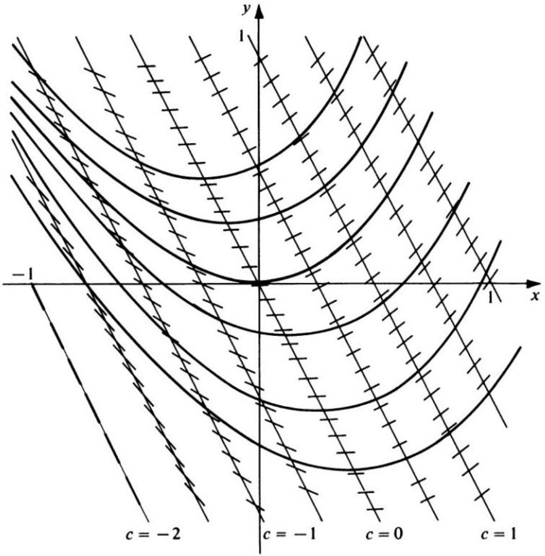


Figure: Isoclines for $\frac{dy}{dx} = 2x + y$

Example 181

Employ the method of isoclines to sketch the approximate integral curves of

$$\frac{dy}{dx} = x^2 + y^2. \tag{153}$$

Solution The isoclines of the differential equation (153) are the concentric circles $x^2 + y^2 = c$. In the figure, the circles for which $c = \frac{1}{16}, \frac{1}{4}, \frac{9}{16}, 1, \frac{25}{16}, \frac{9}{4}, \frac{49}{16}$ and 4 have been drawn with dashes, and several line elements having the appropriate inclination have been drawn along each. For example, for $c = 4$, the corresponding isocline is the circle $x^2 + y^2 = 4$ of radius 2, and along this circle the line elements have inclination $\tan^{-1} 4 = 76^0$.

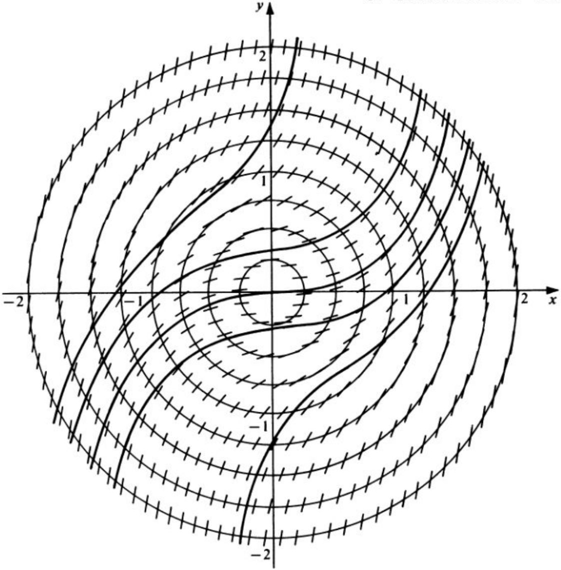


Figure: Isoclines for $\frac{dy}{dx} = x^2 + y^2$.

Source: W. E. Boyce and R. C. DiPrima, Elementary Differential Equations and Boundary Value Problems, John Wiley, 2001

Example 182

Consider the forces that act on the object as it falls:

$$m\frac{dv}{dt} = mg - \gamma v \tag{154}$$

where m is mass of the object, v (m/sec) is velocity of the object, g is the gravity (9.8 m/sec^2), and γ is the drag coefficient. Due to gravity, the object falls. The opposing drag force is assumed to be proportional with the velocity. Therefore, on the righthand side of (154), net downward force is the difference $mg - \gamma v$. Let $m = 10$ kg, and $\gamma = 2$ kg/sec. Then Eq. (154) is rewritten as

$$\frac{dv}{dt} = 9.8 - \frac{v}{5} \tag{155}$$

We investigate the behavior of solutions of Eq. (155) without actually finding its solutions.

Example 182 (cont.)

$$\frac{dv}{dt} = 9.8 - \frac{v}{5} \qquad \text{(cf. 155)}$$

We will proceed by looking at Eq. (155) from a geometrical viewpoint. Suppose that v has a certain value. Then, by evaluating the right side of Eq. (155), we can find the corresponding value of dv/dt.

For instance, if $v = 40$, then $dv/dt = 1.8$. This means that the slope of a solution $v(t)$ has the value 1.8 at any point where $v = 40$. We can display this information graphically in the tv-plane by drawing short line segments, or arrows, with slope 1.8 at several points on the line $v = 40$.

Similarly, if $v = 50$, then $dv/dt = -0.2$, so we draw line segments with slope -0.2 at several points on the line $v = 50$.

We obtain the following figure by proceeding in the same way with other values of v.

Example 182 (cont.)

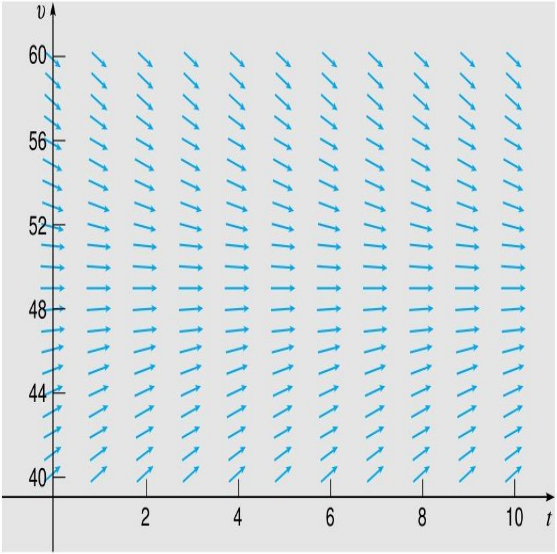


Figure: Direction field for Eq. (155)

Example 182 (cont.)

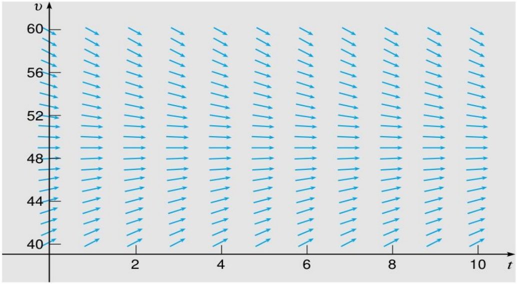
$$\frac{dv}{dt} = 9.8 - \frac{v}{5} \qquad \text{(cf. 155)}$$

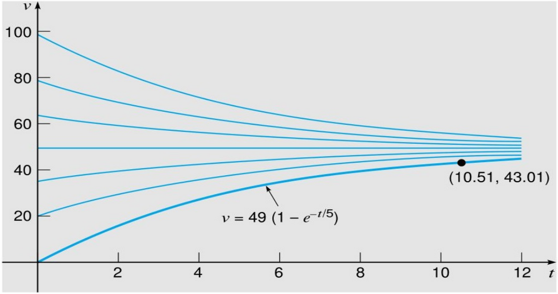
On the graph, if v is less than a certain critical value, then all the line segments have positive slopes, and the speed of the falling object increases as it falls. On the other hand, if v is greater than the critical value, then the line segments have negative slopes, and the falling object slows down as it falls. What is this critical value of v that separates objects whose speed is increasing from those whose speed is decreasing?

It is $v = (5)(9.8) = 49$ m/sec.

Let us substitute $v(t) = 49$ into Eq. (155) and observe that each side of the equation is zero. Because it does not change with time, the solution $v(t) = 49$ is called **an equilibrium solution**. It is the solution that corresponds to a balance between gravity and drag.

Example 182 (cont.)





Solutions of DE sets in the Phase Plane

Example 183

Consider the differential equation

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{156}$$

Its eigenvalues are $r_1 \triangleq -1$ and $r_2 \triangleq -3$. Corresponding eigenvectors are $V^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $V^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Solutions of the d.e. has the form

$$x(t) = c_1 V^{(1)} e^{r_1 t} + c_2 V^{(2)} e^{r_2 t}$$

Using the values we have

$$x(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t}$$

Example 183 (cont.)

$$x(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t}$$

Lets find the solution corresponding to the initial condition

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-0} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3 \cdot 0}$$

$$\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Corresponding solution is

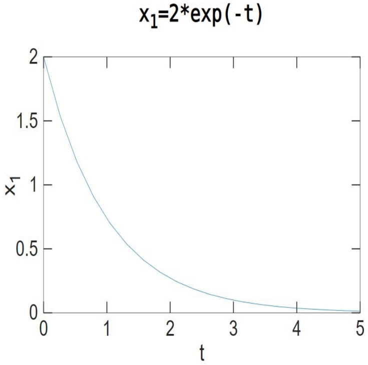
$$x(t) = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + 0 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t}$$

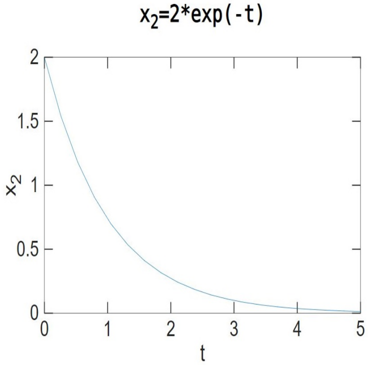
Example 183 (cont.)

Solutions in scalar form are

$$x_1(t) = 2e^{-t}, \;\; x_2(t) = 2e^{-t}$$

Their time graphics are given below:





Example 183 (cont.)

t	$x_1(t)$	$x_2(t)$
0	2	2
0.0445	1.9130	1.9130
0.2345	1.5819	1.5819
1.4077	0.4889	0.4889
3.1805	0.0828	0.0828
5.8435	0.0058	0.0058

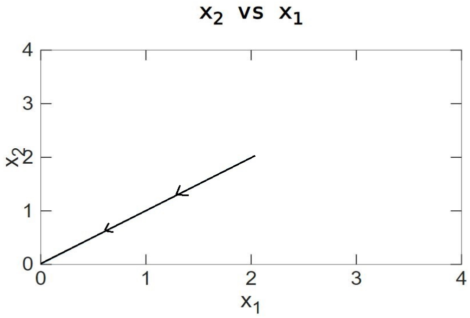
Table: x_1 and x_2 values as a function of time

Example 183 (cont.)

t	$x_1(t)$	$x_2(t)$
0	2	2
0.0445	1.9130	1.9130
0.2345	1.5819	1.5819
1.4077	0.4889	0.4889
3.1805	0.0828	0.0828
5.8435	0.0058	0.0058

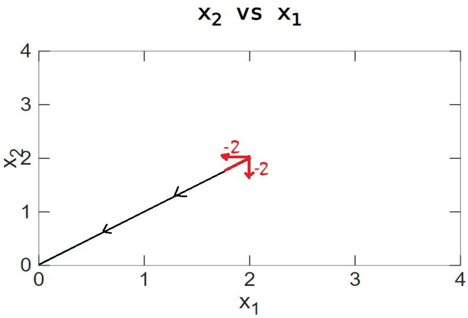
Table: x_1 and x_2 values as a function of time

Instead of plotting each time graphics separately, we may plot x_1 vs. x_2 on the x_1x_2 (phase) plane:



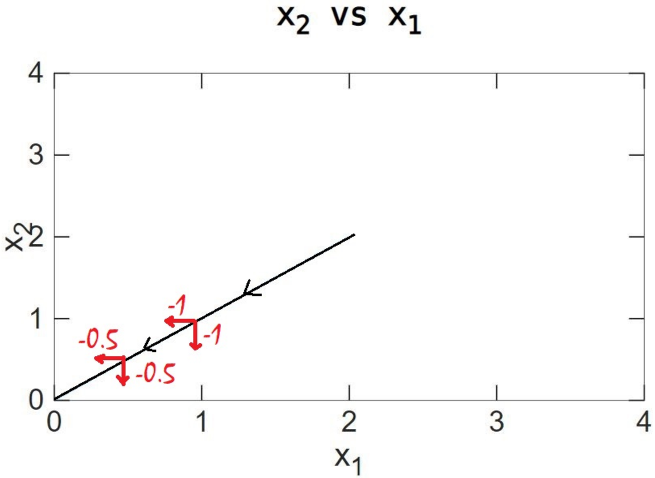
$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \text{(cf. 156)}$$

Example 183 (cont.)



Notice that the trajectory started on the eigenvector line stayed on the line forever. Regarding the d.e. (156), at $(x_1, x_2) = (2, 2)$ we have $(\dot{x}_1, \dot{x}_2) = (-2, -2)$

Example 183 (cont.)



Note that, regarding the d.e. (156), at $(x_1, x_2) = (1, 1)$ we have $(\dot{x}_1, \dot{x}_2) = (-1, -1)$; at $(x_1, x_2) = (0.5, 0.5)$ we have $(\dot{x}_1, \dot{x}_2) = (-0.5, -0.5)$

Example 183 (cont.)

Let us redo the procedure for another i.c. $x(0) = (2, 4)$. The solution corresponding to the initial condition $(2, 4)$:

$$\begin{bmatrix} 2 \\ 4 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-0} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3\cdot 0}$$

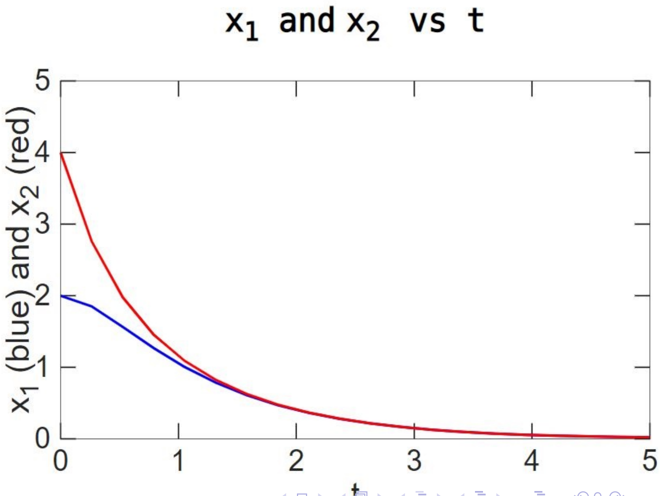
$$\rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Corresponding solution is

$$x(t) = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + 1 \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t}$$

Example 183 (cont.)

$$x(t) = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + 1 \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t}$$



Example 183 (cont.)

t	$x_1(t)$	$x_2(t)$
0	2	4
0.0445	1.9981	3.9111
0.3695	1.8929	3.2750
1.5256	1.1001	1.5342
2.8610	0.4423	0.5557
4.7793	0.0967	0.1133
6.8804	0.0160	0.0181
8.5594	0.0036	0.0040

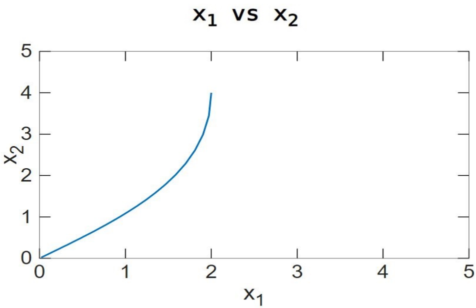
Table: x_1 and x_2 values as a function of time

Example 183 (cont.)

t	$x_1(t)$	$x_2(t)$
0	2	4
0.0445	1.9981	3.9111
0.3695	1.8929	3.2750
1.5256	1.1001	1.5342
2.8610	0.4423	0.5557
4.7793	0.0967	0.1133
6.8804	0.0160	0.0181
8.5594	0.0036	0.0040

Table: x_1 and x_2 values as a function of time

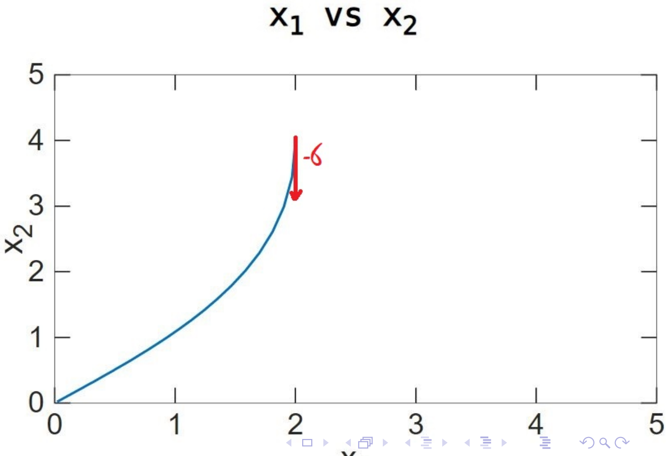
The solution in the phase plane is



Example 183 (cont.)

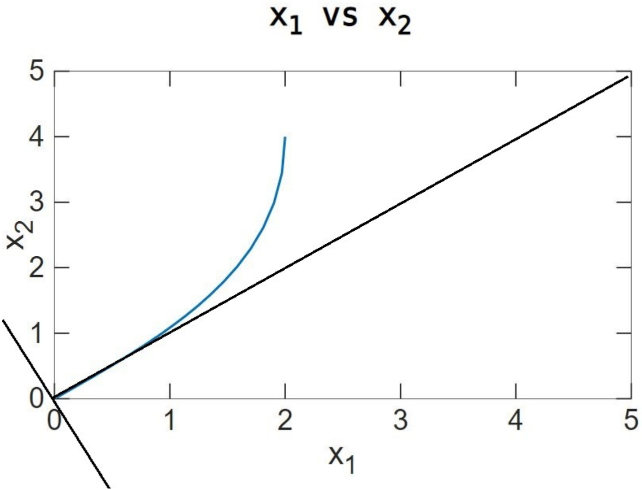
$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \text{(cf. 156)}$$

At $(x_1, x_2) = (2, 4)$, the variable derivatives $(\dot{x}_1, \dot{x}_2) = (0, -6)$:



Example 183 (cont.)

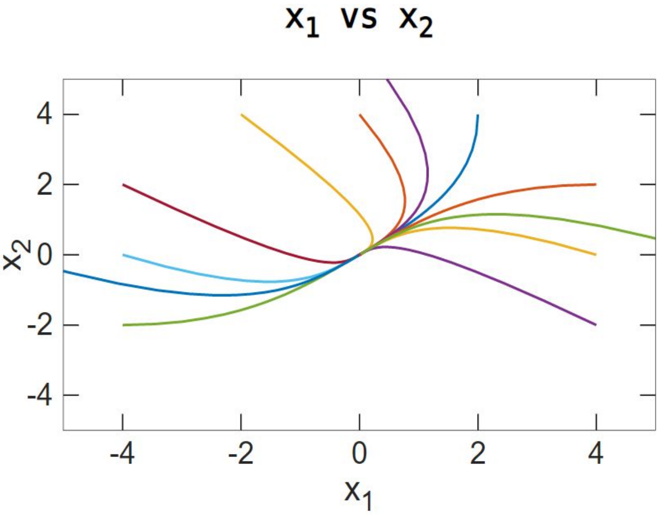
Let us show the eigenvectors on the solution graph:



$$x(t) = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + 1 \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t}$$

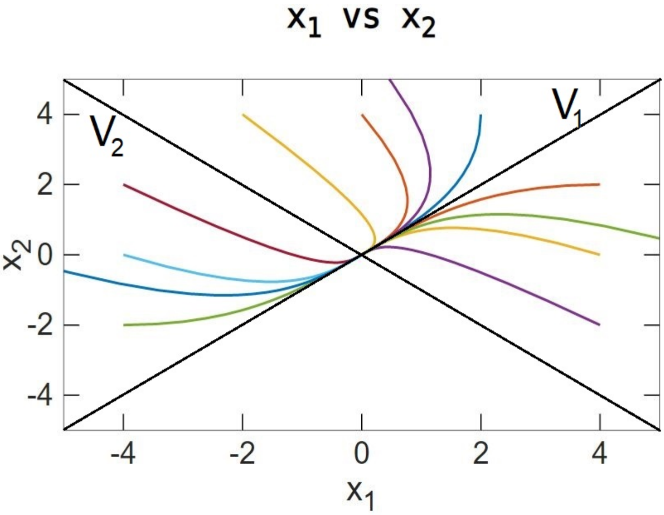
Example 183 (cont.)

Phase portrait for the d.e. (156)



Example 183 (cont.)

Phase portrait for the d.e. (156)with the eigenvectors



Example 184

Consider the differential equation

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad (157)$$

Its eigenvalues are $r_1 \triangleq -1 + 2i$ and $r_2 \triangleq -1 - 2i$. Corresponding eigenvectors are $V^{(1)} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$ and $V^{(2)} = \begin{bmatrix} i \\ 1 \end{bmatrix}$. Solutions of the d.e. have the form

$$x(t) = c_1 e^{-t} \left\{ \cos(2t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \sin(2t) \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right\} + c_2 e^{-t} \left\{ \cos(2t) \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \sin(2t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

Example 184 (cont.)

The solution corresponding to the initial condition $(2, 2)$:

$$\begin{bmatrix} 2 \\ 2 \end{bmatrix} = c_1 e^{-0} \left[\cos(2 \cdot 0) \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \sin(2 \cdot 0) \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right] +$$

$$c_2 e^{-0} \left[\cos(2 \cdot 0) \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \sin(2 \cdot 0 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right]$$

$$\begin{bmatrix} 2 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

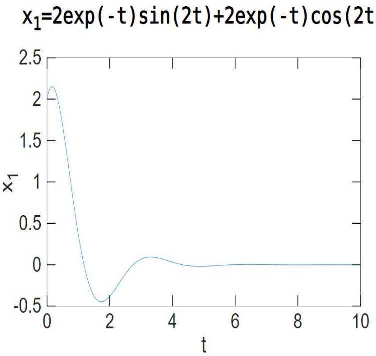
$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

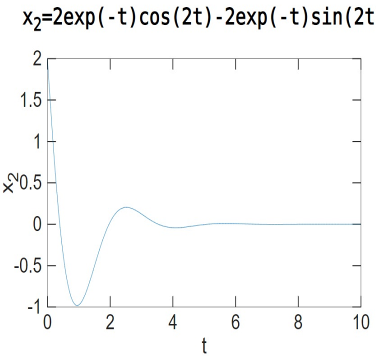
Corresponding solution is

$$x(t) = 2e^{-t} \left[\cos(2t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \sin(2t) \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right] - 2e^{-t} \left[\cos(2t) \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \sin(2t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right]$$

Example 184 (cont.)

Time graphics of the solutions are given below:





Example 184 (cont.)

t	$x_1(t)$	$x_2(t)$
0	2	2
0.0178	2.0334	1.8924
0.0845	2.1194	1.4760
0.1445	2.1452	1.0914
0.3792	1.8330	-0.3443
0.7193	0.6489	-1.6510
1.0376	-0.4913	-1.6988
1.6233	-1.0272	-0.0187
2.4444	0.3850	0.7879

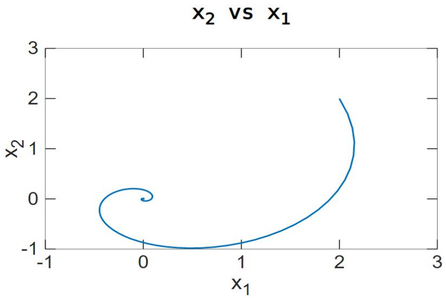
Table: x_1 and x_2 values as a function of time

Example 184 (cont.)

t	$x_1(t)$	$x_2(t)$
0	2	2
0.0178	2.0334	1.8924
0.0845	2.1194	1.4760
0.1445	2.1452	1.0914
0.3792	1.8330	-0.3443
0.7193	0.6489	-1.6510
1.0376	-0.4913	-1.6988
1.6233	-1.0272	-0.0187
2.4444	0.3850	0.7879

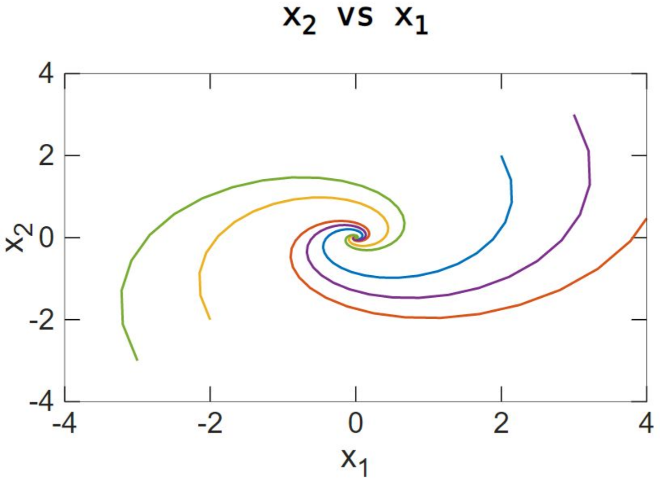
Table: x_1 and x_2 values as a function of time

x_1 vs. x_2 on the phase plane:



Example 184 (cont.)

Phase portrait for the d.e. (157)



Phase Plane Analysis

Since many differential equations cannot be solved conveniently by analytical methods, it is important to consider what qualitative information can be obtained about their solutions without actually solving the equations.

In the sequel we consider the idea of stability of a solution, and employ geometrical methods.

We consider the system

$$\frac{dx}{dt} = Ax \tag{158}$$

where A is a 2×2 constant matrix, and x is a 2×1 vector.

$$\frac{dx}{dt} = Ax \tag{cf. 158}$$

Consider

$$\frac{dy}{dt} = f(t, y) \tag{159}$$

The points where the right side of Eq. (159) is zero are of special importance. Such points correspond to constant solutions, or equilibrium solutions, of Eq. (159), and are often called **critical points**. Similarly, for the system (158), points where

$$Ax = 0$$

correspond to equilibrium (constant) solutions, and they are called critical points. We will assume that A is nonsingular, or that $\det A \neq 0$. It follows that $x = 0$ is the only critical point of the system (158).

$$\frac{dx}{dt} = Ax \qquad \text{(cf. 158)}$$

A solution of Eq. (158) is a vector function $x = \Phi(t)$ that satisfies the differential equation. Such a function can be viewed as a parametric representation for a curve in the $x_1 x_2$ plane. It is often useful to regard this curve as the path, or **trajectory**, traversed by a moving particle whose velocity dx/dt is specified by the differential equation.

The $x_1 x_2$ plane itself is called the **phase plane** and a representative set of trajectories is referred to as a **phase portrait**.

CASE 1: Real Unequal Eigenvalues of the Same Sign

$$\frac{dx}{dt} = Ax \qquad \text{(cf. 158)}$$

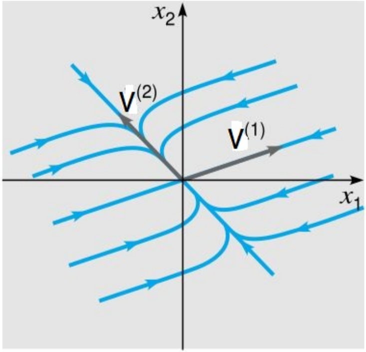
The general solution of Eq. (158) is

$$x = c_1 V^{(1)} e^{r_1 t} + c_2 V^{(2)} e^{r_2 t}$$

In the case of real unequal eigenvalues of the same sign, the eigenvalues r_1 and r_2 are either both positive or both negative. Suppose first that $r_1 < r_2 < 0$, and that the eigenvectors $V^{(1)}$ and $V^{(2)}$ are as shown in figure below. Above solution implies that $||x|| \to 0$ as $t \to \infty$ regardless of the values of c_1 and c_2; in other words, all solutions approach the critical point at the origin as $t \to \infty$.

$$x = c_1 V^{(1)} e^{r_1 t} + c_2 V^{(2)} e^{r_2 t}$$

If the solution starts at an initial point on the line through $V^{(1)}$, then $c_2 = 0$. Consequently, the solution remains on the line through $V^{(1)}$ for all t, and approaches the origin as $t \to \infty$. Similarly, if the initial point is on the line through $V^{(2)}$, then the solution approaches the origin along that line.



$$x = c_1 V^{(1)} e^{r_1 t} + c_2 V^{(2)} e^{r_2 t}$$

It is helpful to write the above equation in the form

$$x = e^{r_2 t} \left(c_1 V^{(1)} e^{(r_1 - r_2)t} + c_2 V^{(2)} \right)$$

Observe that $r_1 - r_2 < 0$. As long as $c_2 \neq 0$, the term $V^{(1)} e^{(r_1 - r_2)t}$ is negligible compared to $c_2 V^{(2)}$ for t sufficiently large.

Note that $e^{(r_1 - r_2)t}$ makes the red colored term smaller as t grows. Obviously, this reduces the parenthetical term to $c_2 V^{(2)}$. Eventually, the overall term becomes $e^{r_2 t} c_2 V^{(2)}$.

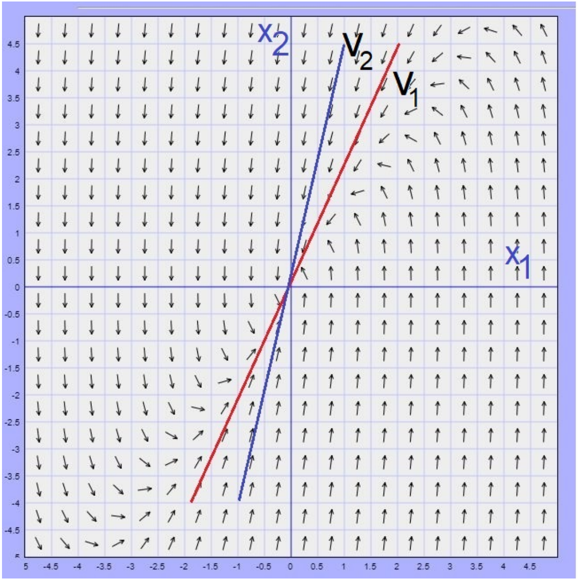
Example 185

$$\frac{dx}{dt} = \begin{bmatrix} 0 & -1 \\ 8 & -6 \end{bmatrix} x$$

The coefficient matrix has eigenvalues -2 and -4. Corresponding eigenvectors are $V^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $V^{(2)} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$. Because both eigenvalues are negative, we know that all the trajectories approach the origin when $t \to \infty$.

For any matrix with two real eigenvalues r_1 and r_2 with two distinct eigenvectors, we have the equilibrium point at the origin, which is called **node**. This point attracts trajectories if eigenvalues are negative and repels them if eigenvalues are positive.

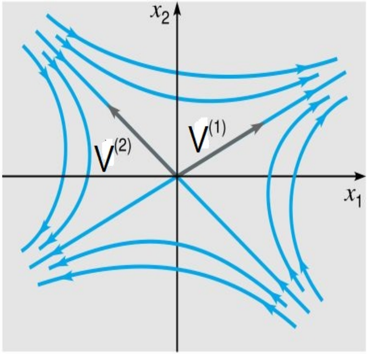
Example 185 (cont'd.)



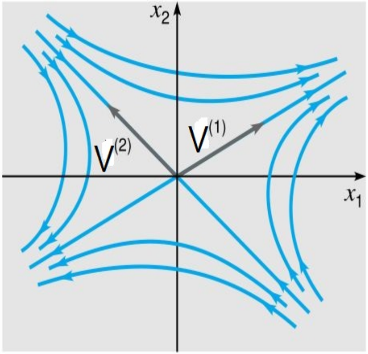
CASE 2: Real Eigenvalues of Opposite Sign

$$x = c_1 V^{(1)} e^{r_1 t} + c_2 V^{(2)} e^{r_2 t}$$

is the general solution, where eigenvalues $r_1 > 0$ and $r_2 < 0$. Suppose that the eigenvectors $V^{(1)}$ and $V^{(2)}$ are as shown in the figure below.



If the solution starts at an initial point on the line through $V^{(1)}$, then it follows that $c_2 = 0$. Consequently, the solution remains on the line through $V^{(1)}$ for all t, and since $r_1 > 0$, $||x|| \to \infty$ as $t \to \infty$. If the solution starts at an initial point on the line through $V^{(2)}$, then the situation is similar except that $||x|| \to 0$ as $t \to \infty$ because $r_2 < 0$. Solutions starting at other initial points follow trajectories such as those shown in the figure below.

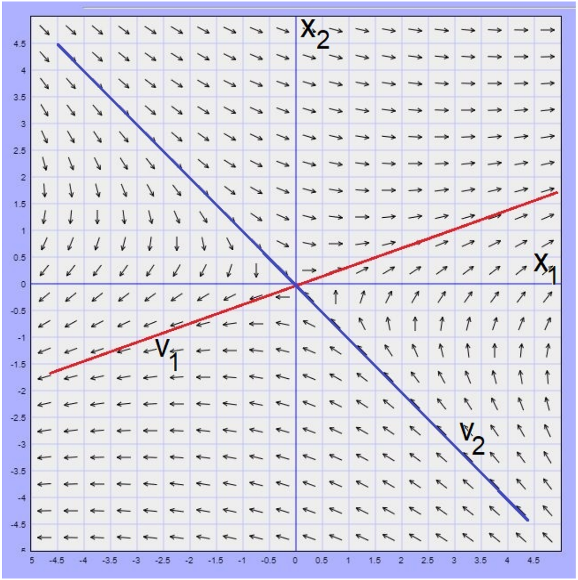


Example 186

$$\frac{dx}{dt} = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix} x$$

The coefficient matrix has eigenvalues 2 and -2. Corresponding eigenvectors are $v_1 \triangleq \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $v_2 \triangleq \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Because it attracts some trajectories and repelssome other trajectories, the point 0 is a **saddle point** of the d.e. If the initial condition is on the v_1 line then its norm goes to infinity along that line. However, if it is on the v_2 line it goes to origin without leaving that line. If the initial condition location is elsewhere then it goes to infinity following the arrows suitable with its position.

Example 186 (cont'd.)



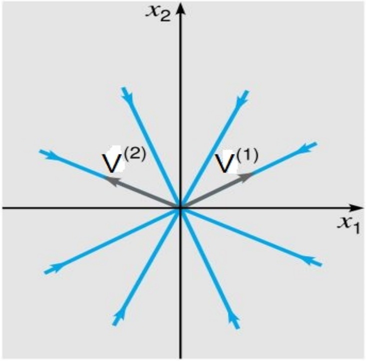
CASE 3: Equal Eigenvalues

We now suppose that $r_1 = r_2 = r$. We consider the case in which the eigenvalues are negative; if they are positive, the trajectories are similar but the direction of motion is reversed. There are two subcases, depending on whether the repeated eigenvalue has two independent eigenvectors or only one.

Subcase a: Two independent eigenvectors Corresponding general solution is

$$x = c_1 V^{(1)} e^{rt} + c_2 V^{(2)} e^{rt}$$

where the eigenvectors $V^{(1)}$ and $V^{(1)}$ are linearly independent. In this case, every trajectory lies on a straight line through the origin, as shown below:



Subcase b: One independent eigenvector Corresponding general solution is

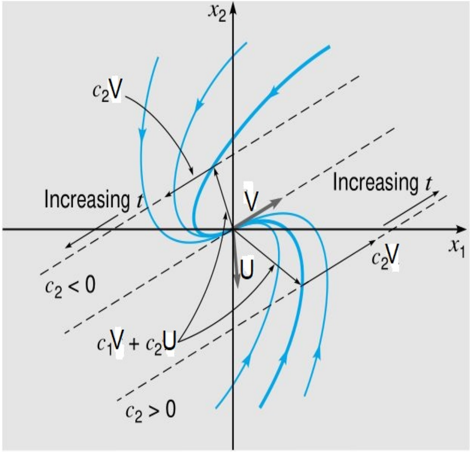
$$x = c_1 V e^{rt} + c_2 (V t e^{rt} + U e^{rt})$$

For large t, the dominant term in the equation above is $c_2 V t e^{rt}$. Thus, as $t \to \infty$, every trajectory approaches the origin tangent to the line through the eigenvector. This is true even if $c_2 = 0$, for then the solution $x = c_1 V e^{rt}$ lies on this line.
We may write the solution as

$$x = [(c_1 V + c_2 U) + c_2 V t] \, e^{rt} \overset{\Delta}{=} y e^{rt}$$

Observe that the vector y determines the direction of x, whereas the scalar quantity e^{rt} affects only the magnitude of x.

When a double eigenvalue has only a single independent eigenvector, the critical point is called an **improper** or **degenerate node**. For this subcase, a phase portrait is given below.



CASE 4: Complex Eigenvalues

Suppose that the eigenvalues are $\lambda \pm i\mu$, where λ and μ are real, $\lambda \neq 0$, and $\mu > 0$. Such systems are typified by

$$\frac{dx}{dt} = \begin{bmatrix} \lambda & \mu \\ -\mu & \lambda \end{bmatrix} x$$

In scalar form

$$x_1' = \lambda x_1 + \mu x_2, \;\; x_2' = -\mu x_1 + \lambda x_2 \tag{160}$$

Introduce polar coordinates:

$$r^2 = x_1^2 + x_2^2, \;\; \tan\theta = x_2/x_1 \tag{161}$$

Differentiate (161)

$$rr' = x_1 x_1' + x_2 x_2', \;\; (\sec^2\theta)\theta' = (x_1 x_2' - x_2 x_1')/x_1^2 \tag{162}$$

$$x_1' = \lambda x_1 + \mu x_2, \quad x_2' = -\mu x_1 + \lambda x_2 \qquad \text{(cf. 160)}$$

$$rr' = x_1 x_1' + x_2 x_2', \quad (\sec^2\theta)\theta' = (x_1 x_2' - x_2 x_1')/x_1^2 \qquad \text{(cf. 162)}$$

Substitute (160) in the first of Eqs. (162), we find that

$$r' = \lambda r$$

$$r = ce^{\lambda t}$$

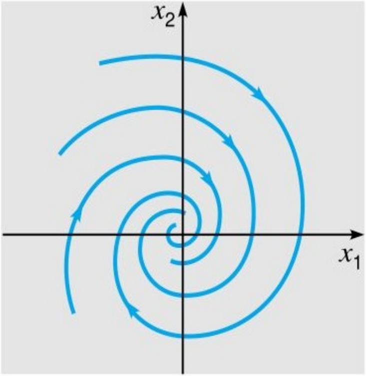
where c is arbitrary constant.
Substitute (160) in the second of Eqs. (162), and using the fact that $\sec^2\theta = r^2/x_1^2$

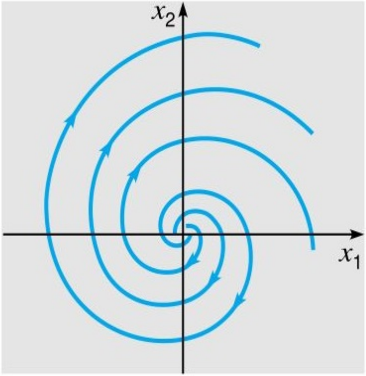
$$\theta' = -\mu$$

$$\theta = -\mu t + \theta_0$$

where θ_0 is arbitrary constant.

Since $\mu > 0$, it follows from $\theta = -\mu t + \theta_0$ that θ decreases as t increases, so the direction of motion on a trajectory is clockwise. As $t \to \infty$, we see from $r = ce^{\lambda t}$ that $r \to 0$ if $\lambda < 0$ and $r \to \infty$ if $\lambda > 0$. The critical point is called a **spiral point** in this case. Typical phase portraits ar shown below:





Complex Arithmetic

Definition 40

A complex number z is an ordered pair $z = (x, y)$ of real numbers x and y with operations of addition and multiplication.

Identify the pairs $(x, 0)$ with real numbers x.
$\therefore$ Complex numbers include the real numbers as a subset.

Complex numbers of the form $(0, y)$ are called imaginary numbers. In $z = (x, y)$, x is known as the real part and y is known as the imaginary part of z.

Set of complex numbers is denoted by $\mathbb{C}$. Consequently, $z \in \mathbb{C}$ means z is a complex number.

Related functions:

$$\mathrm{Re}\,(z) = x, \quad \mathrm{Im}\,(z) = y$$

Let $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$. Define:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2 \ , \ y_1 + y_2)$$

$$(x_1, y_1)(x_2, y_2) = (x_1 x_2 - y_1 y_2 \ , \ y_1 x_2 + x_1 y_2)$$

Note that

$$(x, y) = (x, 0) + \underbrace{(0, 1)(y, 0)}_{(0,y)} \tag{163}$$

Let x denote $(x, 0)$ and let i denote the pure imaginary number $(0, 1)$ we can rewrite (163) as

$$(x, y) = x + iy \tag{164}$$

Note that

$$i^2 = (0,1)(0,1) = (-1,0) = -1$$

In view of expression (164) addition and multiplication can be written as

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

$$(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(y_1x_2 + x_1y_2)$$

Note that

$$(x,y)(a,0) = (ax, ay)$$

$$(a,0)(x,y) = (ax, ay)$$

We therefore define

$$a(x,y) = (ax, ay)$$

$$(x,y)a = (ax, ay)$$

Algebraic Properties

Commutative laws:

$$z_1 + z_2 = z_2 + z_1 \ , \quad z_1 z_2 = z_2 z_1$$

Associative laws:

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3) \ , \quad (z_1 z_2) z_3 = z_1 (z_2 z_3)$$

Distributive law:

$$z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$$

The additive identity $0 = (0, 0)$ and multiplicative identity $1 = (1, 0)$ satisfy

$$z + 0 = z \quad \text{and} \quad z \cdot 1 = z$$

for each complex number z

Additive inverse of z is $(-z)$. That is $z + (-z) = 0$.
Multiplicative inverse z^{-1} of z can be computed as

$$zz^{-1} = 1$$

Let $z = (x, y)$ and $z^{-1} = (u, v)$; then

$$(x, y)(u, v) = 1$$

$$(xu - yv, yu + xv) = (1, 0)$$

$$\rightarrow \quad (xu - yv) = 1 \quad \text{and} \quad (yu + xv) = 0$$

$$\rightarrow \quad u = \frac{x}{x^2 + y^2}\ , \quad v = \frac{-y}{x^2 + y^2}$$

The multiplicative inverse of $z = (x, y)$ is, then,

$$z^{-1} = \left(\frac{x}{x^2 + y^2}\ , \frac{-y}{x^2 + y^2} \right)$$

If a product $z_1 z_2$ is zero, then so is at least one of the factors z_1 and z_2.

For the matrices A and B, the product $AB = \underline{0}$ does not imply $A = \underline{0}$ or $B = \underline{0}$. For instance,

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Suppose that $z_1 z_2 = 0$ and $z_1 \neq 0$. We will show that $z_2 = 0$. The inverse z_1^{-1} exists, and according to the definition of multiplication, any complex number times zero is zero. Hence

$$z_2 = 1 \cdot z_2 = (z_1^{-1} z_1) z_2 = z_1^{-1}(z_1 z_2) = z_1^{-1} \cdot 0 = 0$$

Division by a nonzero complex number is defined as:

$$\frac{z_1}{z_2} = z_1 z_2^{-1}$$

If $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ then

$$\frac{z_1}{z_2} = \left(\frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2}, \frac{y_1 x_2 - x_1 y_2}{x_2^2 + y_2^2} \right), \; z_2 \neq 0$$

The quotient z_1/z_2 is not defined when $z_2 = 0$

Useful Identities

$$\frac{1}{z_1} = z_1^{-1}$$

$$\frac{1}{z_1 z_2} = \frac{1}{z_1}\frac{1}{z_2}$$

$$\frac{z_1 + z_2}{z_3} = \frac{z_1}{z_3} + \frac{z_2}{z_3}$$

$$\frac{z_1 z_2}{z_3 z_4} = \left(\frac{z_1}{z_3}\right)\left(\frac{z_2}{z_4}\right)$$

Example 187

$$\left(\frac{1}{2-3i}\right)\left(\frac{1}{1+i}\right)$$

$$= \frac{1}{5-i} = \frac{1}{5-i}\left(\frac{5+i}{5+i}\right) = \frac{5+i}{26} = \frac{5}{26} + i\frac{1}{26}$$

Exercises

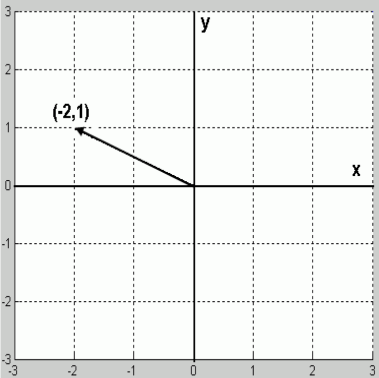
1) Verify that
a) $(\sqrt{2}-i)-i(1-\sqrt{2}i)=-2i$
b) $(2,-3)(-2,1)=(-1,8)$
2) Verify that each of the two numbers $z=1\mp i$ satisfies the equation $z^2-2z+2=0$
3) Solve the equation $z^2+z+1=0$ for $z=(x,y)$ by writing

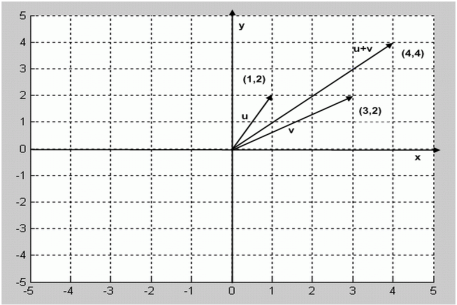
$$(x,y)(x,y)+(x,y)+(1,0)=(0,0)$$

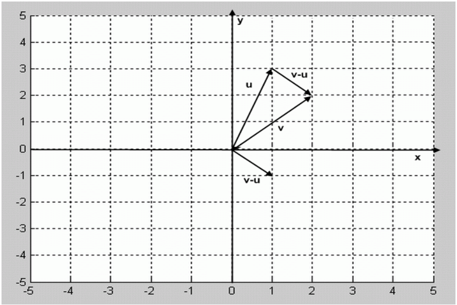
and then solving a pair of simultaneous equation in x and y.
Ans. $z=\left(-\frac{1}{2},\mp\frac{\sqrt{3}}{2}\right)$

Geometric Interpretation

View $z = x + iy$ as a point whose cartesian coordinates (x, y).
Example: The number $-2 + i$ is represented by the point $(-2, 1)$.
The number z can also be thought of as a vector from the origin to the point (x, y).
The xy plane may be called the complex plane, or the z plane.
The x axis is called the real axis, the y axis is called the imaginary axis.







$u + v = (1 + 2i) + (3 + 2i)$ yields $4 + 4i$; and
$v - u = (2 + 2i) - (1 + 3i)$ yields $1 - i$.

The modulus or absolute value of a complex number $z = x + iy$ is defined as the nonnegative real number $\sqrt{x^2 + y^2}$ and is denoted by $|z|$; that is

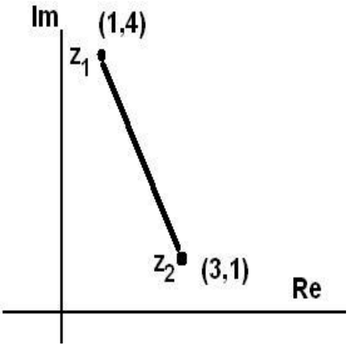
$$|z| = \sqrt{x^2 + y^2}$$

Geometrically the number $|z|$ is the distance between the point

(x, y) and the origin.

$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ is the distance between z_1 and z_2.

Example 188



$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(1-3)^2 + (4-1)^2} = \sqrt{13}$$

Representing Circle in the Complex Plane

The points lying on the circle with center z_0 and radius R satisfy the equation $|z - z_0| = R$.
Note that

$$|(x+iy)-(x_0+iy_0)| = |(x-x_0)+i(y-y_0)| = \sqrt{(x-x_0)^2+(y-y_0)^2} = R$$

Example 189

The points z satisfying equation $|z - 1 + 3i| = 2$ represents the circle whose center is $z_0 = (1, -3)$ and whose radius $R = 2$. The equation may be written as $|z - (1 - 3i)| = 2$.

$$|(x+iy)-(1-3i)| = |x-1+i(y+3)| = \sqrt{(x-1)^2+(y+3)^2} = 2$$

Complex Conjugate

The complex conjugate of $z = x + iy$ is the complex number $x - iy$ and is denoted by $\bar{z}$; that is

$$\bar{z} = x - iy$$

Note that, conjugating an element twice returns the original element:

$$\bar{\bar{z}} = z$$

Useful Identities

$$\overline{\overline{z}} = z\ , \quad \overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}\ , \quad \overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$$

$$\frac{\overline{z_1}}{\overline{z_2}} = \overline{\left(\frac{z_1}{z_2}\right)}\ , \quad z\overline{z} = |z|^2$$

$$|z_1 z_2| = |z_1||z_2|\ , \quad \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}\ , \quad \mathrm{Re}\, z = \frac{z + \overline{z}}{2}$$

$$\mathrm{Im}\, z = \frac{z - \overline{z}}{2i},\ \mathrm{Re}\, z \leq |z|$$

$z\overline{z} = (x + iy)\overline{(x + iy)} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2$

$\mathrm{Re} z = \frac{z+\overline{z}}{2} = \frac{(x+iy)+(x-iy)}{2} = x$

$\mathrm{Re} z = x \leq \sqrt{x^2 + y^2} = |z|$

Triangle Inequality

$$|z_1 + z_2| \leq |z_1| + |z_2| \tag{165}$$

Proof

$$|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1 + z_2}) = (z_1 + z_2)(\overline{z_1} + \overline{z_2})$$

$$= z_1\overline{z_1} + z_1\overline{z_2} + z_2\overline{z_1} + z_2\overline{z_2}$$

$$= |z_1|^2 + (z_1\overline{z_2} + z_2\overline{z_1}) + |z_2|^2$$

But $z_1\overline{z_2} + z_2\overline{z_1} = z_1\overline{z_2} + \overline{z_1\overline{z_2}} = 2\ \mathrm{Re}(z_1\overline{z_2}) \leq 2|z_1\overline{z_2}| = 2|z_1||\overline{z_2}| = 2|z_1||z_2|$

$$\text{and so} \quad |z_1 + z_2|^2 \leq |z_1|^2 + 2|z_1||z_2| + |z_2|^2$$

$$\text{or} \quad |z_1 + z_2|^2 \leq (|z_1| + |z_2|)^2$$

Since moduli are nonnegative the inequality (165) follows.

Generalization of the triangle inequality

$$|z_1 + z_2 + \ldots + z_n| \leq |z_1| + |z_2| + \ldots + |z_n|$$

Exercises

1. Show that $\overline{(2+i)^2} = 3 - 4i$
2. Show that $\left|(2\overline{z}+5)(\sqrt{2}-i)\right| = \sqrt{3}\,|2z+5|$

Polar Form

Let $z = x + iy$ be a complex number. Its **polar** representation is:

$$z = r\,(\cos\theta + i\sin\theta),$$

where r is the **modulus** of z and θ is the **argument** of z. Modulus is not allowed to be negative. The argument is always in radians!!! We have

$$r = \sqrt{x^2 + y^2} \geq 0$$

and θ is any angle such that

$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r} \quad \& \quad \sin\theta = \frac{y}{\sqrt{x^2+y^2}} = \frac{y}{r} \qquad (166)$$

The argument of z is not defined when $z = 0$; equivalently, when $r = 0$.

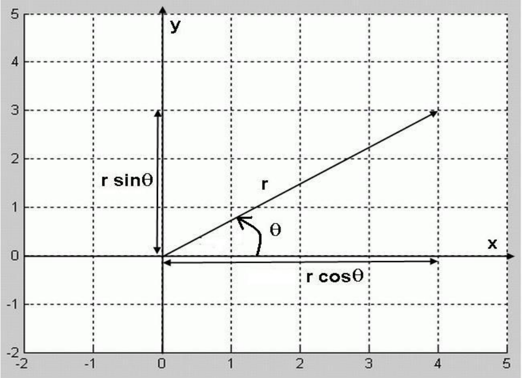


Figure: Polar form illustrations

$r = \sqrt{4^2 + 3^2} = \sqrt{25} = 5$
$\cos\theta = \frac{4}{5}$ and $\sin\theta = \frac{3}{5}$; A solution: $\theta = 0.643$ radians.
$\therefore r = 5$ and $\arg z = \{0.643 + 2k\pi : \ k = 0, \pm 1, \ldots\}$

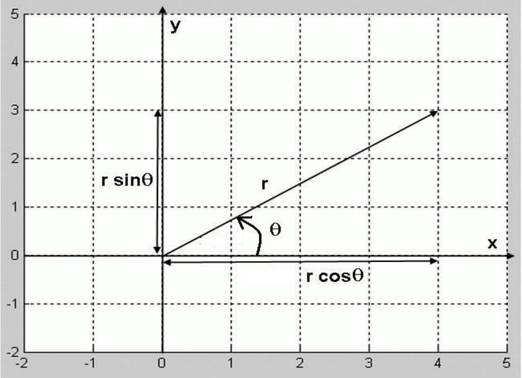


Figure: Polar form illustrations

Notice that arg function generates radians, within the current framework; you cannot say arg $z = 36.86$ degrees!!!

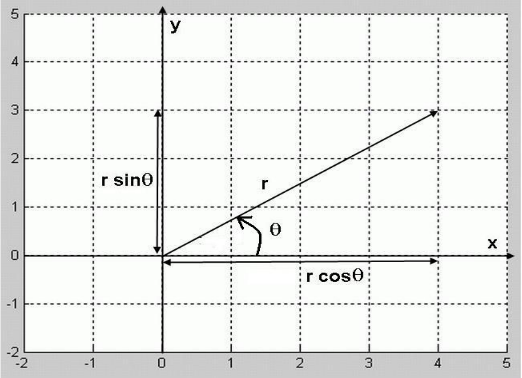


Figure: Polar form illustrations

$$\mathrm{Re}\, z = r\cos\theta\ , \quad \mathrm{Im}\, z = r\sin\theta\ , \quad (166)$$

If θ satisfies (166) then so do $\theta + 2k\pi$ $(k = 0, \mp 1, \mp 2, \ldots)$.
$\therefore$ (166) does not determine a unique value of argument z.

$$\text{Note that } r = \sqrt{x^2 + y^2} = |z| = \sqrt{z\bar{z}}$$

Arg z: Principal value of the argument

If θ is restricted to the interval $-\pi < \theta \leq \pi$, then there is a unique value of θ that satisfies (166).
Called the principal value of the argument and denoted by **Arg z**.

If $z = x + iy$, then

$$-\pi < \mathrm{Arg}\, z \leq \pi \ , \quad \cos(\mathrm{Arg}\, z) = \frac{x}{|z|} \ , \quad \sin(\mathrm{Arg}\, z) = \frac{y}{|z|}$$

The set of all values of the argument will be denoted by

$$\arg z = \{\theta + 2k\pi : k = 0, \mp 1, \mp 2, \ldots\}$$

where θ is any angle that satisfies (166). In particular we have

$$\arg z = \{\operatorname{Arg} z + 2k\pi : k = 0, \mp 1, \mp 2, \ldots\}$$

Unlike Arg z, which is single valued, arg z is multivalued or set valued.

Example 190

Find the modulus, argument, principal value of the argument, and polar form of the given number.

a) $z_1 = 5$ **b)** $z_2 = -3i$ **c)** $z_3 = \sqrt{3} + i$

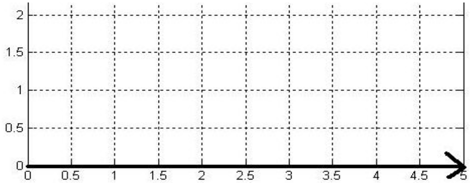
Example 190 (cont.)

a) $z_1 = 5$, $z_1 = 5 + 0i$
$r = |z_1| = \sqrt{5^2 + 0^2} = 5$. An argument of z_1 is 0. Thus

$$\arg z_1 = \{2k\pi : k = 0, \mp 1, \mp 2, \ldots\}$$

Since 0 is in interval $(-\pi, \pi]$, Arg $z_1 = 0$. The polar representation is

$$5 = 5(\cos 0 + i \sin 0)$$



Example 190 (cont.)

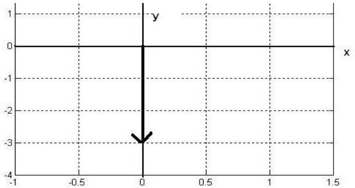
b) $z_2 = -3i, \;\; z_2 = 0 - 3i$

$$r = |z_2| = |0 - 3i| = \sqrt{0^2 + (-3)^2} = 3$$

$$\arg z_2 = \{\frac{3\pi}{2} + 2k\pi : k = 0, \mp 1, \mp 2, \ldots\}$$

Arg $z_2 = \frac{-\pi}{2}$; it is the element of arg z_2 that lies in $(-\pi, \pi]$.

$$-3i = 3\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) = 3\left(\cos\left(\frac{-\pi}{2}\right) + i\sin\left(\frac{-\pi}{2}\right)\right)$$



Example 190 (cont.)

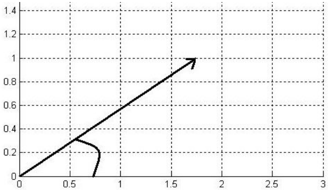
c) $z_3 = \sqrt{3} + i$

$$r = |z_3| = |\sqrt{3} + i| = \sqrt{(\sqrt{3})^2 + 1^2} = 2$$

$$\cos\theta = \frac{x}{r} = \frac{\sqrt{3}}{2} \quad \text{and} \quad \sin\theta = \frac{y}{r} = \frac{1}{2}; \text{ A solution: } \theta = \frac{\pi}{6}$$

$$\arg z_3 = \{\frac{\pi}{6} + 2k\pi : k = 0, \mp 1, \mp 2, \ldots\}, \quad \operatorname{Arg} z_3 = \frac{\pi}{6}$$

$$\sqrt{3} + i = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$



Arithmetic in Polar Form

Let

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \text{ and } z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

$$z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_1)\, r_2(\cos\theta_2 + i\sin\theta_2)$$

$$= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]$$

$$= r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

Thus:

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2 = \{\theta_1 + \theta_2 + 2k\pi : k = 0, \mp 1, \mp 2, \ldots\}$$

$$|z_1 z_2| = |z_1||z_2|$$

When we multiply two complex numbers in polar form, we multiply their moduli and add their arguments.

Inverse of $z = r(\cos\theta + i\sin\theta)$ is

$$z^{-1} = \frac{1}{r}(\cos(-\theta) + i\sin(-\theta)) = \frac{1}{r}(\cos\theta - i\sin\theta)$$

Because it satisfies $zz^{-1} = 1$.

When $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

Example 191

Let $z_1 = 3\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$ and $z_2 = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

$$\begin{aligned} z_1 z_2 &= 2 \cdot 3\left[\cos\left(\frac{\pi}{4} + \frac{5\pi}{6}\right) + i\sin\left(\frac{\pi}{4} + \frac{5\pi}{6}\right)\right] \\ &= 6\left[\cos\frac{13\pi}{12} + i\sin\frac{13\pi}{12}\right] \end{aligned}$$

Example 192

$$z_1 = 5\left[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right], \quad z_2 = 2\left[\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right]$$

$$\frac{z_1}{z_2} = \frac{5}{2}\left[\cos\left(\frac{3\pi}{4} - \frac{\pi}{2}\right) + i\sin\left(\frac{3\pi}{4} - \frac{\pi}{2}\right)\right]$$

$$= \frac{5}{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \frac{5}{2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$$

Exponential Form

$$\underbrace{e^{i\theta} = \cos\theta + i\sin\theta}_{Euler's\ formula} \quad \rightarrow \quad z = re^{i\theta}$$

This formula establishes a relationship between the trigonometric functions and the complex exponential function.
Validity of this formula will be proven in the sequel.
Next we see some of its consequences.

$$e^{i\theta} = \cos\theta + i\sin\theta \quad \rightarrow \quad z = re^{i\theta}$$

Some identities: Let $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$

$$z_1 z_2 = r_1 r_2 \, e^{i(\theta_1+\theta_2)}$$

$$z^{-1} = \frac{1}{r} e^{i(-\theta)}$$

$$zz^{-1} = re^{i\theta}\frac{1}{r}e^{i(-\theta)} = \underbrace{r\frac{1}{r}}_{1}\underbrace{e^{i\theta}e^{i(-\theta)}}_{1} = 1$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\, e^{i(\theta_1-\theta_2)}\,, \quad z_2 \neq 0$$

The circle $|z - z_0| = R$, whose center is z_0 and whose radius is R has the parametric representation

$$z = z_0 + R\, e^{i\theta} \qquad 0 \leq \theta \leq 2\pi$$

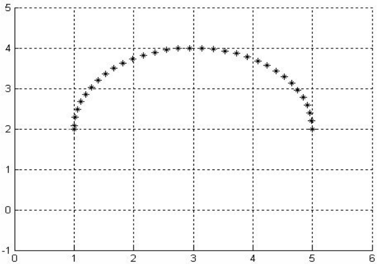
Example 193

$$\left\{z = 3 + 2i + 2e^{i\theta} :\ 0 \le \theta \le \pi\right\}$$

θ	z
0	$5.0000 + 2.0000i$
0.1	$4.9900 + 2.1997i$
0.2	$4.9601 + 2.3973i$
0.3	$4.9107 + 2.5910i$
0.4	$4.8421 + 2.7788i$
0.5	$4.7552 + 2.9589i$
0.6	$4.6507 + 3.1293i$
0.7	$4.5297 + 3.2884i$
0.8	$4.3934 + 3.4347i$
0.9	$4.2432 + 3.5667i$
1.0	$4.0806 + 3.6829i$
1.1	$3.9072 + 3.7824i$
1.2	$3.7247 + 3.8641i$

Example 193 (cont.)

θ	z
1.3	$3.5350 + 3.9271i$
1.4	$3.3399 + 3.9709i$
$\vdots$	$\vdots$
2.9	$1.0581 + 2.4785i$
3.0	$1.0200 + 2.2822i$
3.1	$1.0017 + 2.0832i$
π	$1 + 2i$



Powers and Roots

Integral powers of a nonzero complex number $z = re^{i\theta}$ are given by

$$z^n = r^n e^{in\theta}\ , \quad n = 2, 3, \ldots$$

De Moivre's Formula

$$(e^{i\theta})^n = e^{in\theta} \ \rightarrow \ (\cos\theta + i\sin\theta)^n = \cos(n\theta) + isin(n\theta) \quad n = 2, 3, \ldots$$

Note that

$$(\cos x + i\sin x)^2 = \cos^2 x + 2i\sin x\cos x - \sin^2 x$$

$$= (\cos^2 x - \sin^2 x) + i(2\sin x\cos x) = \cos(2x) + i\sin(2x)$$

Formula holds true for n=2. Use induction to show that it also holds true for larger integers.

Assume that it holds true for a positive integer k, that is,

$$(\cos x + i \sin x)^k = \cos(kx) + i \sin(kx).$$

Then

$$(\cos x + i \sin x)^{k+1} = (\cos x + i \sin x)^k (\cos x + i \sin x)$$

$$= [\cos(kx) + i \sin(kx)] (\cos x + i \sin x)$$

$$= \cos(kx) \cos x - \sin(kx) \sin x + i [\cos(kx) \sin x + \sin(kx) \cos x]$$

$$= \cos[(k+1)x] + i \sin[(k+1)x]$$

Therefore,

$$(\cos \theta + i \sin \theta)^n = \cos(n\theta) + isin(n\theta) \quad n = 2, 3, \ldots$$

or, equivalently,

$$(e^{i\theta})^n = e^{in\theta}$$

Example 194

Let us solve the equation

$$z^6 = 1$$

Write $z = re^{i\theta}$ and look for values of r and θ such that

$$(re^{i\theta})^6 = 1$$

or

$$r^6 e^{i6\theta} = 1e^{i(0+2k\pi)}$$

$$r^6 = 1 \quad \text{and} \quad 6\theta = 0 + 2k\pi \ , \quad k = 0, \pm 1, \ldots$$

Consequently $r = 1$ and $\theta = 2k\pi/6$ and it follows that the complex numbers

$$z = e^{i\frac{2k\pi}{6}} \ , \quad k = 0, \pm 1, \ldots$$

are 6-th roots of unity.

Example 194 (cont.)

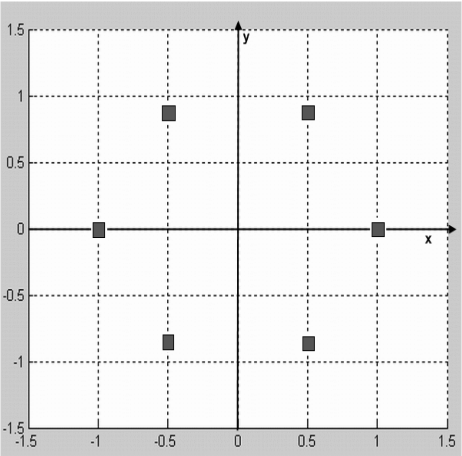


Figure: Roots of $z^6 = 1$

$z^n = 1$ has n distinct roots.

Example 195

Find all values of $(-8i)^{\frac{1}{3}}$. Let $z = (-8i)^{\frac{1}{3}}$ It is equivalent to solving $z^3 = -8i$. Or $(re^{i\theta})^3 = 8e^{i(\frac{-\pi}{2}+2k\pi)}$, $\quad k = 0, \pm 1, \ldots$. That is,

$$r^3 e^{i3\theta} = 8e^{i(\frac{-\pi}{2}+2k\pi)}, \qquad k = 0, \pm 1, \ldots$$

$$r^3 = 8 \text{ and } 3\theta = \frac{-\pi}{2} + 2k\pi, \quad k = 0, \pm 1, \ldots$$

$$\text{The roots are } z_k = 2e^{i(\frac{-\pi}{6}+\frac{2k\pi}{3})}, \quad k = 0, \pm 1, \ldots$$

Exercises

1) Find one value of arg z when z is

a) $\dfrac{-2}{1+\sqrt{3}i}$ **b)** $\dfrac{i}{-2-2i}$ **c)** $(\sqrt{3}-i)^6$

2) By writing the individual factors on the left in exponential form, performing the needed operations, and finally changing back to cartesian coordinates, show that

a) $i(1-\sqrt{3}i)(\sqrt{3}+i) = 2(1+\sqrt{3}i)$

b) $5i/(2+i) = 1+2i$

c) $(-1+i)^7 = -8(1+i)$

d) $(1+\sqrt{3}i)^{-10} = 2^{-11}(-1+\sqrt{3}i)$

3) In each case find all the roots in cartesian form, exhibit them geometrically

a) $(2i)^{1/2}$ b) $(-1-\sqrt{3}i)^{1/2}$ c) $(-16)^{1/4}$

4) Find the four roots of the equation $z^4+4=0$ and use them to factor z^4+4 into quadratic factors with real coefficients.
Ans. $(z^2+2z+2)(z^2-2z+2)$