Introduction

Recall

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For the course "**Communications**"

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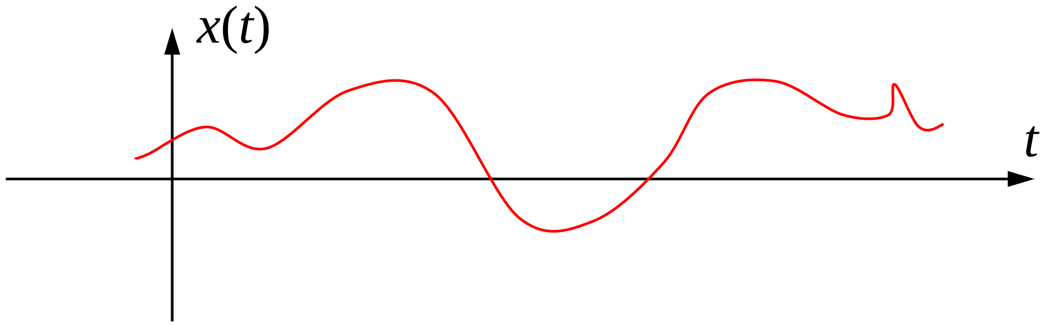
We have seen that the information is represented by symbols, and symbols are represented by signals (mainly by changing some properties of signals)

In Electronic Communication, when we say “signal” we are usually referring to

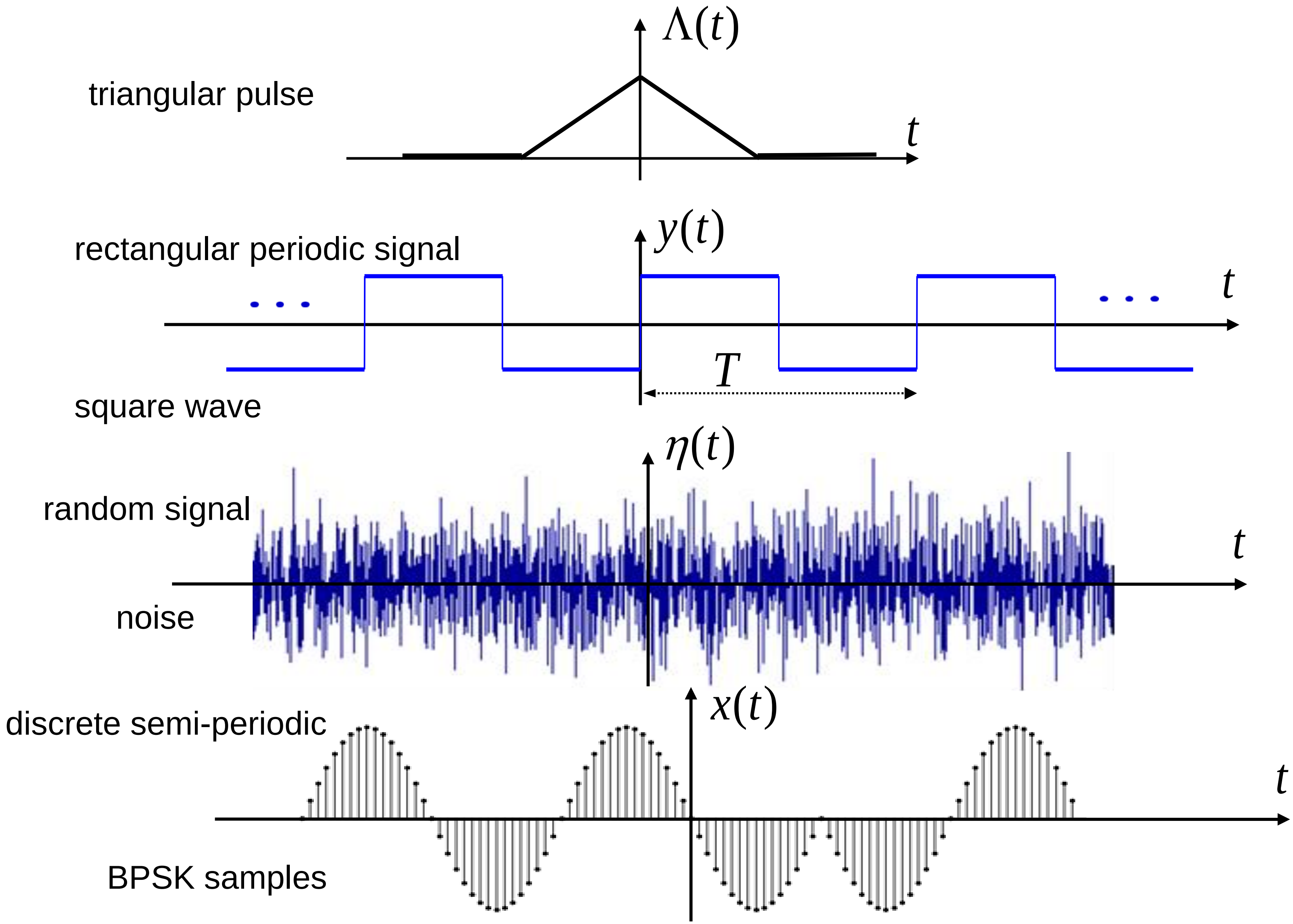
a. Electromagnetic wave, possibly carrying some information.
b. Electric part of the EM wave when it travels on a conductor.
c. A specific part of the electrical wave in b.

depending on the context.

We usually draw plot of instantaneous voltage value of a signal against time while describing/analyzing the signal.



Various Signals



Average / Expected Values

Average value of a continuous signal $m_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{+T/2} x(t)dt$

Average value of discrete samples $x_{avg} = \frac{1}{N} \sum_{i=1}^{N} x_i$

X is a random process, x is its generated values.

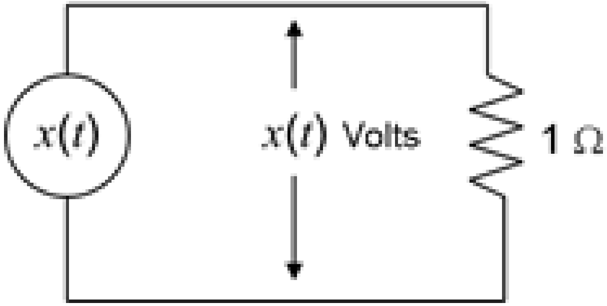
(we will define *random process* later)

Expected value of a continuous random process $E\{X\} = \int_{-\infty}^{\infty} x f(x) dx$

where $f(x)$ is the *probability density function* of x.

Average and expected values are equal when observation duration (or number of samples) is infinite

Energy of a Signal



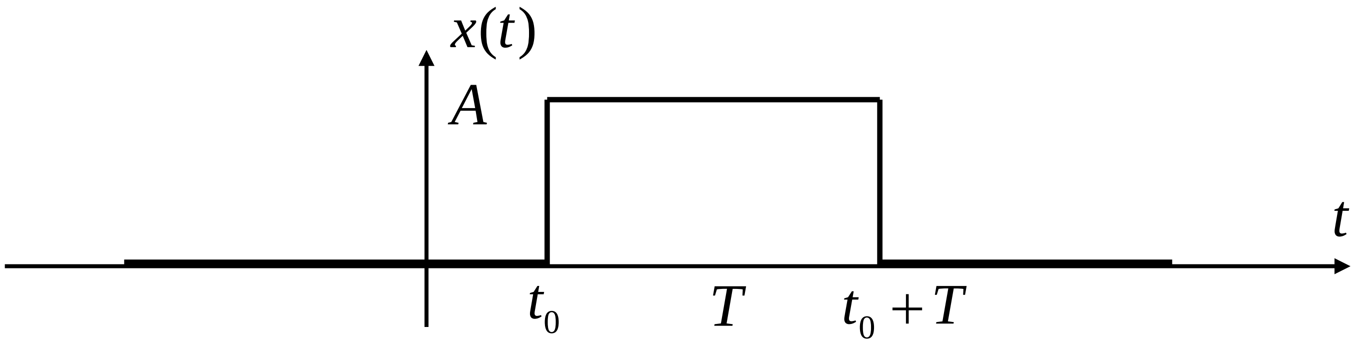
Energy of a signal $x(t)$ is defined as the energy spent on a 1 Ohm load

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

Its unit is $\text{Volts}^2/\text{Ohm}$ = Joules

Example

Energy of a rectangular pulse

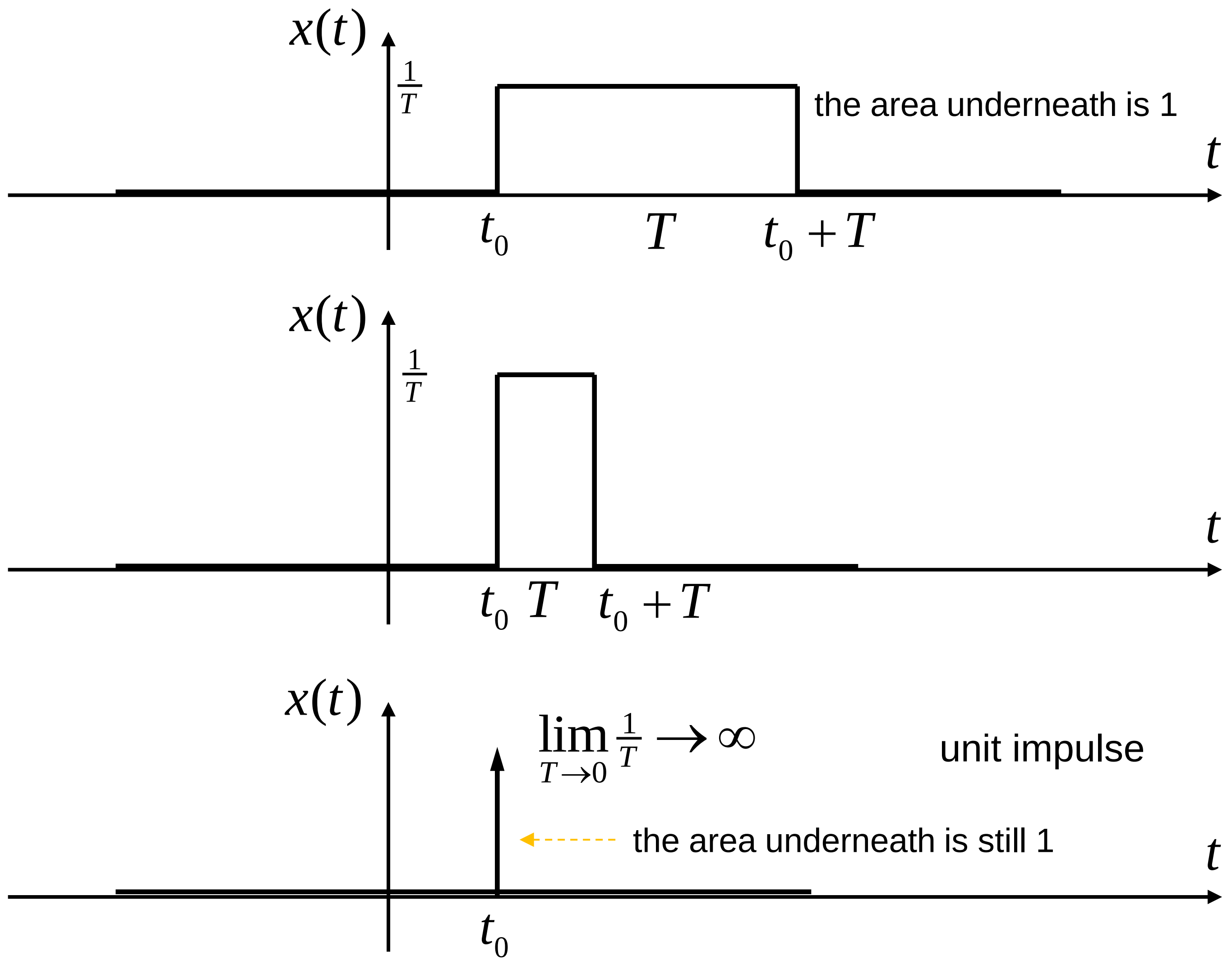


$$E_x = \int_{-\infty}^{\infty} |x(t)|^2\, dt \;=\; \int_{t_0}^{t_0+T} A^2 dt = A^2\, t\Big|_{t_0}^{t_0+T} = A^2 T$$

independent of the position on time axis (valid for all signals)

Example

Let $AT = 1 \Rightarrow A = \frac{1}{T}$



Power of a Signal

If the energy is infinite, then we talk about the energy spent in unit time.

$$P_x = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{+T/2} |x(t)|^2 \, dt$$

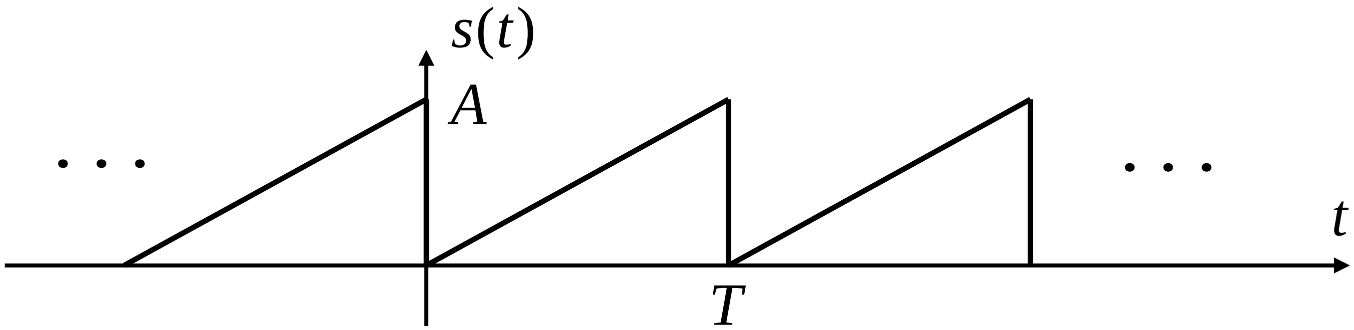
For periodic signals

$$P_x = \frac{1}{T} \int_{\alpha}^{\alpha + T_0} |x(t)|^2 \, dt$$

Its unit is Watts

Example

Find the power of the saw-tooth signal



$$P_s = \frac{1}{T}\int_0^T |s(t)|^2\,dt = \frac{1}{T}\int_0^T \left|\frac{At}{T}\right|^2 dt = \frac{A^2}{3T^3}\,t^3\Big|_0^T$$

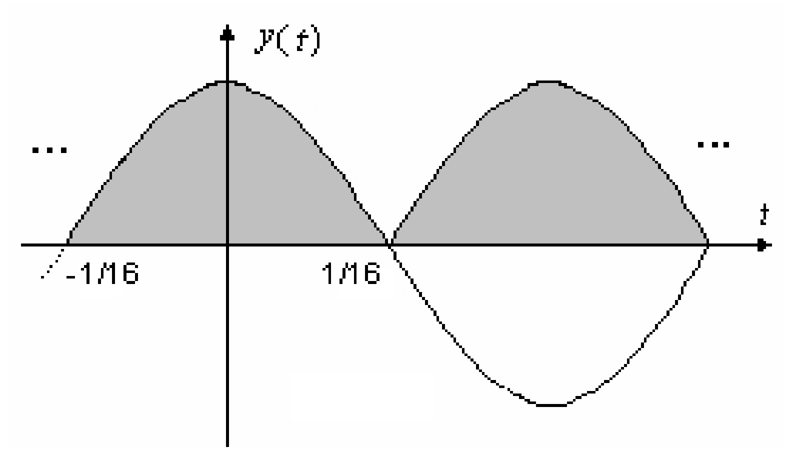
$$P_s = \frac{A^2}{3}$$

power is independent of the period / frequency / time-shift

Example

Find the power and energy of the waveform $y(t) = |\cos(8\pi t)|$

$$P = \frac{1}{T}\int_a^{a+T} |x(t)|^2 \, dt$$



$$P = \frac{1}{1/8}\int_{-1/16}^{1/16} \cos^2(8\pi t)dt$$

$$P = 8\left[\frac{t}{2} + \frac{\sin(16\pi t)}{32\pi}\right]_{-1/16}^{1/16} = \frac{1}{2}$$ (verify!)

Since $\frac{1}{2} < \infty$ it is a power signal. Therefore it is not an energy signal. So, $E = \infty$

Since power is independent of the phase & frequency, it is ok to shift & squeze the waveform & take advantage of symmetry in order to have easier integration

Power and Energy Temporal Densities

The definition of energy $E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$

Total Energy

Density

Defines the distribution of energy along time axis

Given the definition of power $P_x = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{+T/2} |x(t)|^2 \, dt$

and for periodic signals $P_x = \frac{1}{T} \int_{\alpha}^{\alpha + T_0} |x(t)|^2 \, dt$

$P(t) = \frac{1}{T} |x(t)|^2$ should be named as power density/distribution within a period

$P(t)$ is instantaneous power

if the signal is energy signal ($T \to \infty$), temporal power density should be zero.

Generally, if energy is finite then the power is zero (energy signal).
If power is finite, energy is infinite (power signal).

Example Let us have two sinusoidal signals

$$x(t) = \sin(2\pi t / T) \quad \text{and} \quad y(t) = \cos(2\pi t / T)$$

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \infty \qquad E_y = \int_{-\infty}^{\infty} |y(t)|^2 \, dt = \infty$$

energies are both infinite (periodic signals)

$$P_x = \tfrac{1}{T} \int_{\alpha}^{\alpha+T_0} |x(t)|^2 \, dt = \tfrac{1}{T} \int_{0}^{T} |\sin(2\pi t / T)|^2 \, dt = \frac{1}{2}$$

$$P_y = \tfrac{1}{T} \int_{\alpha}^{\alpha+T_0} |y(t)|^2 \, dt = \tfrac{1}{T} \int_{0}^{T} |\cos(2\pi t / T)|^2 \, dt = \frac{1}{2}$$

powers are the same, too

$$m_y = m_x = \lim_{T \to \infty} \tfrac{1}{T} \int_{-T/2}^{+T/2} \sin(2\pi t / T) dt = 0$$ averages are the same

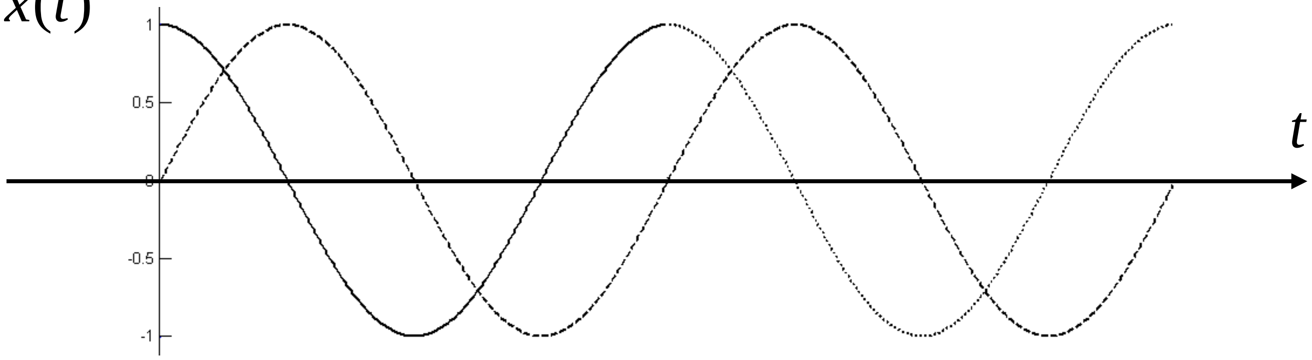
They are both sinusoidal **question** : what is different?

Similarity / Dissimilarity Measure

Similarity of signals is measured using an inner product

$$\langle y(t), x(t) \rangle = \int_{-\infty}^{\infty} y(t)x(t)dt$$

$y(t), x(t)$



It is obvious that the integral, except for finite duration signals, will be infinite. Therefore, we need to have some kind of normalization.

$$< x(t), y(t) >= \int_{0}^{T} \sin(2\pi t / T)\cos(2\pi t / T)dt = 0!$$

Does this mean these two are dissimilar?

We need to check similarity for shifted versions of signals too.

Example

Cross-Correlation

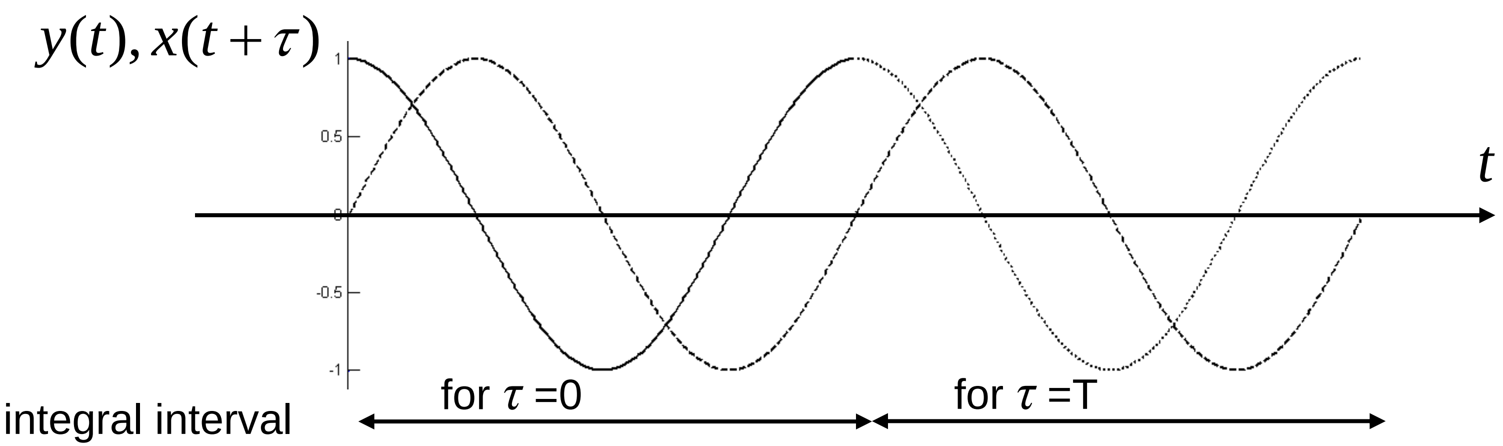
Similarity of shifted versions of signals

$$R''_{xy}(\tau) = \int_{-\infty}^{\infty} x(t)\,y(t+\tau)\,dt$$

called Cross-Correlation Function, where τ represents the time/spatial shift

$$R'_{xy}(\tau) = \int_{0}^{T} x(t)\,y(t+\tau)\,dt \qquad \text{for periodic signals}$$

$$R_{xy}(\tau) = R_{xy}(\tau) / R_{\max} \qquad \text{(normalized)}$$

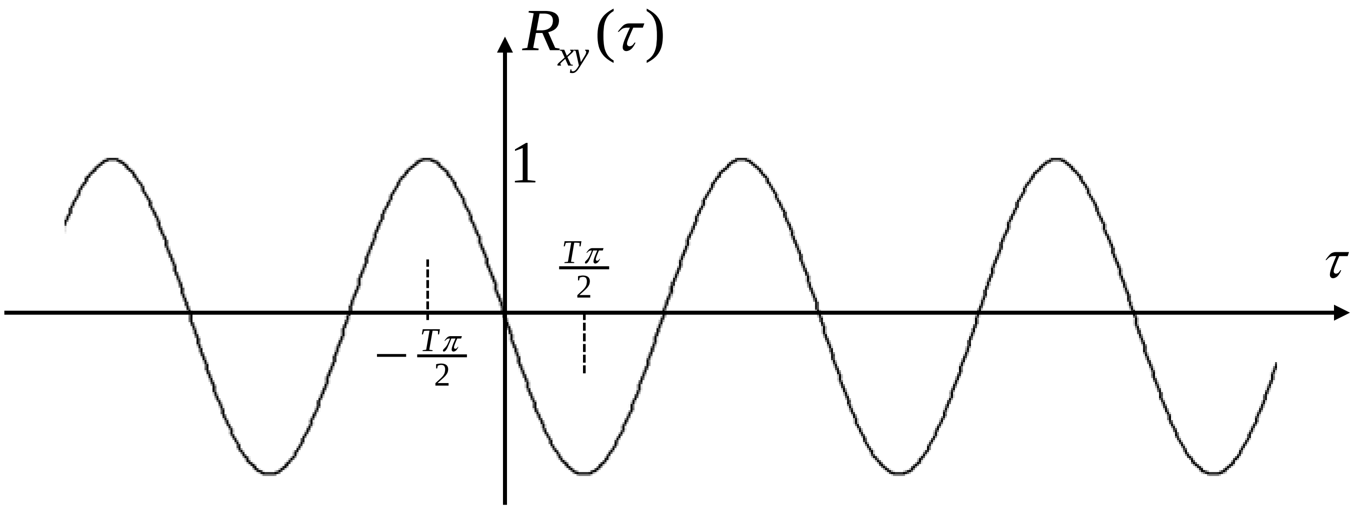


Example

Since our signals are periodic, $T_x = T_y = T$

we can select an integral interval of T

$$R'_{xy} = \int_0^T \sin(2\pi t/T)\cos(2\pi(t+\tau)/T)dt = -\tfrac{T}{2}\sin(2\pi\tau/T)$$



The signals are similar to each other on periodic intervals

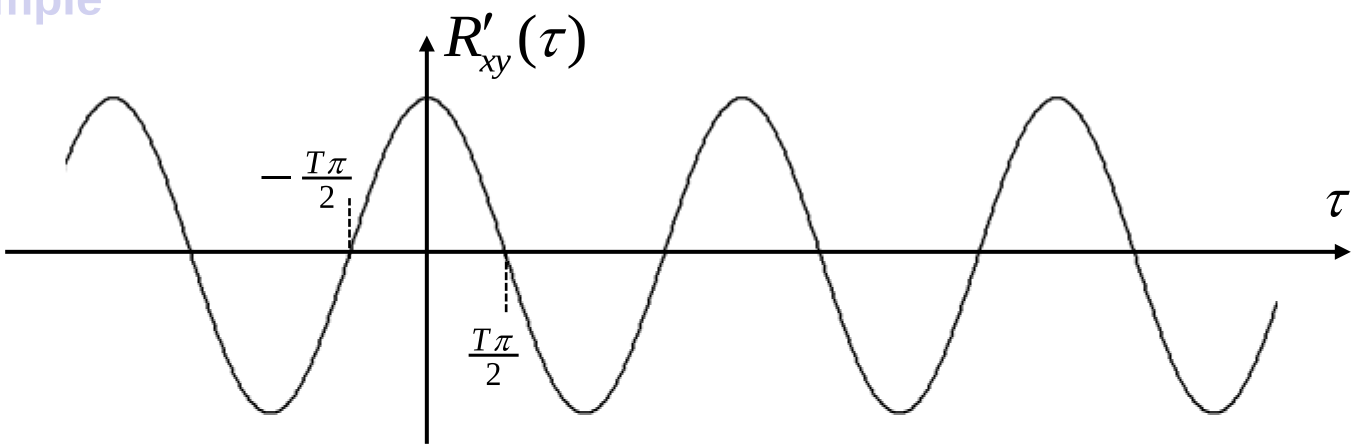
Autocorrelation

if both signals are same; $y(t) = x(t)$

the similarity is named as autocorrelation (function)

$$R''_{xy}(\tau) = \int_{-\infty}^{\infty} x(t)x^*(t+\tau)dt$$

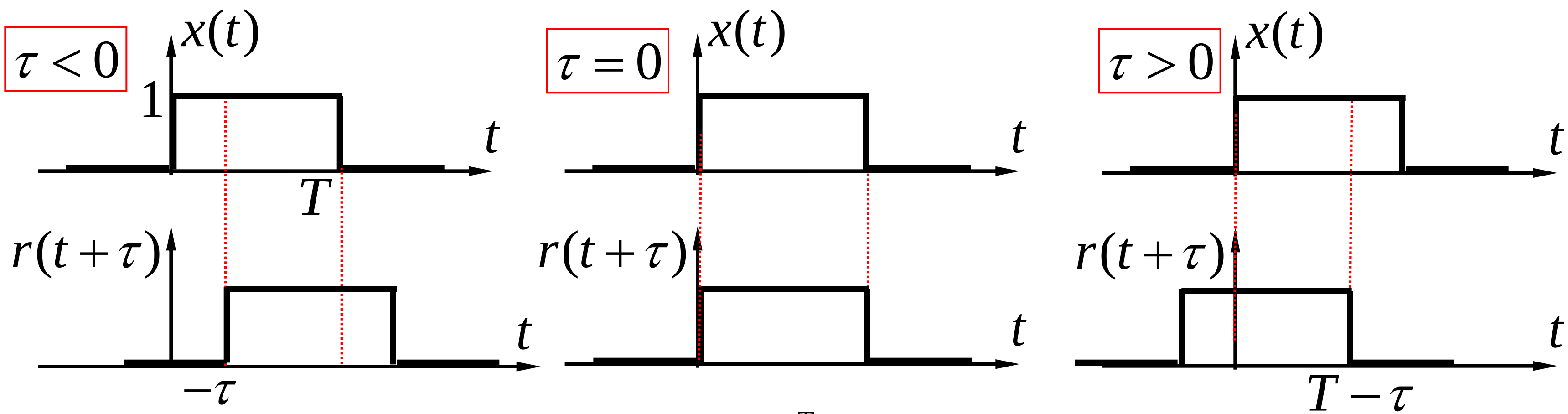
Example



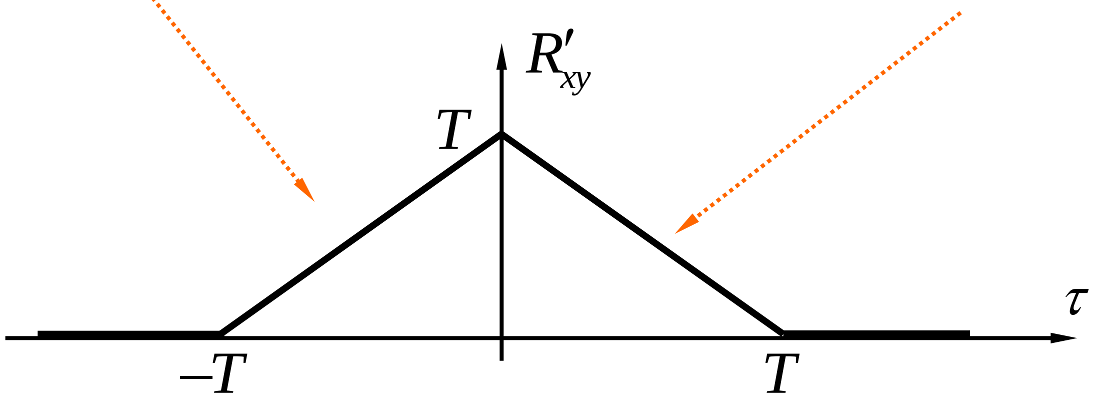
signal x is similar to itself on periodic intervals

Example

$$r(t)=\begin{cases}1 & , 0<t<T \\ 0 & , otherwise\end{cases} \quad \text{and} \quad x(t)=r(t)$$

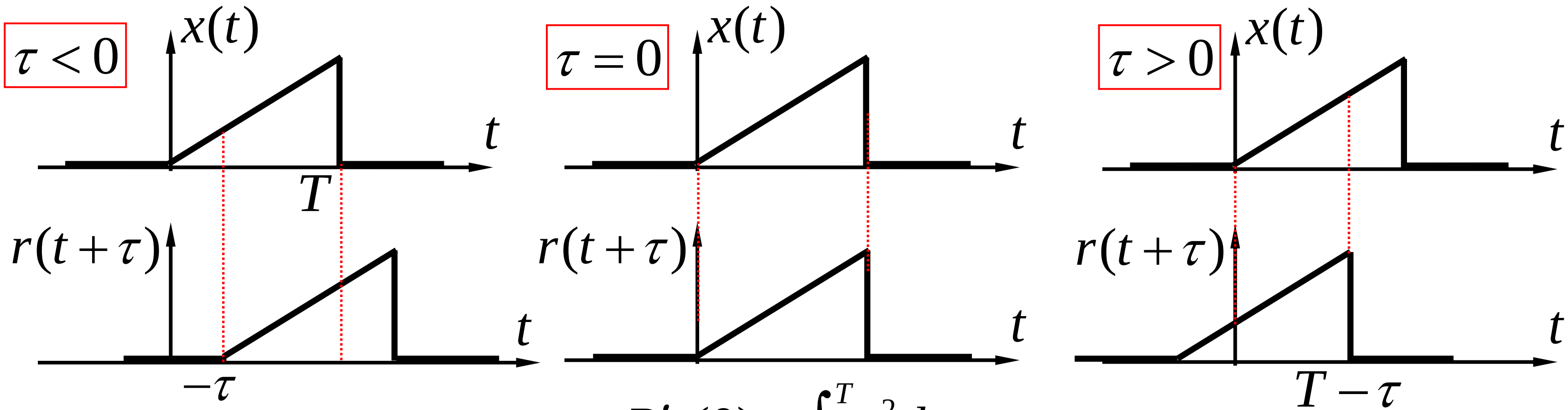


$$R'_{xy}(\tau)=\int_{-\tau}^{T} dt = T+\tau \qquad R'_{xy}(0)=\int_{0}^{T} dt = T \qquad R'_{xy}(\tau)=\int_{0}^{T-\tau} dt = T-\tau$$



Example

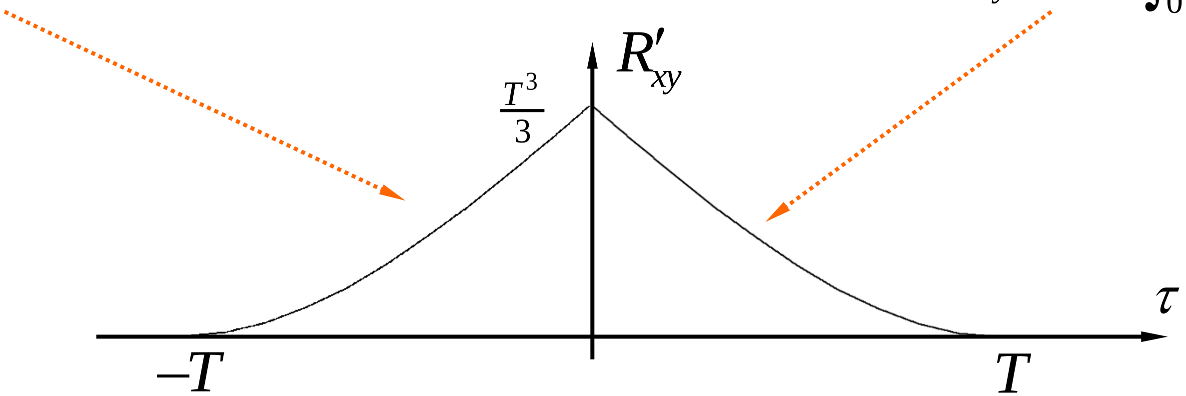
$$r(t)=\begin{cases} t & ,\ 0<t<T \\ 0 & ,\ otherwise \end{cases} \quad \text{and} \quad x(t)=r(t)$$



$$R'_{xy}(\tau)=\int_{-\tau}^{T} t(t+\tau)dt$$

$$R'_{xy}(0)=\int_{0}^{T} t^2 dt$$

$$R'_{xy}(\tau)=\int_{0}^{T-\tau} t(t+\tau)dt$$

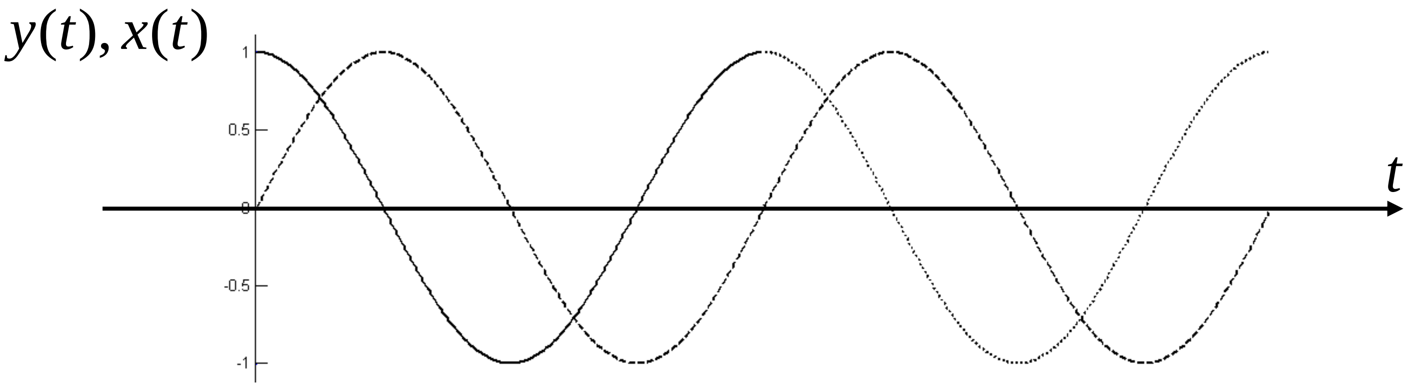


Orthogonal Signals

Inner product also tells us if the signals are orthogonal

If $\langle y(t), x(t) \rangle = \int_{-\infty}^{\infty} y(t)x(t)dt = 0$

then $y(t), x(t)$ are said to be orthogonal

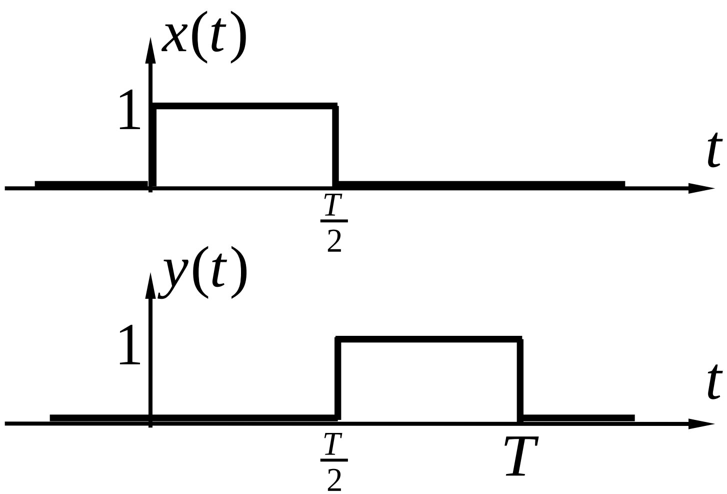


$$< x(t), y(t) >= \int_0^T \sin(2\pi t / T)\cos(2\pi t / T)dt = 0$$

meaning that $y(t)$ does not have any component of $x(t)$ within

(shifted versions of signals may not be orthogonal)

Example



$$< x(t), y(t) >= \int_0^T (u(t) - u(t - \tfrac{T}{2}))(u(t - \tfrac{T}{2}) - u(t - T))dt = 0$$

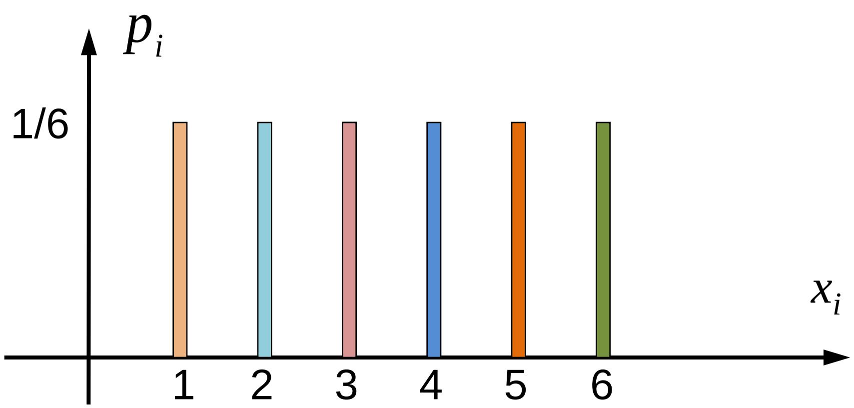
Given a set of waveforms $x_i(t)$, we can find an orthogonal waveform set $\psi_k(t)$ so that $x_i(t)$ can be written as a weighted linear sum of $\psi_k(t)$

$$x_i(t) = \sum_{k=0}^{M-1} c_{k,i} \psi_k(t)$$

Hmw : study this subject (*orthogonalization*) from the referenced sources

Probability

Die throwing experiment



$$\sum_i p(x_i) = 1$$

Random variable : An event or value which is measured

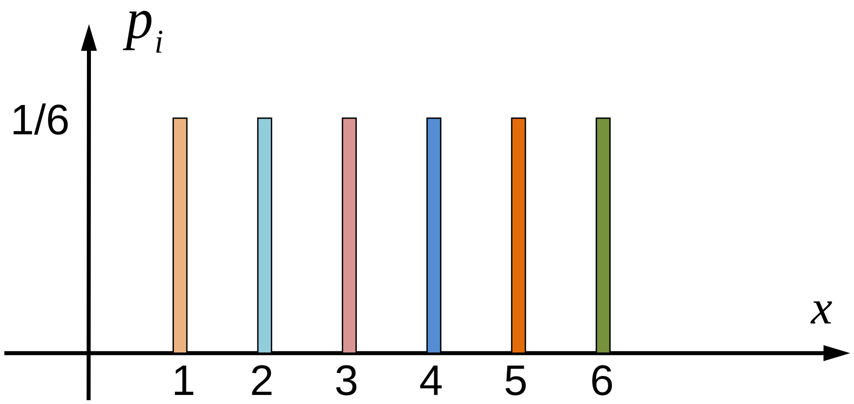
x : Value read on die after throwing event (random variable)
p_i : probability of x ($p(x_i)$)

Expected Value of a Discrete Experiment

$$E(X) = \sum_i x_i p(x_i)$$

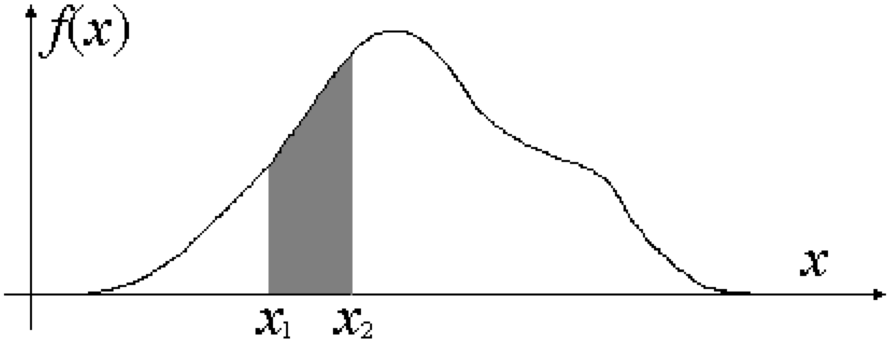
The name "expected value" does not imply that it is expected to happen

Expected value of die throwing experiment = 3.5 which will never happen



This graph is called Probability Mass Function (pmf)

Probability Density Function



$$\int_{-\infty}^{+\infty} f(x)dx = 1.0$$

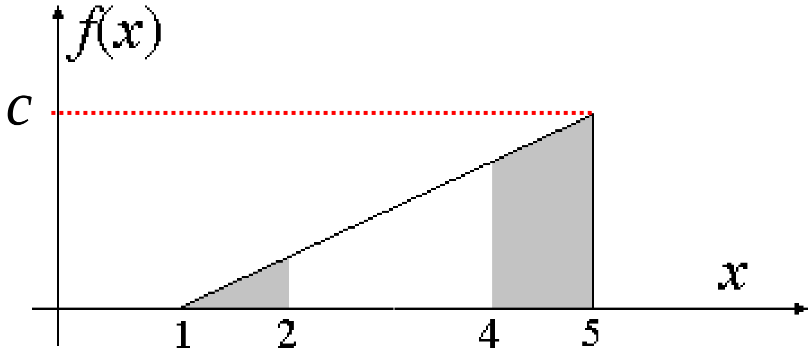
$$P(x_1 \le x < x_2) = \int_{x_1}^{x_2} f(x)dx$$

if this is a "total of something" then this is a "density"

shaded area gives the probability of $P(x_1 \le x < x_2)$

Example

What is c?



$$\int_{-\infty}^{+\infty} f(x)dx = 1.0 \Rightarrow \int_1^5 \frac{c}{4}(x-1)dx = \frac{c}{8}(x-1)^2\Big|_1^5 = 2c \Rightarrow c = \frac{1}{2}$$

$$E(X) = \int_1^5 xf(x)dx = \frac{1}{8}\int_1^5 (x^2 - x)dx = \frac{1}{8}\left[\frac{1}{3}x^3 - \frac{1}{2}x^2\right]_1^5 \simeq 3.67$$

$$p(x \le 2) = \int_1^2 \frac{1}{8}(x-1)dx = \frac{1}{16} \qquad p(x > 4) = \int_4^5 \frac{1}{8}(x-1)dx = \frac{7}{16}$$

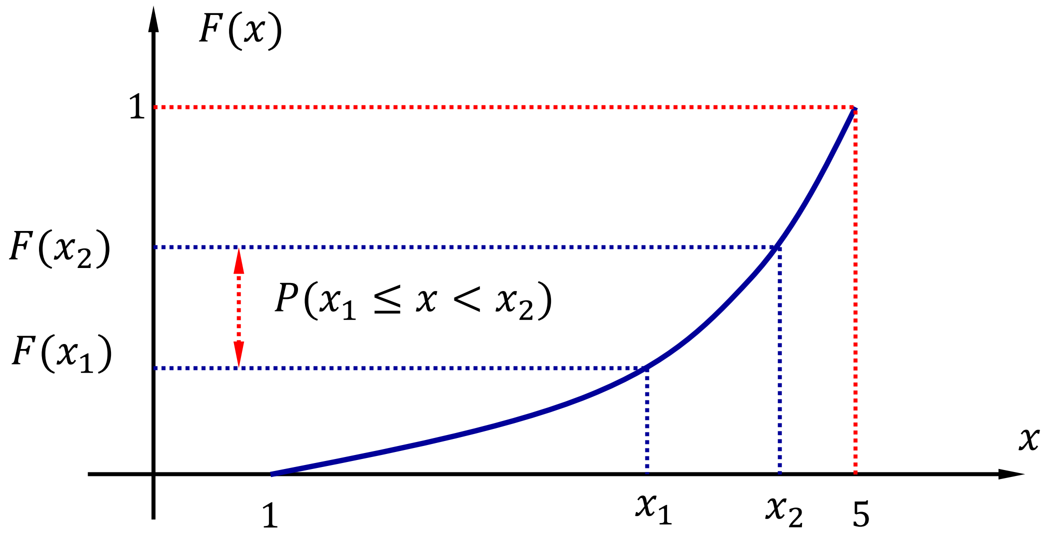
Q : if x is a periodic function, what are the possibilities of $x(t)$?

Cumulative Distribution Function

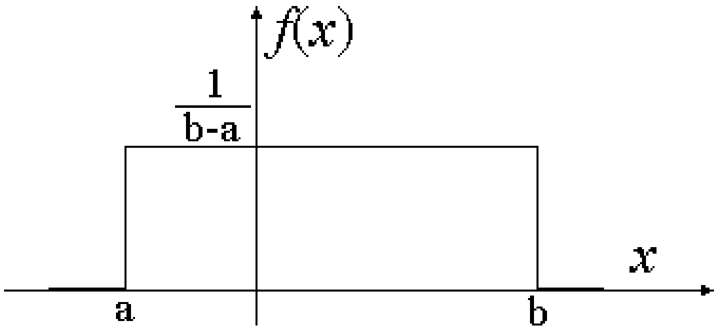
$$F(x) = \int_{-\infty}^{x} f(u)du$$

so that

$$P(x_1 \leq x < x_2) = F(x_2) - F(x_1)$$

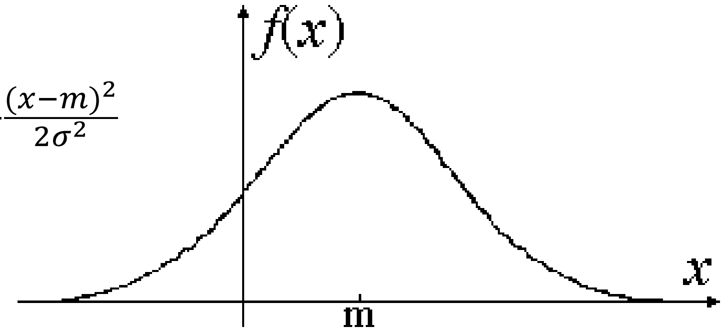


Well Known Distributions



uniform pdf

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}}$$

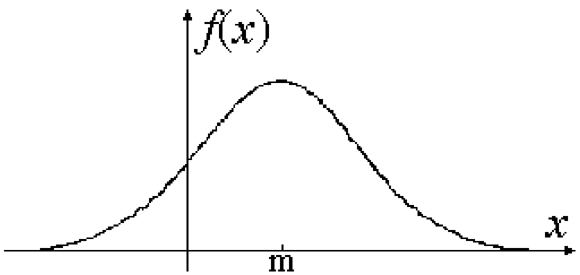


Gaussian (normal) pdf

$$\sigma = \sqrt{E((x-m)^2)}$$

Standard deviation

Example :



is given with $m = 4$ and $\sigma = 2$

What is the probability of having an experiment output to be larger than 10?

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}}$$

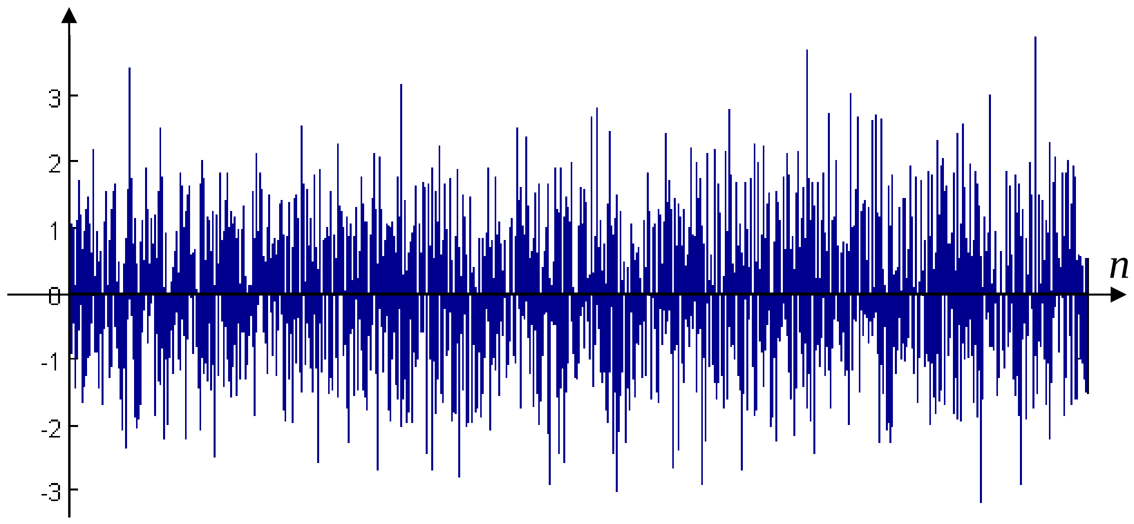
$$P(x > 10) = \int_{10}^{\infty} f(x)dx = \frac{1}{2\sqrt{2\pi}} \int_{10}^{\infty} e^{-\frac{(x-4)^2}{8}} dx = \frac{1}{2\sqrt{2\pi}} \int_{x=6}^{\infty} e^{-\frac{(x)^2}{8}} dx$$

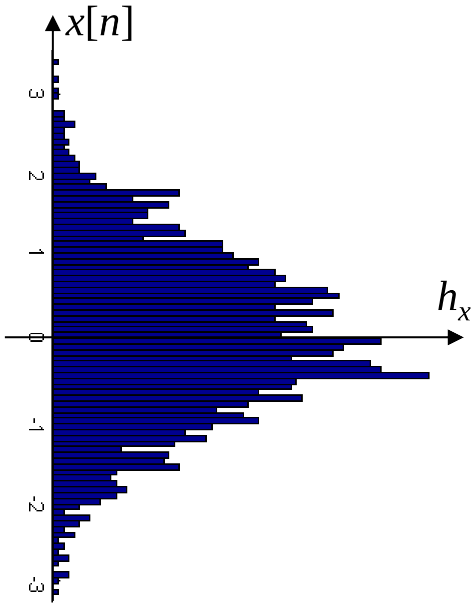
Let $\frac{-x^2}{8} = \frac{-u^2}{2}$, then $\frac{x}{\sqrt{8}} = \frac{u}{\sqrt{2}}$ or $x = 2u$, $dx = 2du$, and $x = 6 \rightarrow u = 3$

$$P(x > 10) = \frac{1}{2\sqrt{2\pi}} \int_{u=3}^{\infty} e^{-\frac{u^2}{2}} 2du = \frac{1}{\sqrt{2\pi}} \int_{u=3}^{\infty} e^{-\frac{u^2}{2}} du$$

we made it look like the Q function, because we have tables for $Q(x)$ where $\sigma = 1$ and $m = 0$

Histogram





Histogram : A graph showing the #occurrences of r.v. within each bin.
for a random experiment repeated N times.

For the given signal : See the resemblance to Gaussian r.v.

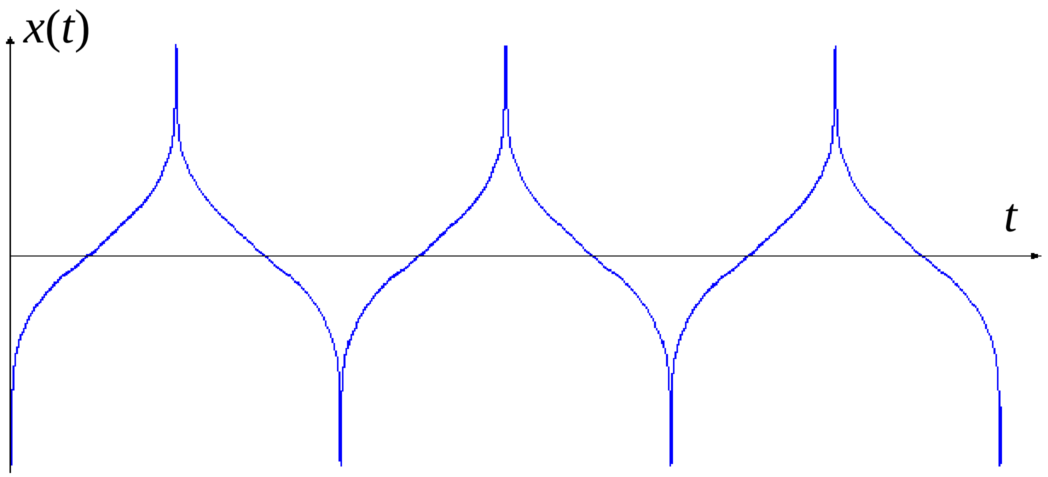
Histogram shows the distribution for the measured results.

pdf is the expected distribution (may not have done yet)

For large number of experiments, histogram (h_x) becomes a representation of pdf (f_x)

pdf shows only the probabilistic distribution, not the time function itself

For example, pdf of the following periodic function looks like a Gaussian



There may be infinite number of functions that have the same pdf

(we might have an example on how to determine pdf for a given periodic waveform)

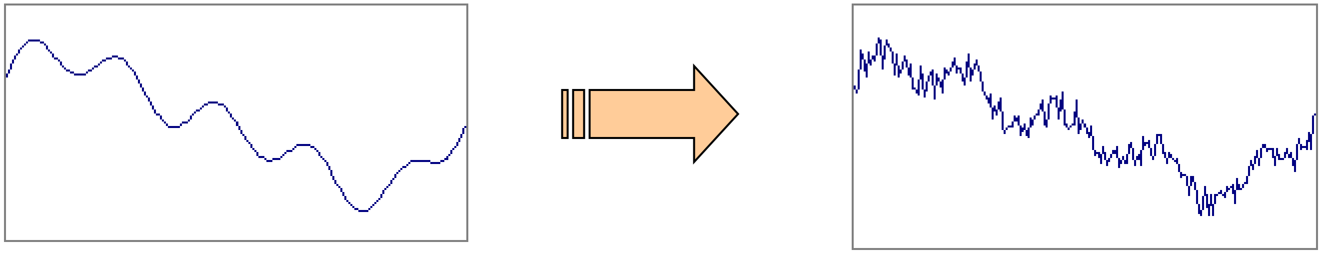
Noise

Any signal other than our structured signal, but intentionally or unintentionally added onto our signal, is categorized as noise.

Noise Sources

- Electronic / Thermal noise
- Electrical discharges in the atmosphere / nearby devices
- Interference / Crosstalk between channels and multipath effects
- Solar / Cosmic effects
- Distortion from nonlinearities of the electronics / media

(quantization noise and granular noise are evaluated in different contexts)



Channel input Channel output

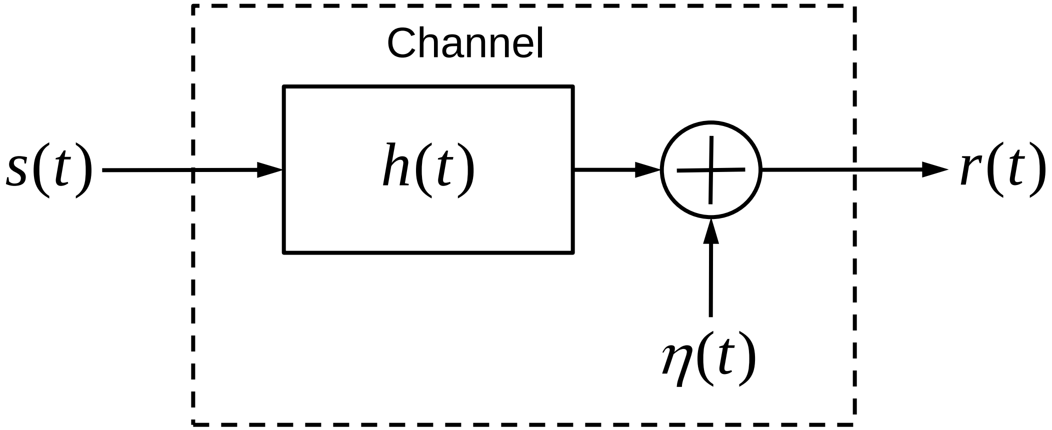
AWGN

Noise is usually assumed to be (this assumption is not baseless)

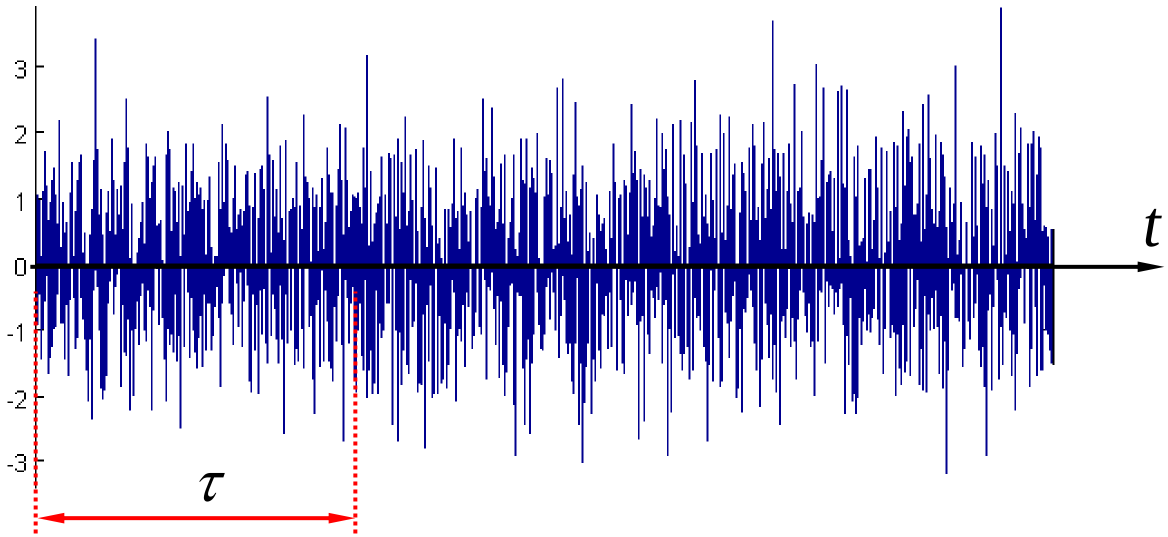
Additive : $S_o = kS_i + N$

White : $|\mathcal{F}(N)| = c$ (has the same power at all frequencies)

Gaussian : Probability distribution function is Gaussian



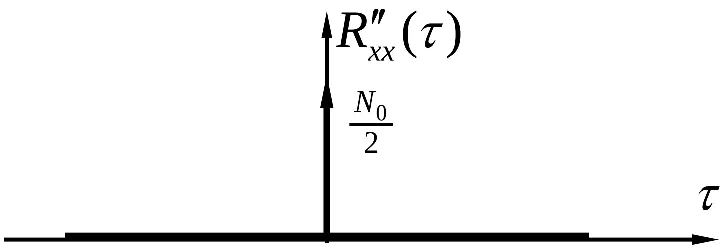
Autocorrelation of White Noise



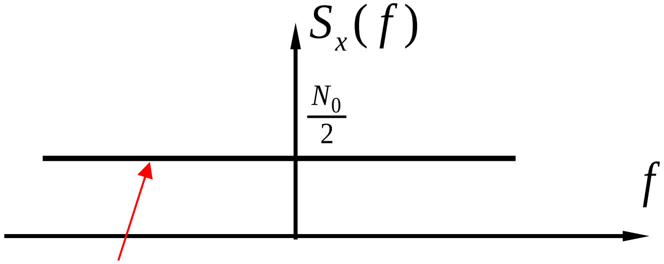
pdf might be Gaussian

$$R''_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)x^*(t+\tau)dt = \frac{N_0}{2}\delta(\tau)$$

what does "white" mean?

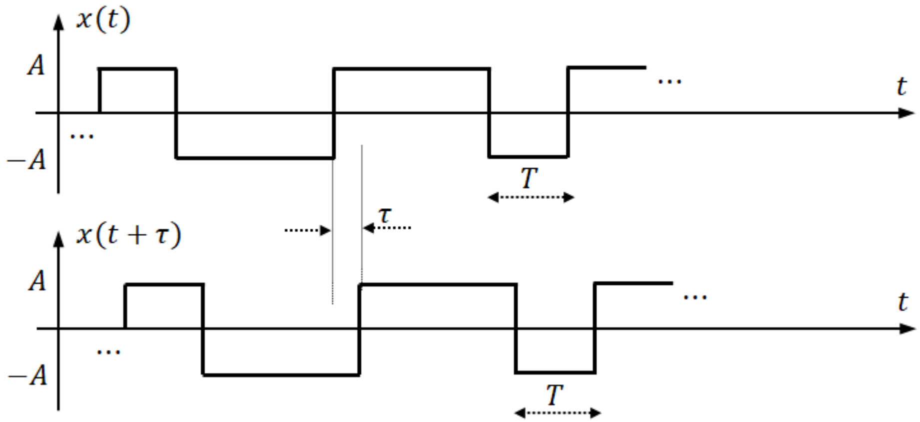


it is uncorrelated



flat: same power at all frequencies

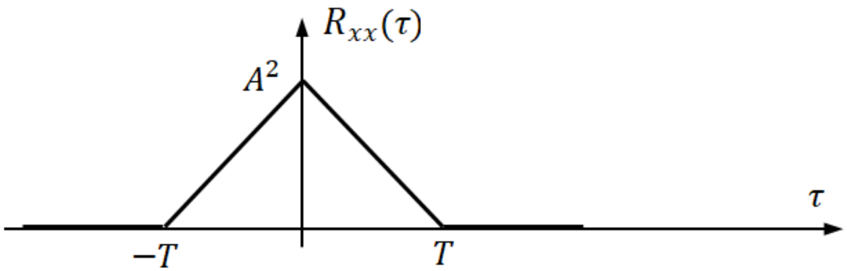
Autocorrelation of Random Rectangular Pulse Train



For τ=0 => $x(t+\tau) = x(t)$ => $R_{xx}(0) = max = A^2$ $\qquad$ $R_{xx}(0) = P_x = \frac{1}{T}\int_0^T |x(t)|^2\, dt = A^2$

For $\tau > T$ => P(match)=P(unmatch)=0.5 => $R_{xx}(\tau) = 0$

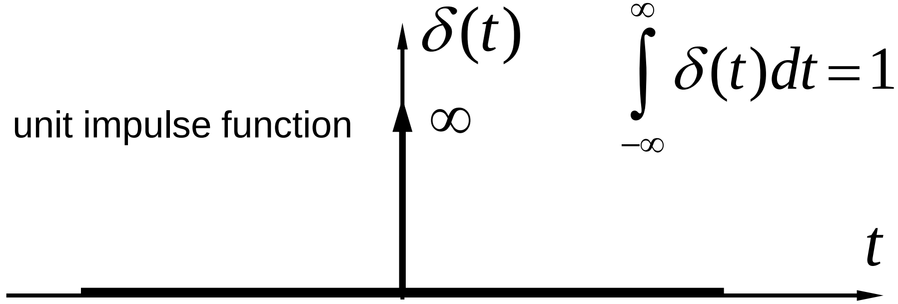
For $0 < |\tau| < T$ => linear decrease from A^2 to 0 as τ increases



Autocorrelation of RBRPT

Impulse

unit impulse function



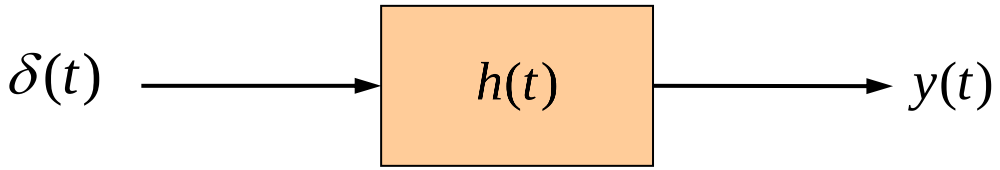
$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

$$E_x = \int_{-\infty}^{\infty} |\delta(t)|^2 dt = ?$$ (undefined)

$$\int_{-\infty}^{\infty} \delta(t)x(t)\,dt = x(0)$$

as a result $\int_{-\infty}^{\infty} \delta(t-\tau)x(t)\,dt = x(\tau)$ where τ is the position of impulse

Impulse Response of a System



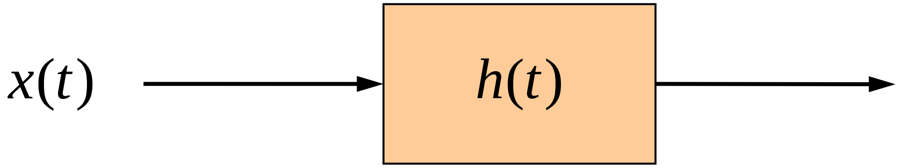
$$y(t) = \int_{-\infty}^{\infty} \delta(t-\tau)h(\tau)\,dt = h(t)$$

$y(t) = h(t)$ when input is $\delta(t)$

Convolution

Think of $x(t)$ as an infinite sum of $\int_{-\infty}^{\infty} \delta(\tau - t) x(\tau) dt = x(t)$

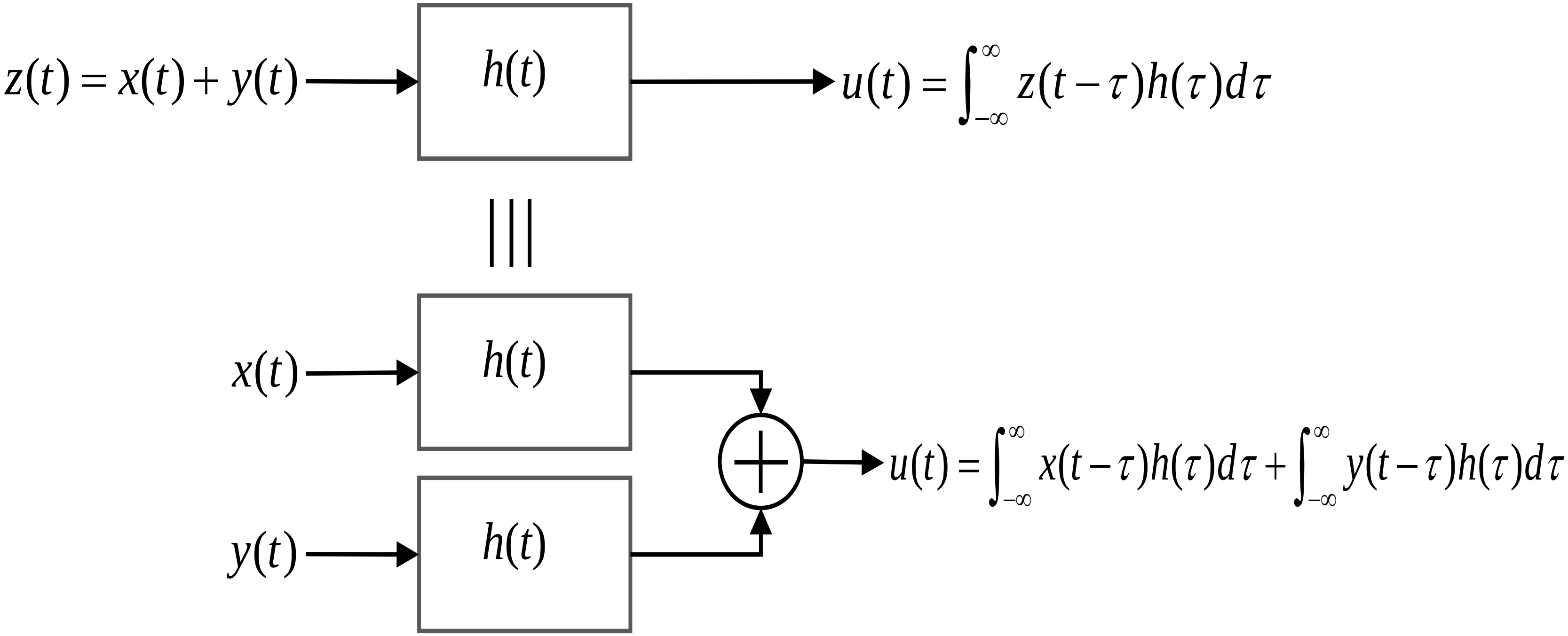
The output of the system will be an infinite sum of responses to each weighted impulse



$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$$

For this infinite summation to hold, the system must be a Linear System

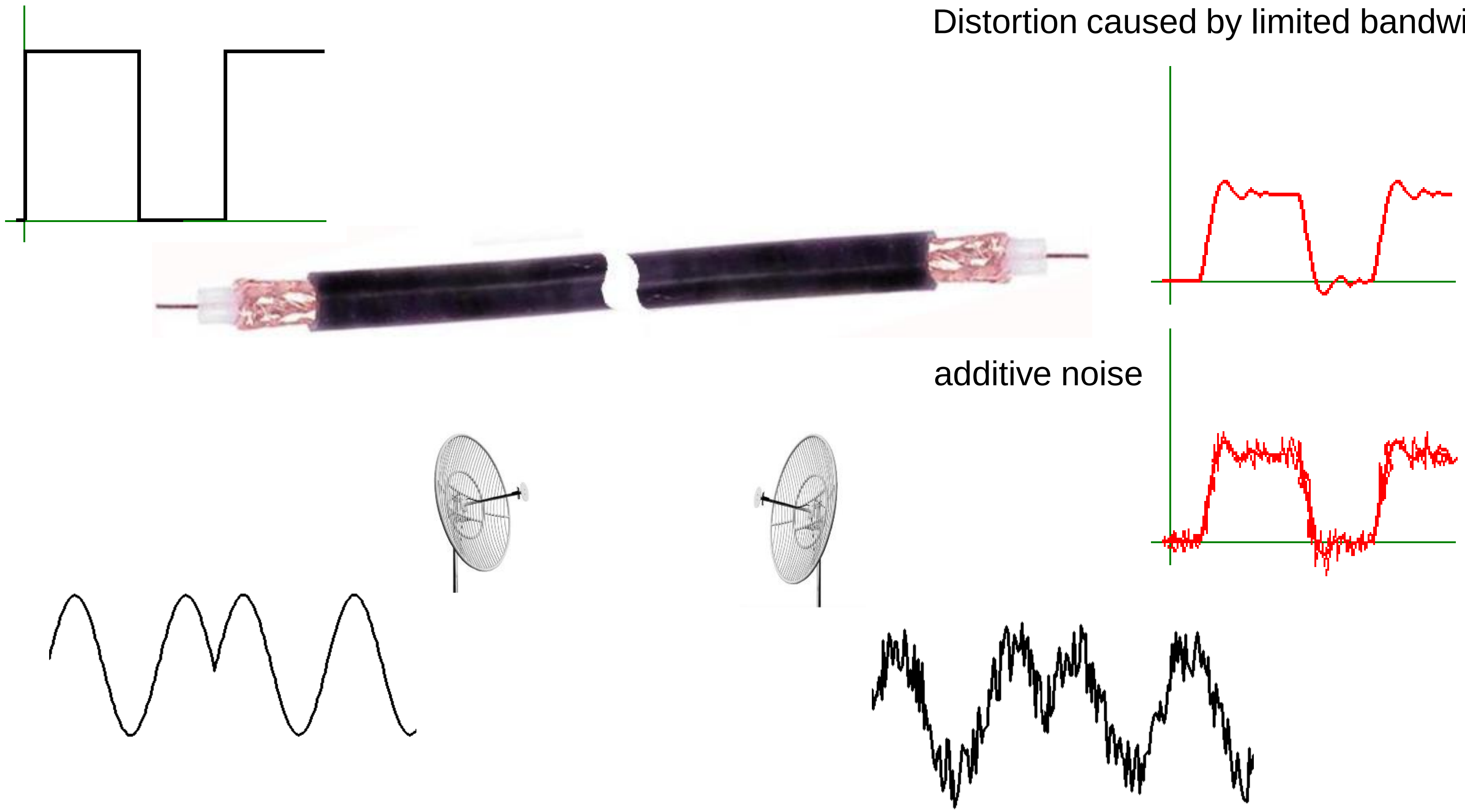
Linearity

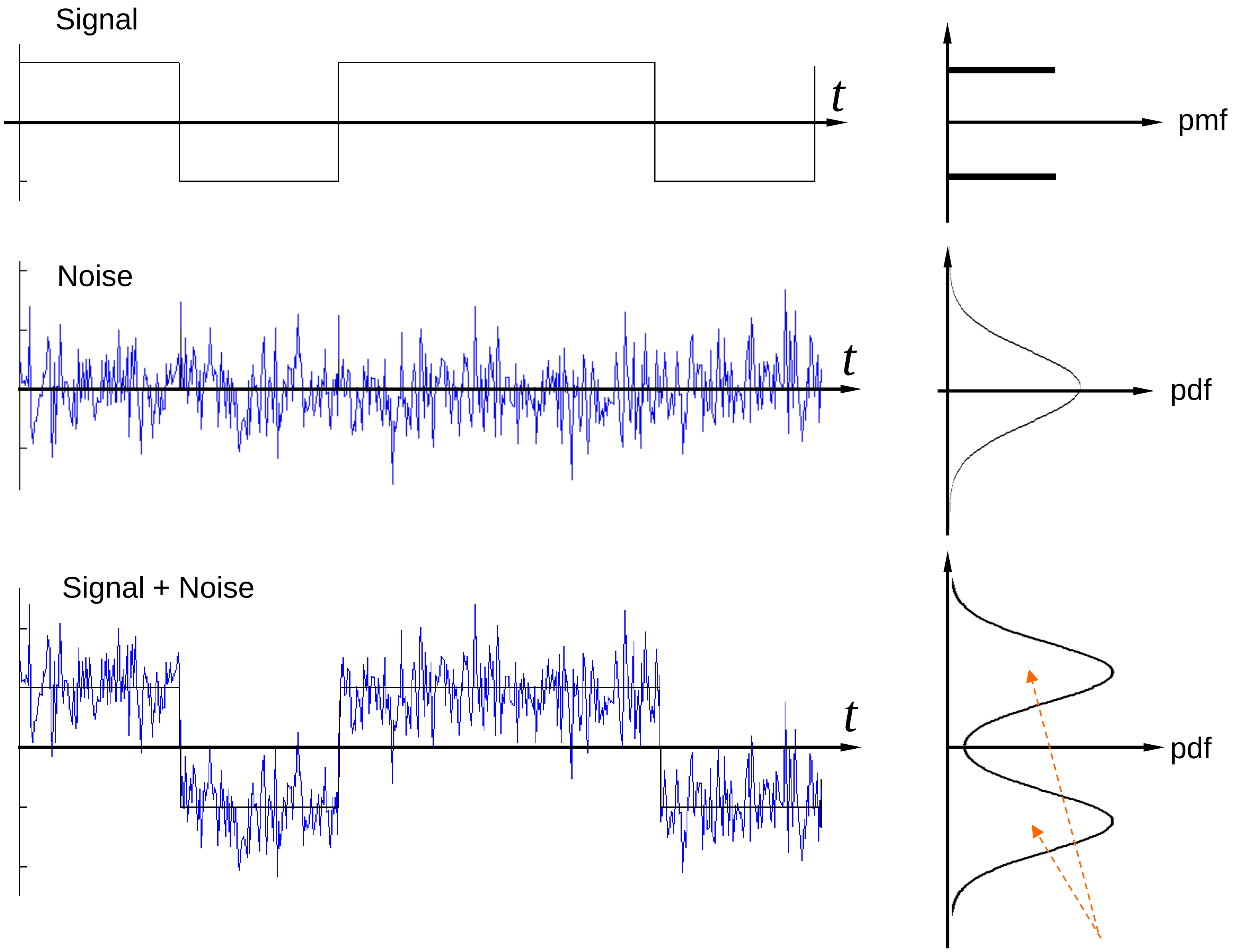


$h(t)$ is a Linear Time Invariant (LTI) system

Time Invariant : the system $h(t)$ does not change by time

Signal + Noise





Note that : If Signal/Noise decreases, it becomes more difficult to distinguish these hills

END