Fourier

Spectrum

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For the course "**Communications**"

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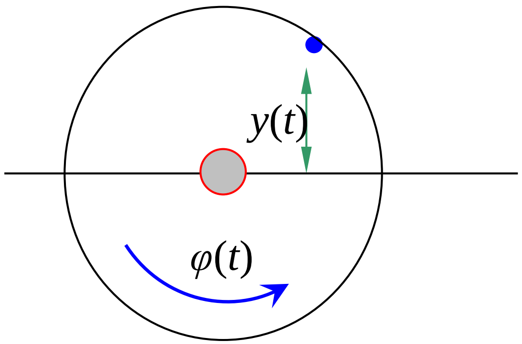
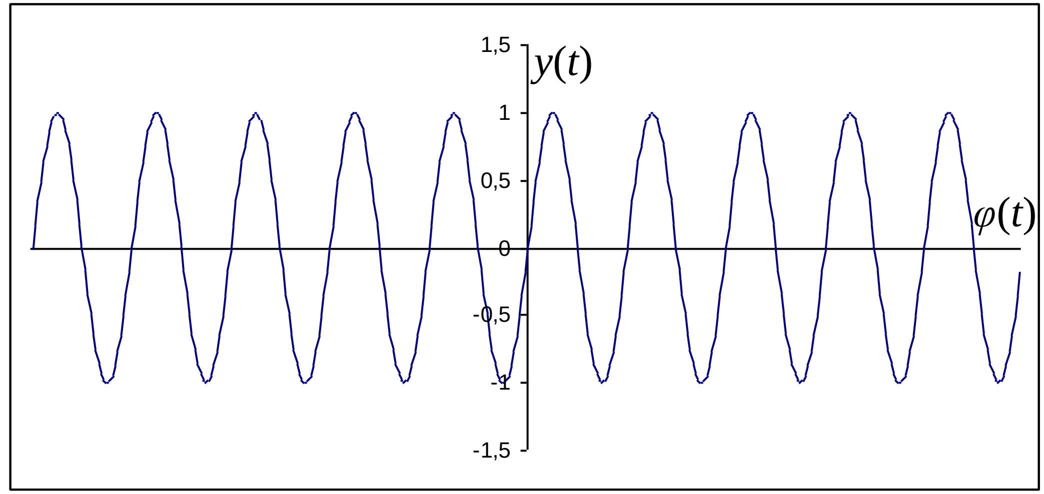
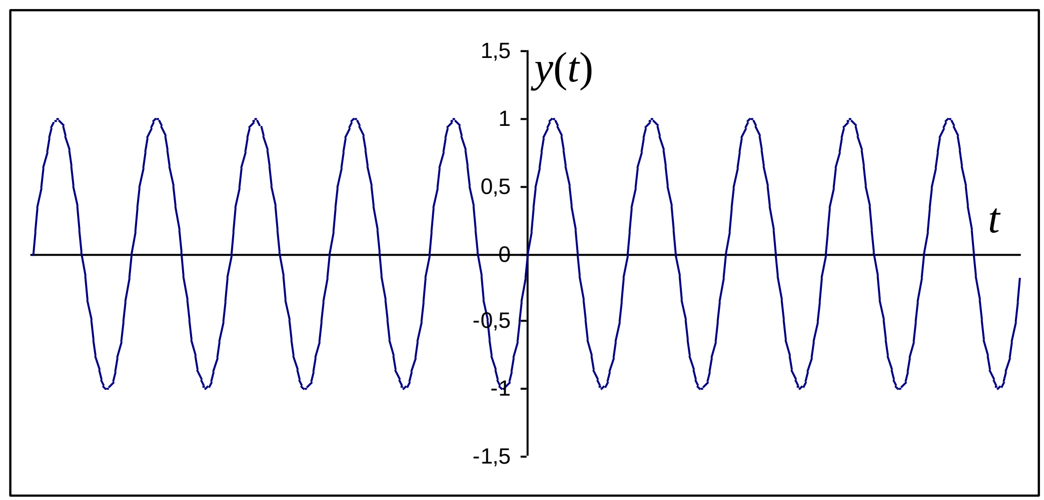
Origin of a sinusoid



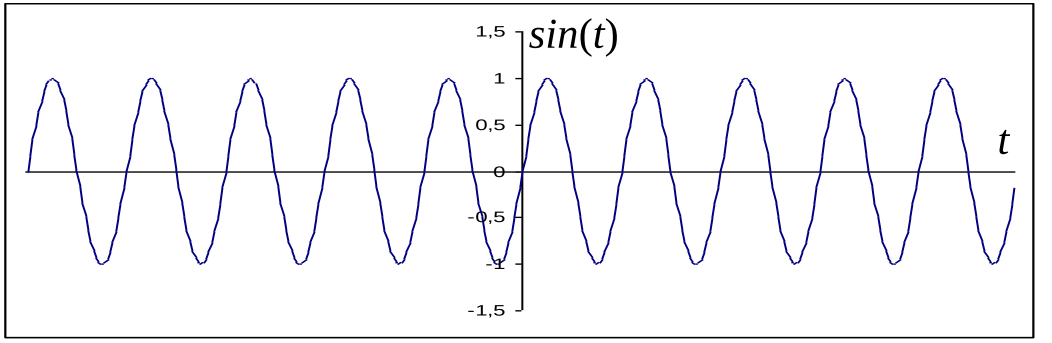


if the angular velocity of the disk is constant (c [rad/s]) then we can have another graph of sinusoid

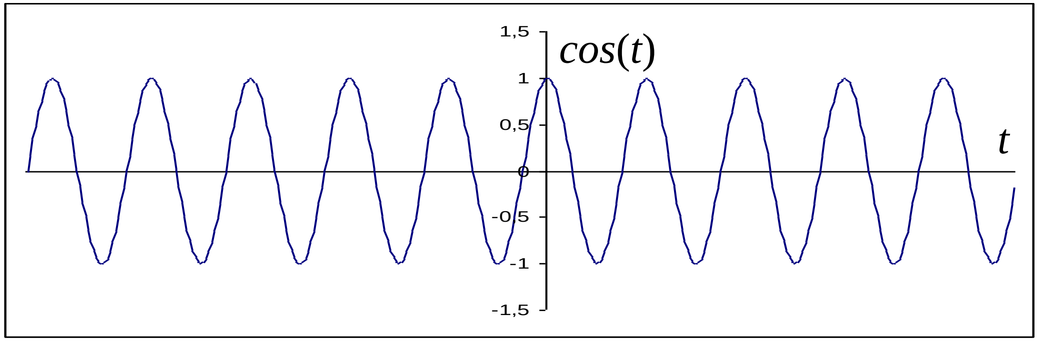
The number of revolutions of the disk per unit time [rev/s] can be called the frequency of the $y(t)$, and it would be a constant also. The unit is cycles/sec or, since 1970s, Hertz (named after Heinrich Rudolf Hertz, the German electromagnetizm scientist)



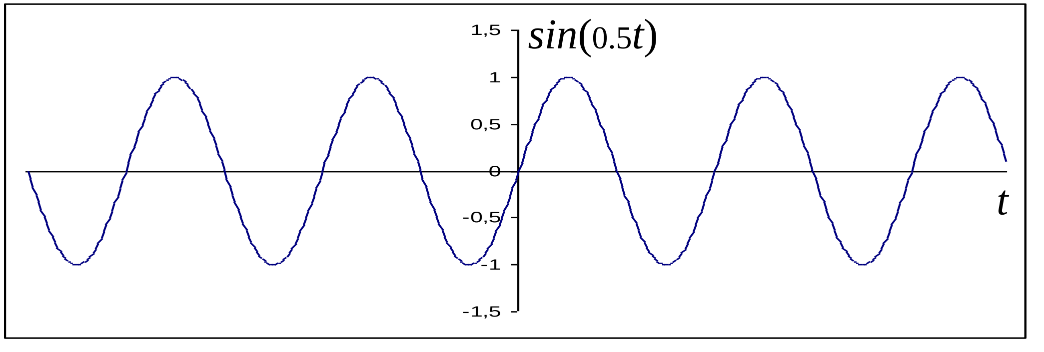
we give a special name to this function : sinusoidal or shortly *sin*



if we measure the angle from the top of the disk we get a 90° phase shifted version of *sin* function which we call *cos*.



This is what we get when we rotate the disk at half the speed of the original. Frequency is halved of course.

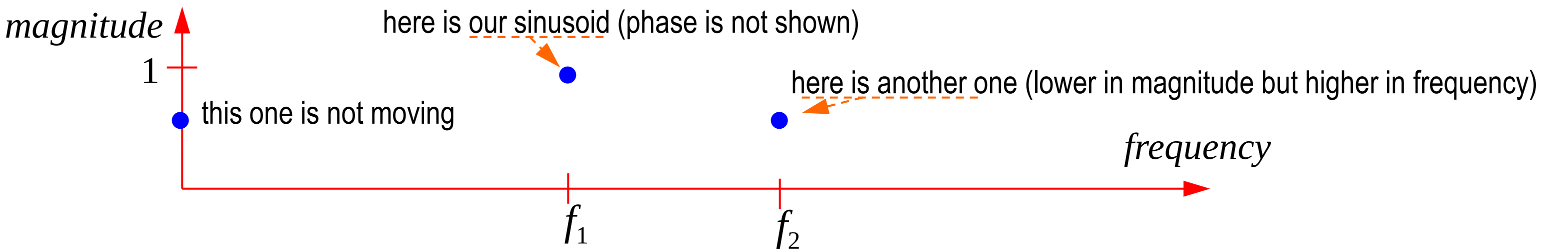


The distinctive properties of such sinusoids are:

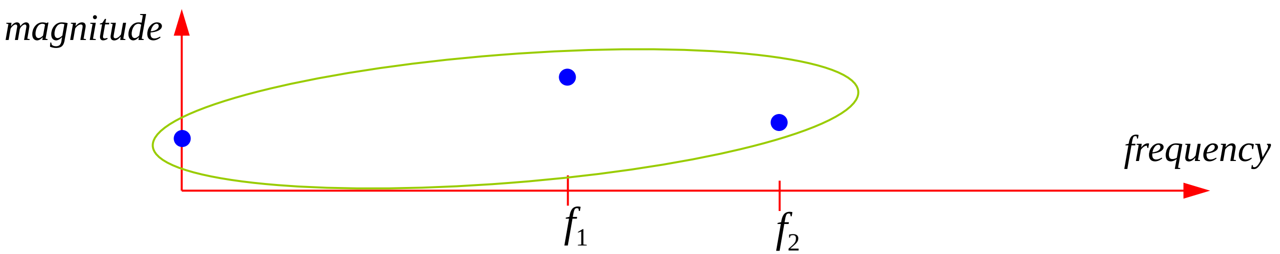
1. frequency (rotations per second)
2. magnitude (radius of the disk)
3. phase (location of the mark on the edge of the disk)

That is, if we have these three parameters, we know everything about $y(t)$

So, we can compare different sinusoids by marking them on a magnitude vs. frequency plane



Q: Assuming that *magnitude*s are electrical quantities (like voltage), can we add them up?



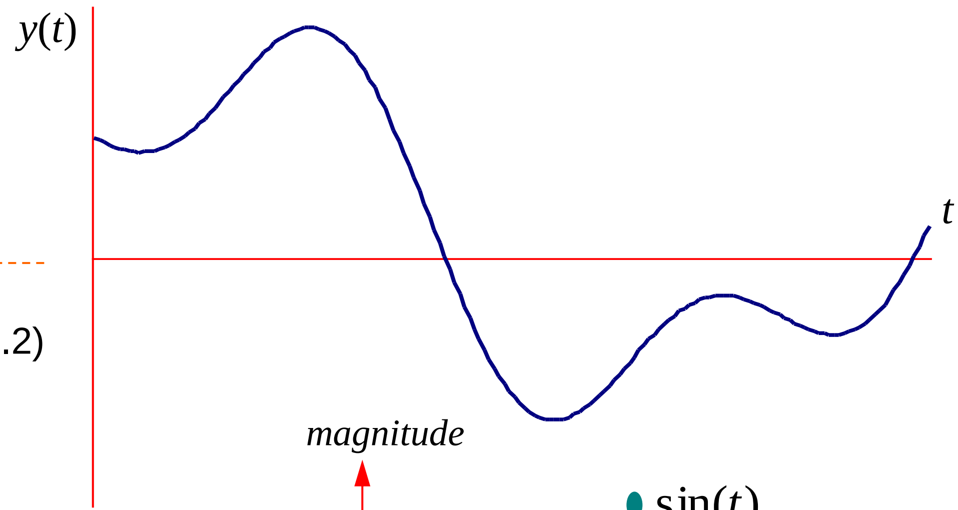
$y_1(t)+y_2(t)+y_3(t)=?$

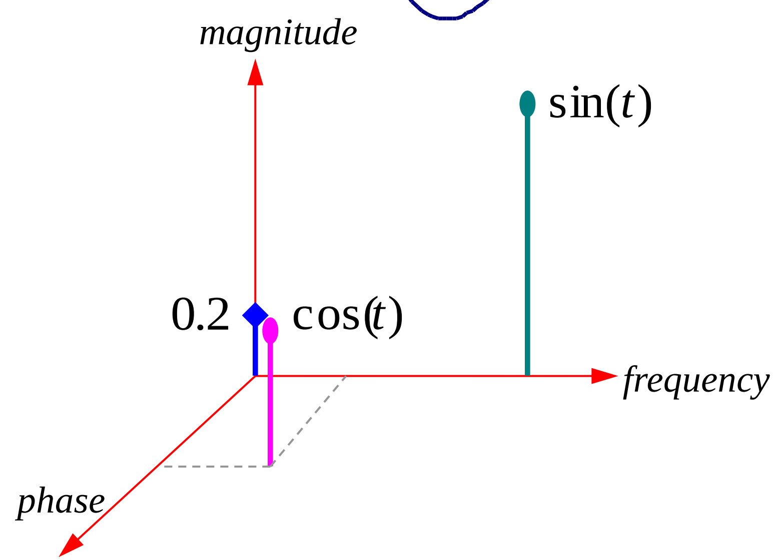
it turns out we can…

$$y(t) = \sin(t) + 0.5\cos(\tfrac{1}{4}t) + 0.2$$

90° shifted and lower frequency (magnitude is 0.5)

zero frequency (magnitude is 0.2)





since it is difficult to illustrate 3D graphs we usually have *freq-mag* and *freq-phase* graphs

The question is : Can we obtain any waveform by summing up sinusoids with different *frequency*, *magnitude* and *phase*?

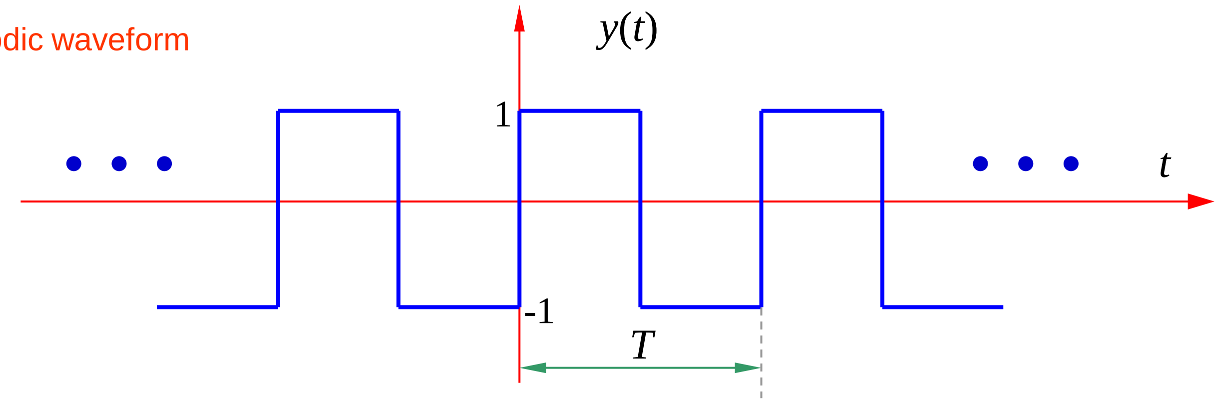
Fourier Series (Jean Baptiste Joseph Fourier 1768-1830)

It turns out that any periodic waveform (with some limits, of course) can be obtained by an infinite sum of *sin* and *cos* signals

$$y(t) = \sum_{n=0}^{\infty} b_n \cos(n\omega_o t) + \sum_{n=1}^{\infty} c_n \sin(n\omega_o t) \quad n = 0,1,\ldots,\infty \quad \omega_o = 2\pi f_o \text{ and } f_o = \frac{1}{T}$$

where $n\omega_o$ is called the n^{th} harmonic of the fundamental frequency ω_o

Example periodic waveform



The coefficients b_n and c_n can be calculated using

$$b_o = \frac{1}{T}\int_{-T/2}^{T/2} y(t)dt \qquad b_n = \frac{2}{T}\int_{-T/2}^{T/2} y(t)\cos(n\omega_o t)dt \qquad c_n = \frac{2}{T}\int_{-T/2}^{T/2} y(t)\sin(n\omega_o t)dt$$

$$n = 1,2,3,\cdots$$

$$b_o = \frac{1}{T}\int_{-T/2}^{T/2} y(t)dt \qquad n = 1,2,3,\cdots$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} y(t)\cos(n\omega_o t)dt$$

$$c_n = \frac{2}{T}\int_{-T/2}^{T/2} y(t)\sin(n\omega_o t)dt$$

Note: it is obvious that these integrals actually yield the correlation function between the given harmonic and the waveform.

$$b_o = \frac{1}{T}\left(\int_0^{T/2} dt + \int_{T/2}^{T}(-1)dt\right) = 0$$ (mean value is zero, just as seen in the figure)

$$b_n = \frac{2}{T}\left(\int_0^{T/2}\cos(n\omega_o t)dt - \int_{T/2}^{T}\cos(n\omega_o t)dt\right) = \frac{2}{Tn\omega_o}\left(\left[\sin(n\omega_o t)\right]_0^{T/2} - \left[\sin(n\omega_o t)\right]_{T/2}^{T}\right)$$

$$= \frac{1}{n\pi}\left(\left[\sin(n2\pi t/T)\right]_0^{T/2} - \left[\sin(n2\pi t/T)\right]_{T/2}^{T}\right) = \frac{1}{n\pi}\left(\sin(n\pi) - \sin(0) - \sin(2n\pi) + \sin(n\pi)\right)$$

$$b_n = 0$$ (we see this from the figure, thus no need for integration. It is an odd function)

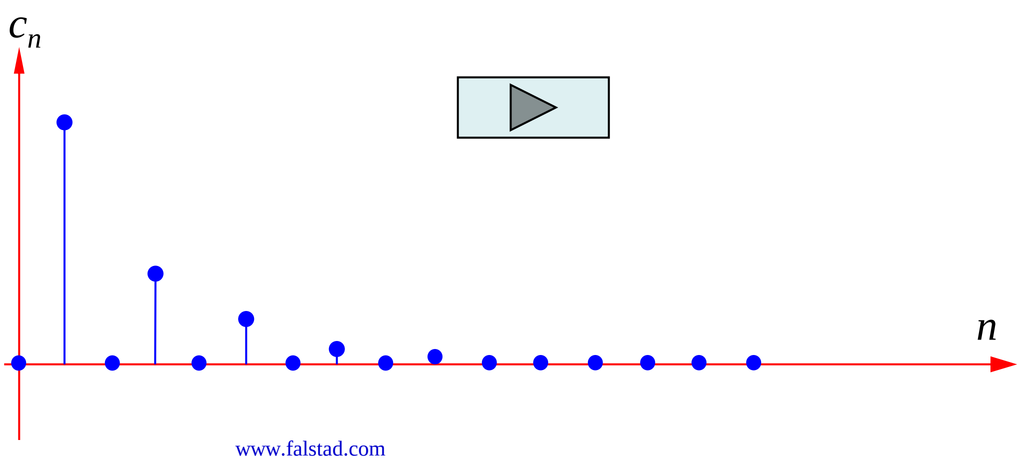
$$c_n = \frac{2}{T}\left(-\int_{-T/2}^{0}\sin(n\omega_o t)dt + \int_{0}^{T/2}\sin(n\omega_o t)dt\right) = \frac{1}{n\pi}\left(\left[\cos(n2\pi t/T)\right]_{-T/2}^{0} - \left[\cos(n2\pi t/T)\right]_{0}^{T/2}\right)$$

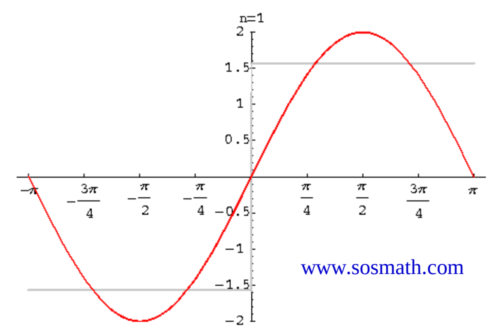
$$c_n = \frac{1}{n\pi}\left(1-\cos(n\pi)-\cos(n\pi)+1\right) = \frac{2}{n\pi}\left(1-\cos(n\pi)\right) \quad = \frac{2}{n\pi}\left(1-(-1)^n\right) = \begin{cases} 0 & ,n\,is\,even \\ \dfrac{4}{n\pi} & ,n\,is\,odd \end{cases}$$

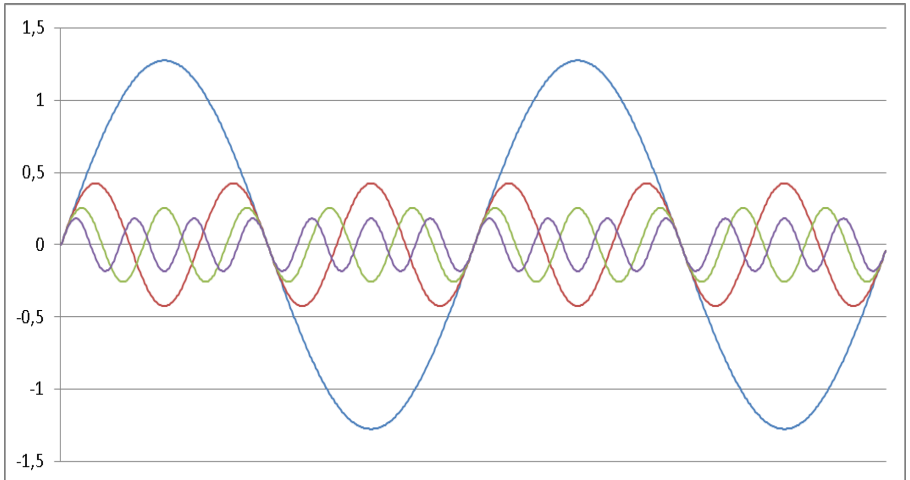
Therefore

$$y(t) = 4\sum_{n=1,3\ldots}^{\infty}\frac{\sin(n\omega_o t)}{n\pi}$$

interpretation: the infinite sum of odd harmonics of fundamental frequency. The magnitude of the sin-waves decreases inversely with the harmonic number

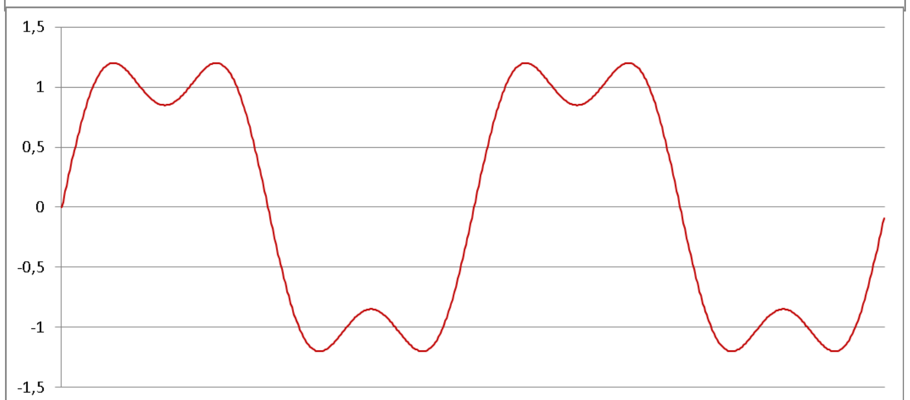






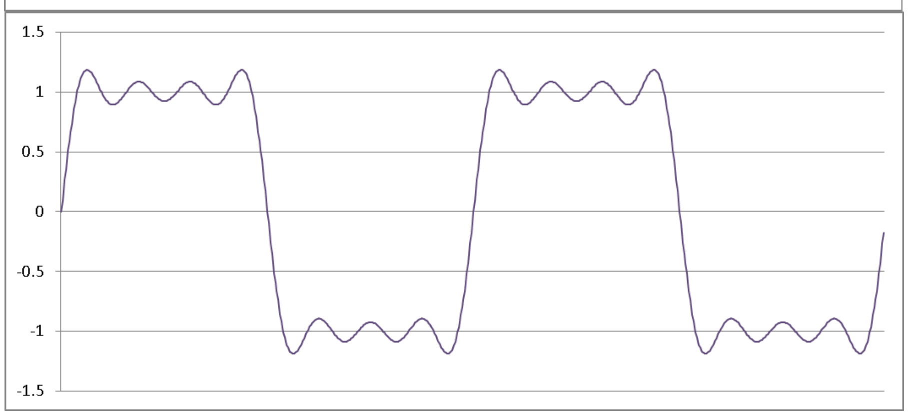
Sinusoidals for

$n = 1, 3, 5, 7$



Sum of sinusoidals for

$n = 1, 3$



Sum of sinusoidals for

$n = 1, 3, 5, 7$

Euler eq.'s

$$\cos(n\omega_o t) = \frac{e^{jn\omega_o t} + e^{-jn\omega_o t}}{2}$$

$$\sin(n\omega_o t) = \frac{e^{jn\omega_o t} - e^{-jn\omega_o t}}{j2}$$

$$y(t) = \sum_{n=0}^{\infty} b_n \cos(n\omega_o t) + \sum_{n=0}^{\infty} c_n \sin(n\omega_o t)$$

exponential Fourier series

$$y(t) = \sum_{n=-\infty}^{\infty} a_n e^{jn\omega_o t}$$

coefficients of exponential Fourier series

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} y(t)\cos(n\omega_o t)dt$$

$$c_n = \frac{2}{T}\int_{-T/2}^{T/2} y(t)\sin(n\omega_o t)dt$$

$$a_n = \frac{1}{T}\int_{-T/2}^{T/2} y(t)e^{-jn\omega_o t}dt$$

$\omega_o = \frac{2\pi}{T_o}$: Fundamental Angular Frequency

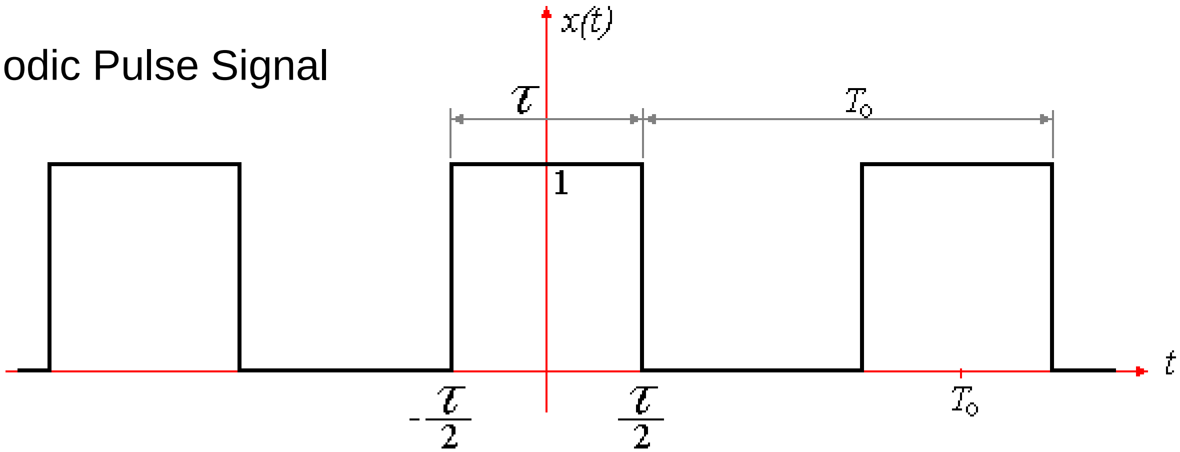
$n\omega_o$: Harmonics

a_o : Zero frequency component or DC value (or mean)

Any periodic signal which satisfies Dirichlet conditions can be represented by a weighted sum of (possibly infinite number of) sinusoids with different magnitude and delay (phase)

Example

Periodic Pulse Signal



a single pulse $\Pi(t) = \begin{cases} 1 \;, & |t| < \frac{\tau}{2} \\ ½ \;, & |t| = \frac{\tau}{2} \\ 0 \;, & \text{otherwise} \end{cases} = u(t + \frac{\tau}{2}) - u(t - \frac{\tau}{2})$

a pulse train $x(t) = \sum_{n=-\infty}^{\infty} \Pi\left(\frac{t - nT_o}{\tau}\right)$

The coefficients $a_n = \frac{1}{T_0}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} x(t) e^{-j\frac{2\pi n t}{T_0}} dt = \frac{1}{-j2\pi n} e^{-j\frac{2\pi n t}{T_0}}\Big|_{-\frac{\tau}{2}}^{\frac{\tau}{2}} = \frac{1}{j2\pi n}\left(e^{j\frac{\pi n \tau}{T_0}} - e^{-j\frac{\pi n \tau}{T_0}}\right)$

$$\frac{1}{\pi n}\frac{\left(e^{j\frac{\pi n \tau}{T_0}} - e^{-j\frac{\pi n \tau}{T_0}}\right)}{2j} = \frac{1}{\pi n} sin\left(\frac{\pi n \tau}{T_0}\right) = \frac{\tau}{T_0} sinc\left(\frac{n\tau}{T_0}\right)$$

$$sinc(x) = \frac{sin(\pi x)}{\pi x}$$

Ubiquitous Sinc Function

$sinc(x)$ →

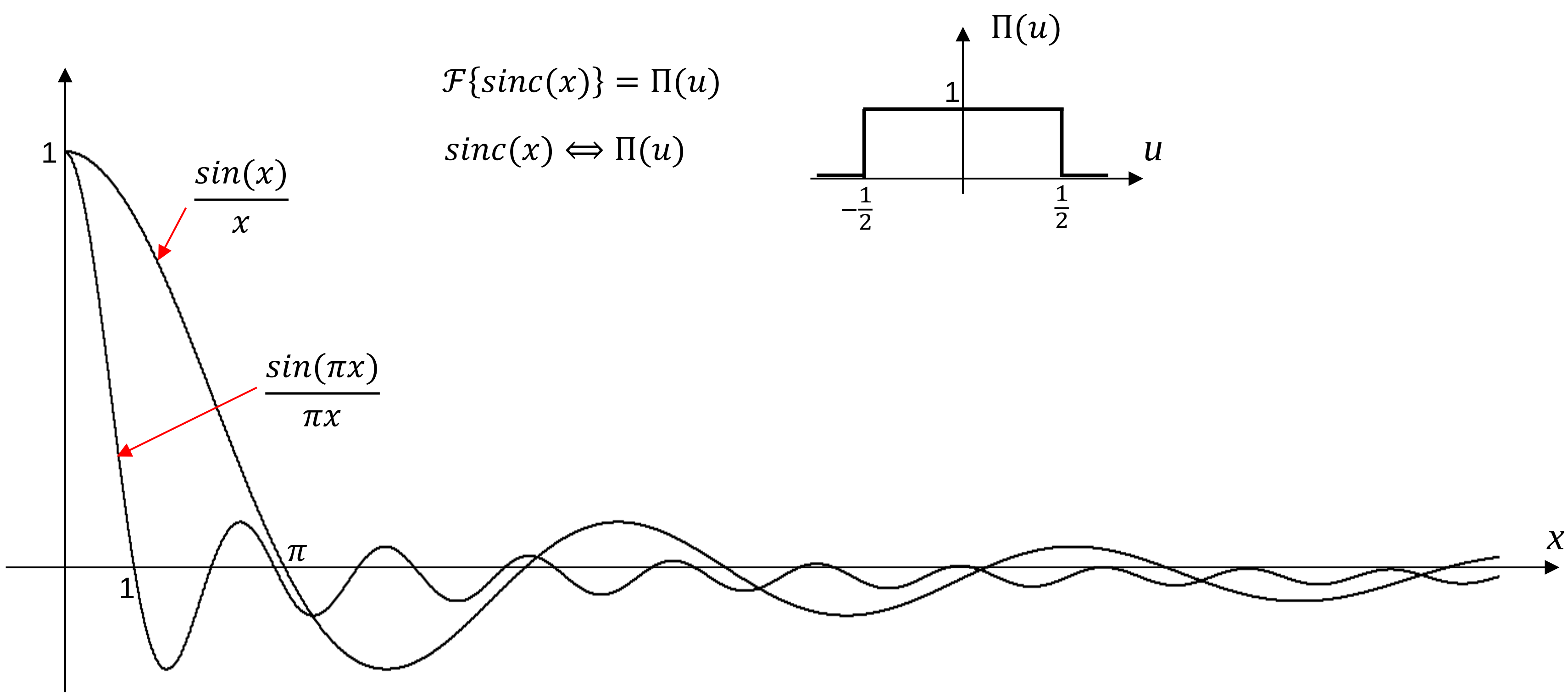
- $\frac{sin(x)}{x}$ used mostly by mathematicians (sometimes called $Sa(x)$)
- $\frac{sin(\pi x)}{\pi x}$ used mostly by signal processing guys (normalized $sinc$)

$sinc(0) \triangleq 1$ for both

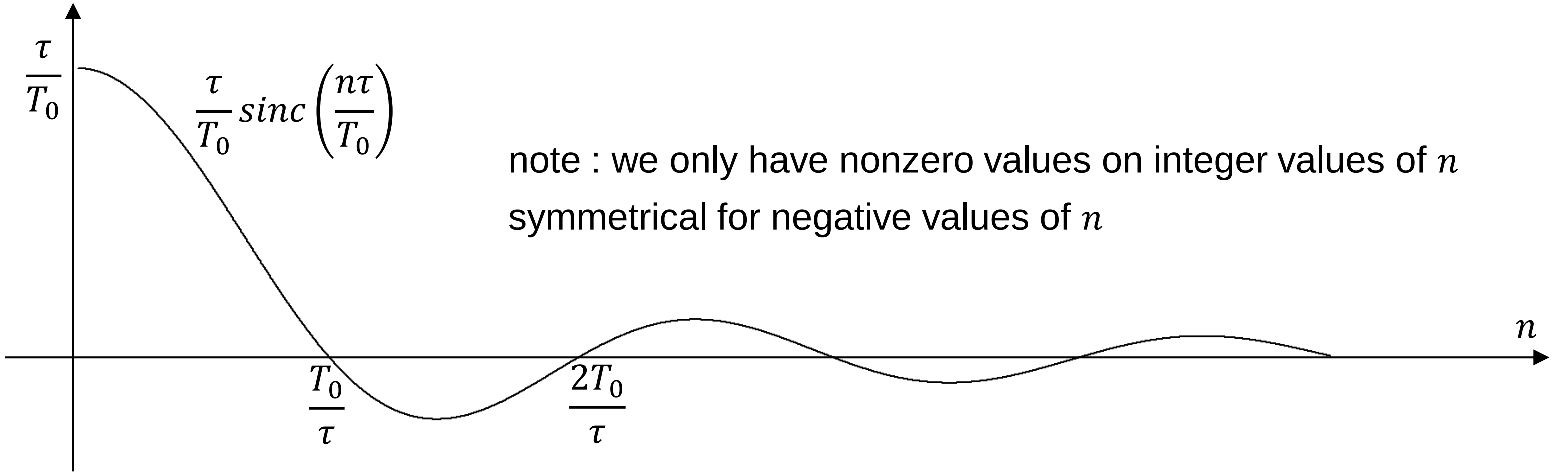
obviously they are different, we will be using the normalized version

$$\mathcal{F}\{sinc(x)\} = \Pi(u)$$

$$sinc(x) \Leftrightarrow \Pi(u)$$



Fourier Series of $x(t)$ is then $$x(t) = \sum_{n=-\infty}^{\infty} \frac{\tau}{T_0} sinc\left(\frac{n\tau}{T_0}\right) e^{j\frac{2\pi n}{T_0}t}$$



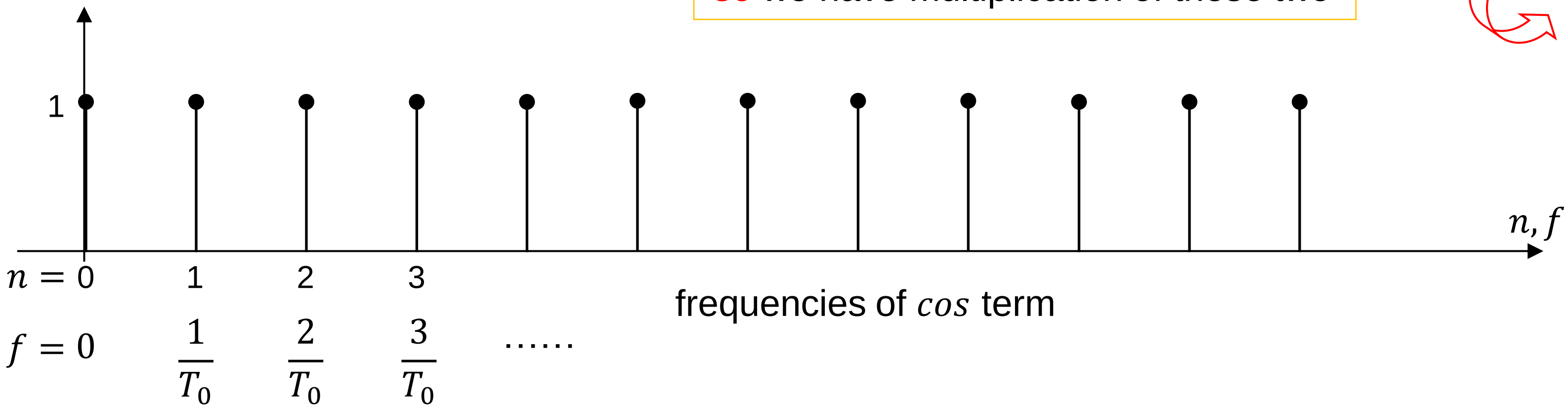
note : we only have nonzero values on integer values of n

symmetrical for negative values of n

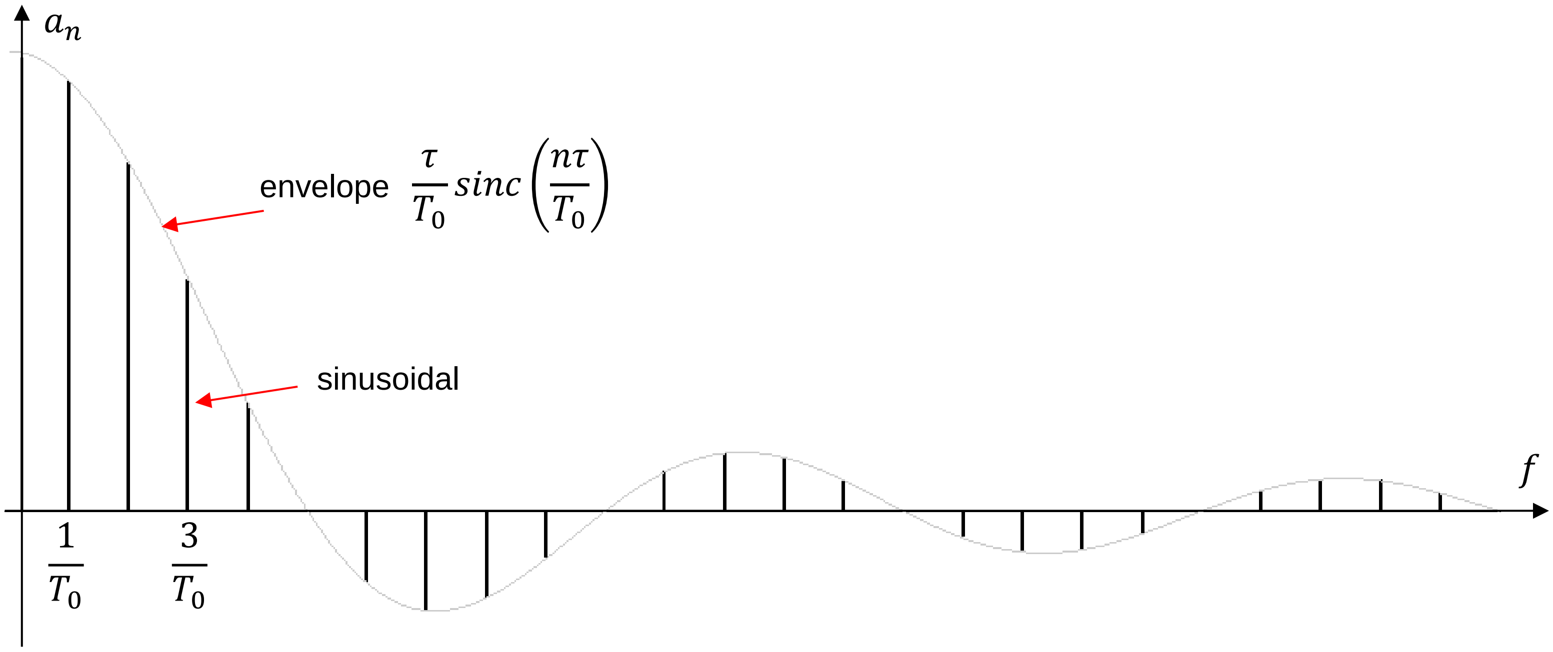
$\pm n$ values make up a sinusoidal since $$\frac{e^{j\frac{2\pi n}{T_0}t} + e^{-j\frac{2\pi n}{T_0}t}}{2} = cos\left(\frac{2\pi n}{T_0}t\right)$$

magnitudes of cos term

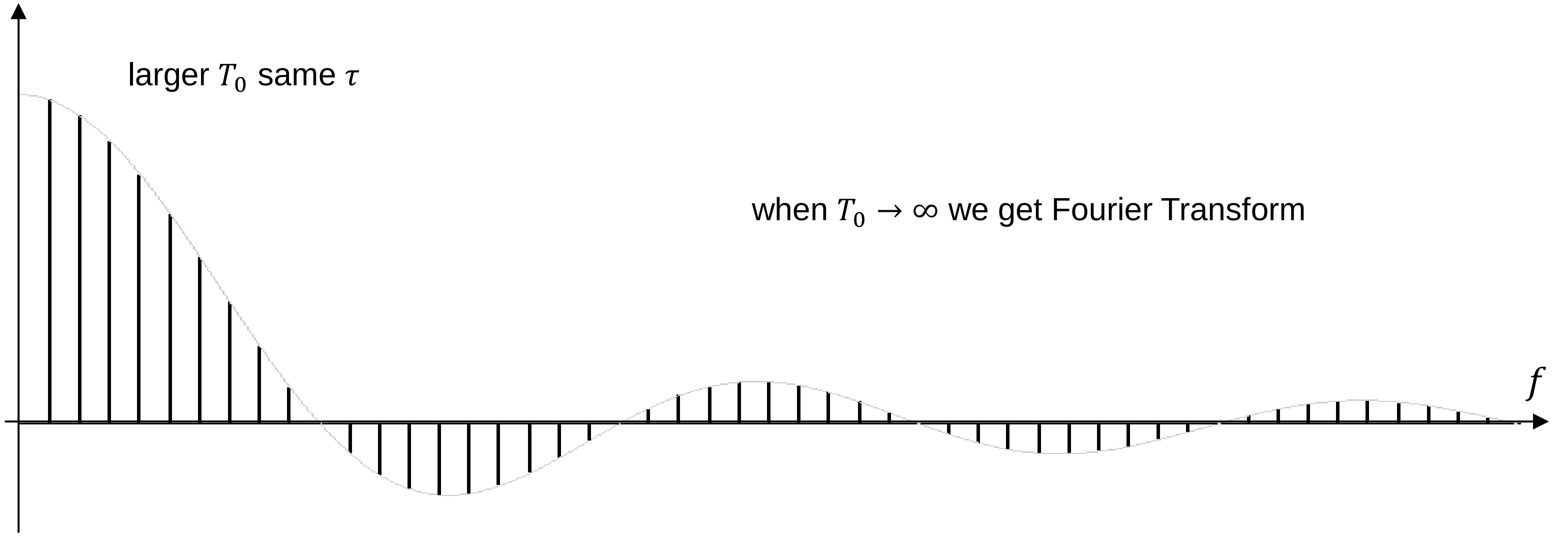
so we have multiplication of these two



frequencies of cos term



a_n
envelope $\frac{\tau}{T_0} sinc\left(\frac{n\tau}{T_0}\right)$
sinusoidal
f
$\frac{1}{T_0}$
$\frac{3}{T_0}$



larger T_0 same τ
when $T_0 \to \infty$ we get Fourier Transform
f

Fourier Transform

Making the period T infinity in order to handle arbitrary (not periodic) waveforms

As $T \to \infty$ $\omega_o = \frac{2\pi}{T} \to 0$ and the spectrum covers everywhere (continuous)

$$a_n = \frac{1}{T}\int_{-T/2}^{T/2} y(t)e^{-jn\omega_o t}dt \longrightarrow Y(\omega) = \int_{-\infty}^{\infty} y(t)e^{-j\omega t}dt$$

We no longer have coefficients for linear sum but continuous function for linear integral, so

$$y(t) = \sum_{n=-\infty}^{\infty} a_n e^{jn\omega_o t} \longrightarrow y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\omega)e^{j\omega t}d\omega$$

The notation is $X(\omega) = \mathcal{F}\{x(t)\}$ Forward transform

or $x(t) = \mathcal{F}^{-1}\{X(\omega)\}$ Inverse transform

or $x(t) \Leftrightarrow X(\omega)$ Transform pair

Some Properties of Fourier Transform

Linearity — if $x(t) = c_1 x_1(t) + c_2 x_2(t)$ then $X(\omega) = c_1 X_1(\omega) + c_2 X_2(\omega)$

Time Shift

$$\mathcal{F}\{x(t-t_o)\} = e^{-j\omega t_o}\mathcal{F}\{x(t)\}$$

Scaling

$$\mathcal{F}\{x(at)\} = \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

Convolution

$$\mathcal{F}\{x(t) * y(t)\} = \mathcal{F}\{x(t)\} \cdot \mathcal{F}\{y(t)\} = X(f) \cdot Y(f)$$

$$x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau) y^*(t-\tau) d\tau$$

Differentiation and Integration

$$\mathcal{F}\left\{\frac{d}{dt} x(t)\right\} = j\omega X(\omega)$$

$$\mathcal{F}\left\{\int_{-\infty}^{t} x(r) dr\right\} = \frac{X(\omega)}{j\omega} + \pi G(0)\delta(\omega)$$

Autocorrelation

$$R_x(\tau) = \int_{-\infty}^{\infty} x(t) x^*(t-\tau) dt$$

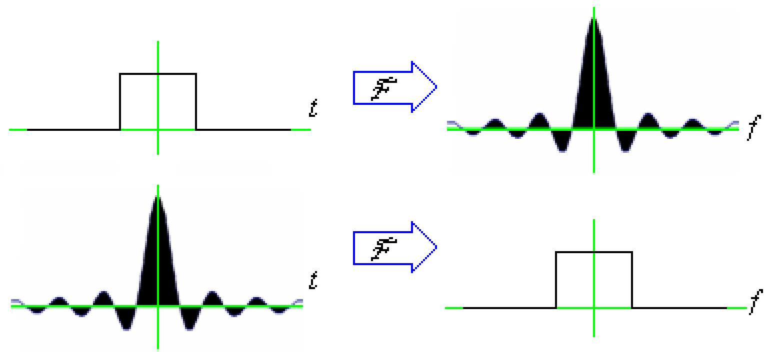
$$\mathcal{F}\{R_x(\tau)\} = |X(\omega)|^2$$

Modulation

$$\mathcal{F}\{x(t)\cos(\omega_o t)\} = \tfrac{1}{2} X(\omega - \omega_o) + \tfrac{1}{2} X(\omega + \omega_o)$$

Some Properties of Fourier Transform

Duality



if $x(t) \Leftrightarrow X(\omega)$

then $X(t) \Leftrightarrow 2\pi x(-\omega)$

Parseval's relation $$\int_{-\infty}^{\infty} x(t)y^*(t)dt = \int_{-\infty}^{\infty} X(f)Y^*(f)df$$

Rayleigh's property $$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

Additionaly

For Real Signals

$$\mathcal{R}e\{X(-\omega)\} = \mathcal{R}e\{X(\omega)\} \qquad |X(-\omega)| = |X(\omega)|$$

$$\mathcal{I}m\{X(-\omega)\} = -\mathcal{I}m\{X(\omega)\} \qquad \angle X(-\omega) = -\angle X(\omega)$$

If $x(t)$ is real and even then $X(\omega)$ is real and even

If $x(t)$ is real and odd then $X(\omega)$ is imaginary and odd

Time flip if $x(f) = \mathcal{F}\{X(-t)\}$ then $x(-f) = \mathcal{F}\{X(t)\}$

Some Transform Pairs

$$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \Longleftrightarrow 2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega_k \omega_0)$$

$$e^{j\omega_0 t} \Longleftrightarrow 2\pi\delta(\omega + \omega_0)$$

$$\cos\omega_0 t \Longleftrightarrow \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$

$$\sin\omega_0 t \Longleftrightarrow \frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$$

$$x(t) = 1 \Longleftrightarrow 2\pi\delta(\omega)$$

$$\delta(t - t_0) \Longleftrightarrow e^{j\omega t_0}$$

$$e^{-at}u(t), \mathrm{Re}\{a\} > 0 \Longleftrightarrow \frac{1}{a + j\omega}$$

$$u(t) \Longleftrightarrow \frac{1}{j\omega} + \pi\delta(\omega)$$

$$\delta(t) \Longleftrightarrow 1$$

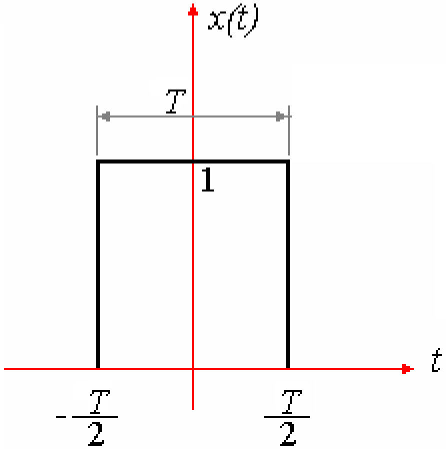
$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| \le \frac{T_0}{2} \end{cases} \Longleftrightarrow \sum_{k=-\infty}^{+\infty} \frac{2\sin k\omega_0 T_1}{k}\delta(\omega_k \omega_0)$$

$$\sum_{n=-\infty}^{+\infty} \delta(t - nT) \Longleftrightarrow \frac{2\pi}{T}\sum^{+\infty} k = -\infty\delta\left(\omega - \frac{2\pi k}{T}\right)$$

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases} \Longleftrightarrow 2T_1 \sin c\left(\frac{\omega T_1}{\pi}\right) = \frac{2\sin\omega T_1}{\omega}$$

Example

Determine the FT of the gate signal



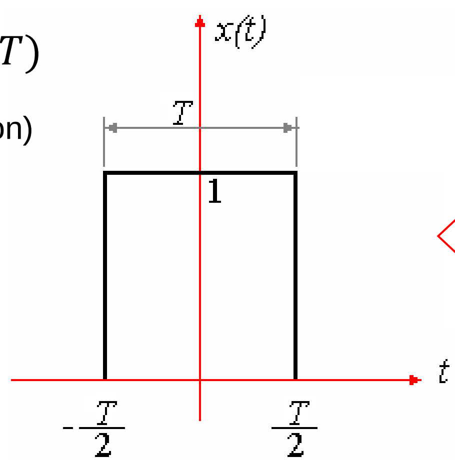
$$\Pi\left(\frac{t}{T}\right)=\begin{cases}1 & , \quad |t|<\frac{T}{2}\\ \tfrac{1}{2} & , \quad |t|=\frac{T}{2}\\ 0 & , \quad |t|>\frac{T}{2}\end{cases}$$

$$X(\omega)=\mathcal{F}\left(\Pi\left(\frac{t}{T}\right)\right)=\int_{-\infty}^{\infty}\Pi\left(\frac{t}{T}\right)e^{-j\omega t}dt=\int_{-\frac{T}{2}}^{\frac{T}{2}}e^{-j\omega t}dt=\left.\frac{e^{-j\omega t}}{-j\omega}\right|_{-T/2}^{T/2}$$

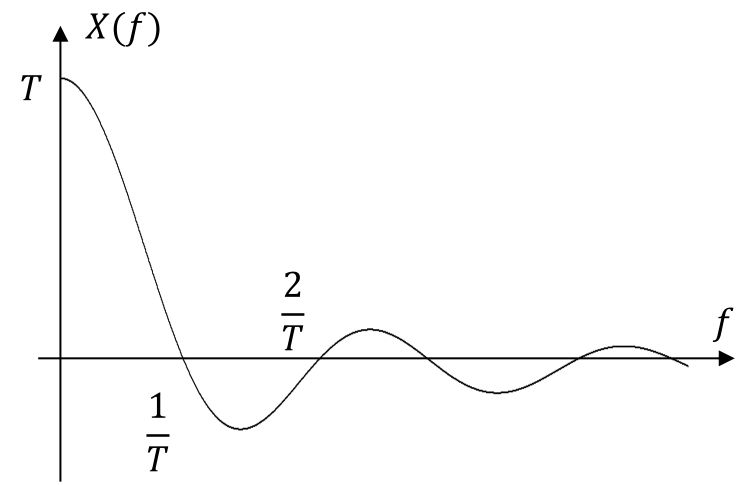
$$=\frac{1}{j\omega}\left(e^{j\omega T/2}-e^{-j\omega T/2}\right)=\frac{2}{j\omega 2}\left(e^{j\omega T/2}-e^{-j\omega T/2}\right)=2\frac{sin(\omega T/2)}{\omega}=T\frac{sin(\pi f T)}{\pi f\ T}$$

$$X(f)=Tsinc(fT)$$

(normalized *sinc* function)

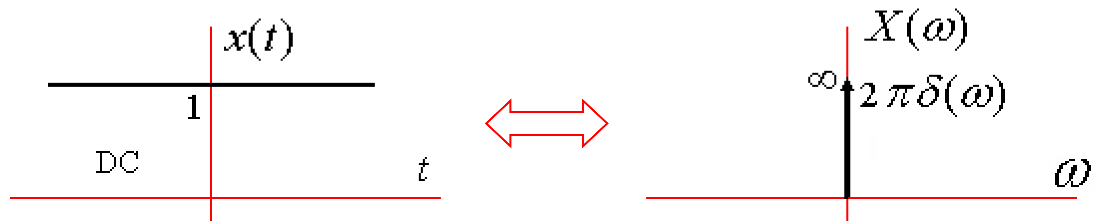


$\Longleftrightarrow$

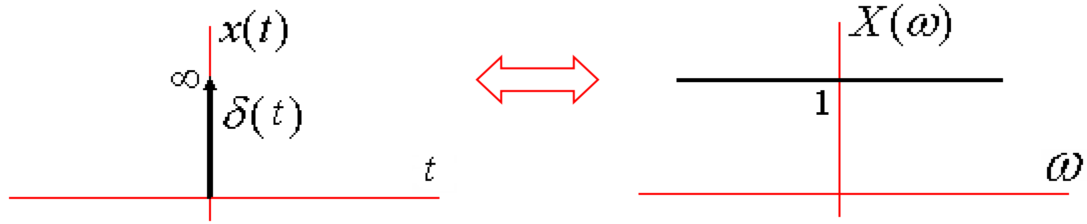


Extreme Cases

$T \to \infty$



$T \to 0$



Example

Find the energy of the *sinc* signal $x(t) = \text{sinc}(t)$

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} \text{sinc}^2(t) dt$$!! hard case !!

we found that $\mathcal{F}\left\{\Pi\left(\frac{t}{T}\right)\right\} = sinc(fT)$ (let $T = 1$ to get $\mathcal{F}\{\Pi(t)\} = sinc(f)$)

using duality prop. of FT : if $x(t) \Leftrightarrow X(f)$ then $X(t) \Leftrightarrow x(-f)$

we get $sinc(t) \Leftrightarrow \Pi(-f) = \Pi(f)$ (symmetric)

using Rayleigh's property

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df = \int_{-\infty}^{\infty} \left(\Pi(f)\right)^2 df$$

$$E_x = \int_{-1/2}^{1/2} df = 1$$

Power and Energy Spectral Densities

According to Rayleigh's property $$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

and the definition of energy $$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$E_x = \int_{-\infty}^{\infty} |X(f)|^2 df$$

Energy

$$\Psi_x(f) = |X(f)|^2$$

Energy Spectral Density

Similarly, the Power Spectral Density for a periodic signal is defined by the equation

$$P_x = \int_{-\infty}^{\infty} G_x(f) df$$

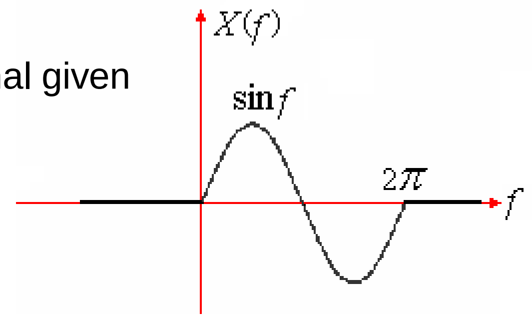
Power

Power Spectral Density

And for a non-periodic signal $$G_x(f) = \lim_{T \to \infty} \frac{1}{T} |X_T(f)|^2$$

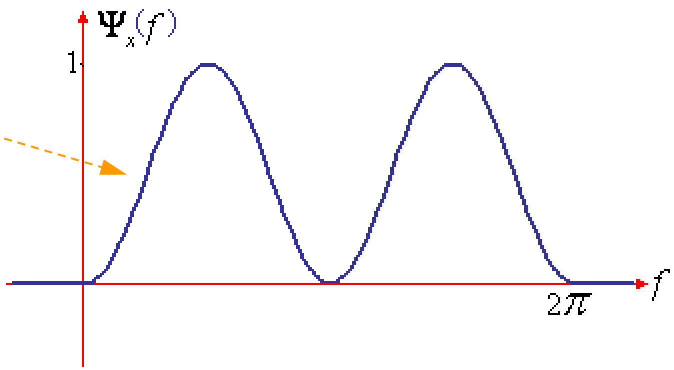
Example

Draw the energy spectral density and find the energy of the signal given



$$\Psi_x(f) = |X(f)|^2 = |\sin(f)|^2 = \sin^2(f)$$

$$(0 \le f < 2\pi)$$



$$E_x = \int_{-\infty}^{\infty} \Psi_x(f)df = 2\int_0^{\pi} \sin^2(f)df = \int_0^{\pi} (1 - \cos(2f))df$$

$$E_x = \pi - \frac{1}{2}\sin(2f)\Big|_0^{\pi} = \pi$$

Power Spectral Density of Random Rectangular Pulse Train

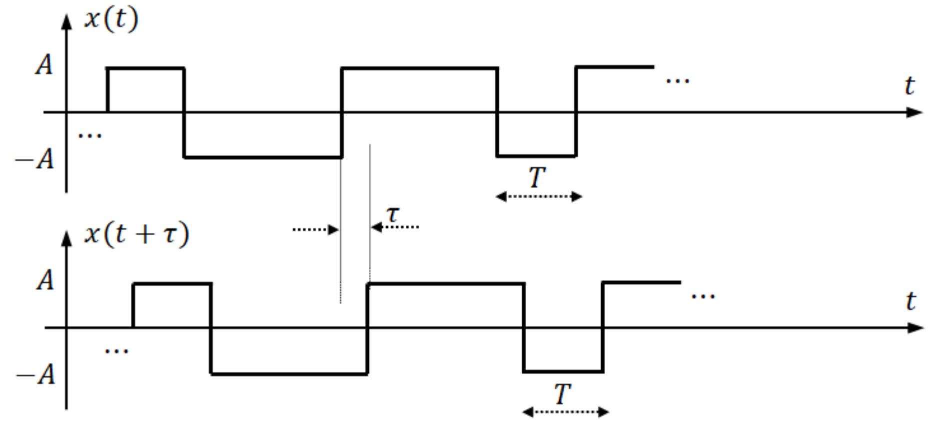
Remember that

$\mathcal{F}\{R_x(\tau)\} = |X(\omega)|^2$

or $\mathcal{F}\{R_x(\tau)\} = |X(f)|^2$

(Wiener–Khinchin theorem)

So, we need to find $R_x(\tau)$

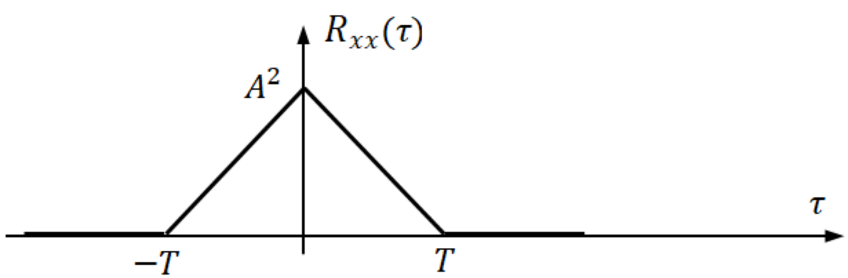


For τ=0 => $x(t+\tau) = x(t)$ => $R_{xx}(0) = max = A^2$

$$R_{xx}(0) = P_x = \frac{1}{T}\int_0^T |x(t)|^2\, dt = A^2$$

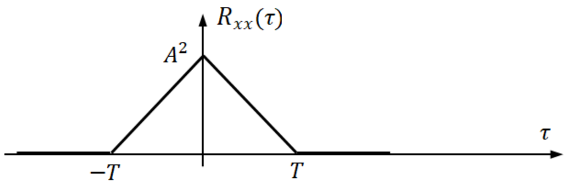
For $\tau > T$ => P(match)=P(unmatch)=0.5 => $R_{xx}(\tau) = 0$

For $0 < |\tau| < T$ => linear decrease from A^2 to 0 as τ increases



Autocorrelation of RBRPT

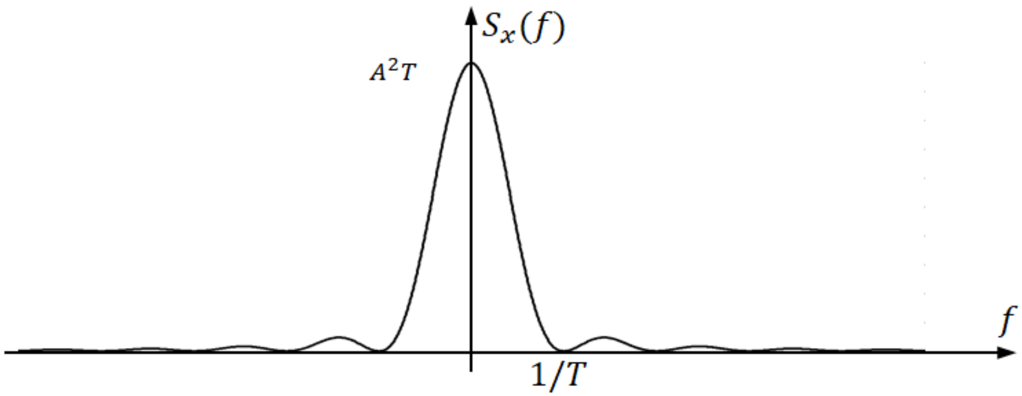
We know that Fourier Transform of triangular pulse is the square of Fourier Transforms of rectangular pulse



1. autocorrelation and self-convolution are same for a rectangular pulse
2. self-convolution (autocorrelation) of rectangular pulse is a triangular pulse.
3. convolution in time domain is multiplication in Fourier domain
4. Fourier Transform of rectangular pulse is *sinc* shaped function

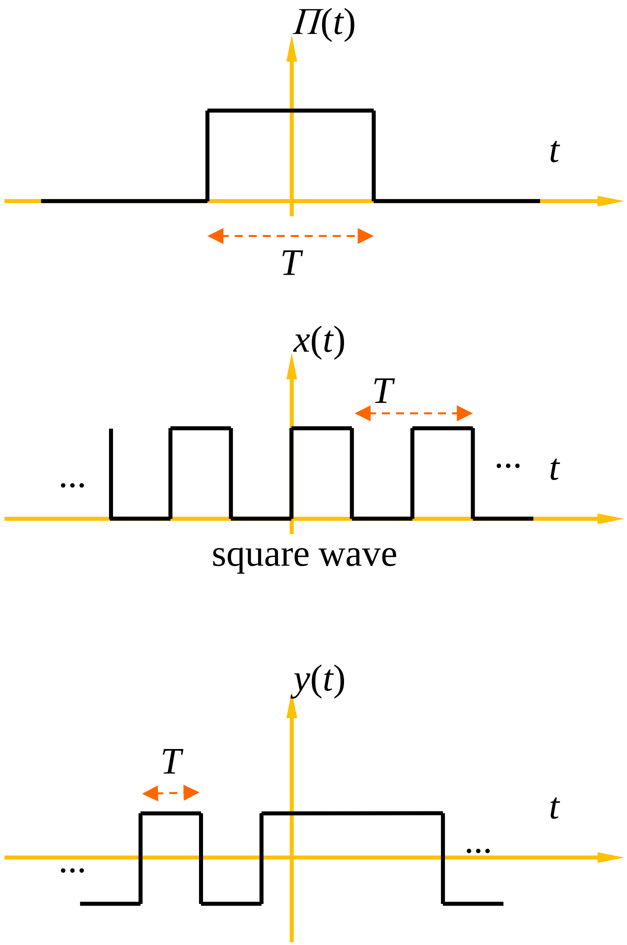
Hmw: you need to do calculations at home to convince yourself

We deduce that $$S_x(f) = A^2 \frac{sin^2(\pi f T)}{\pi f T} = A^2 T sinc^2(fT)$$

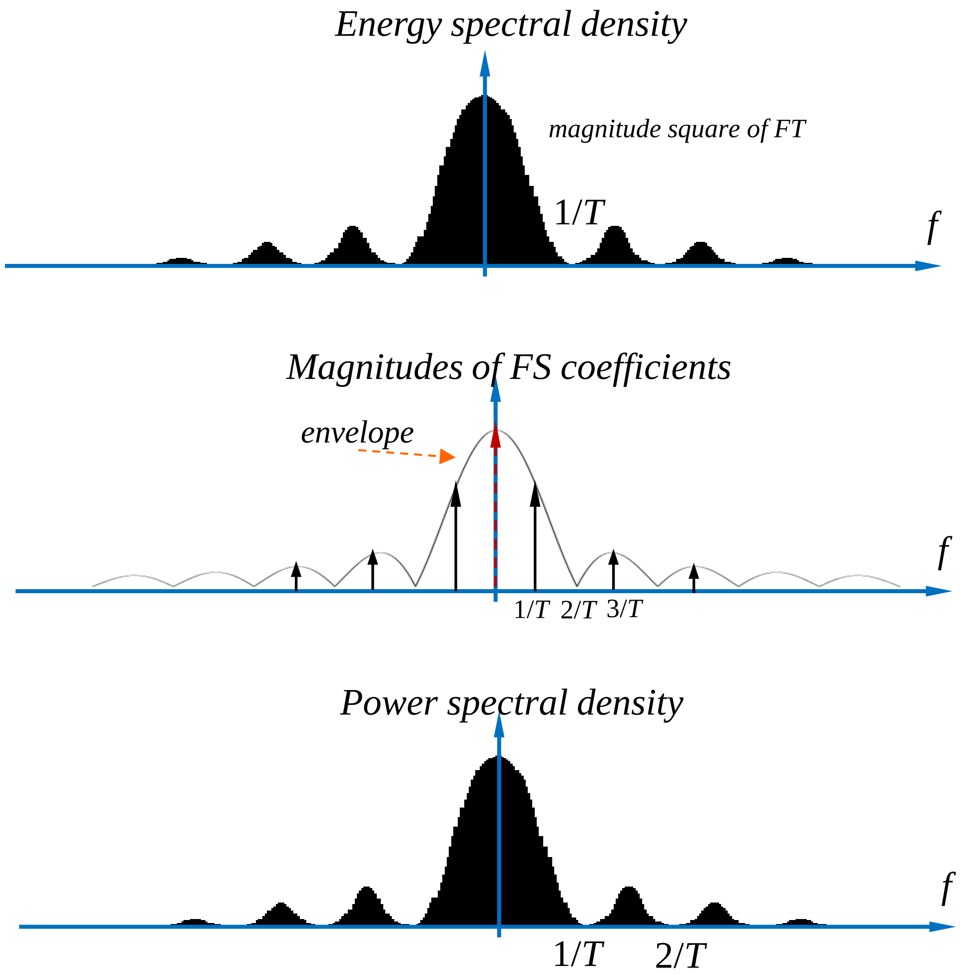


Power Spectral Density (psd) of RBRPT

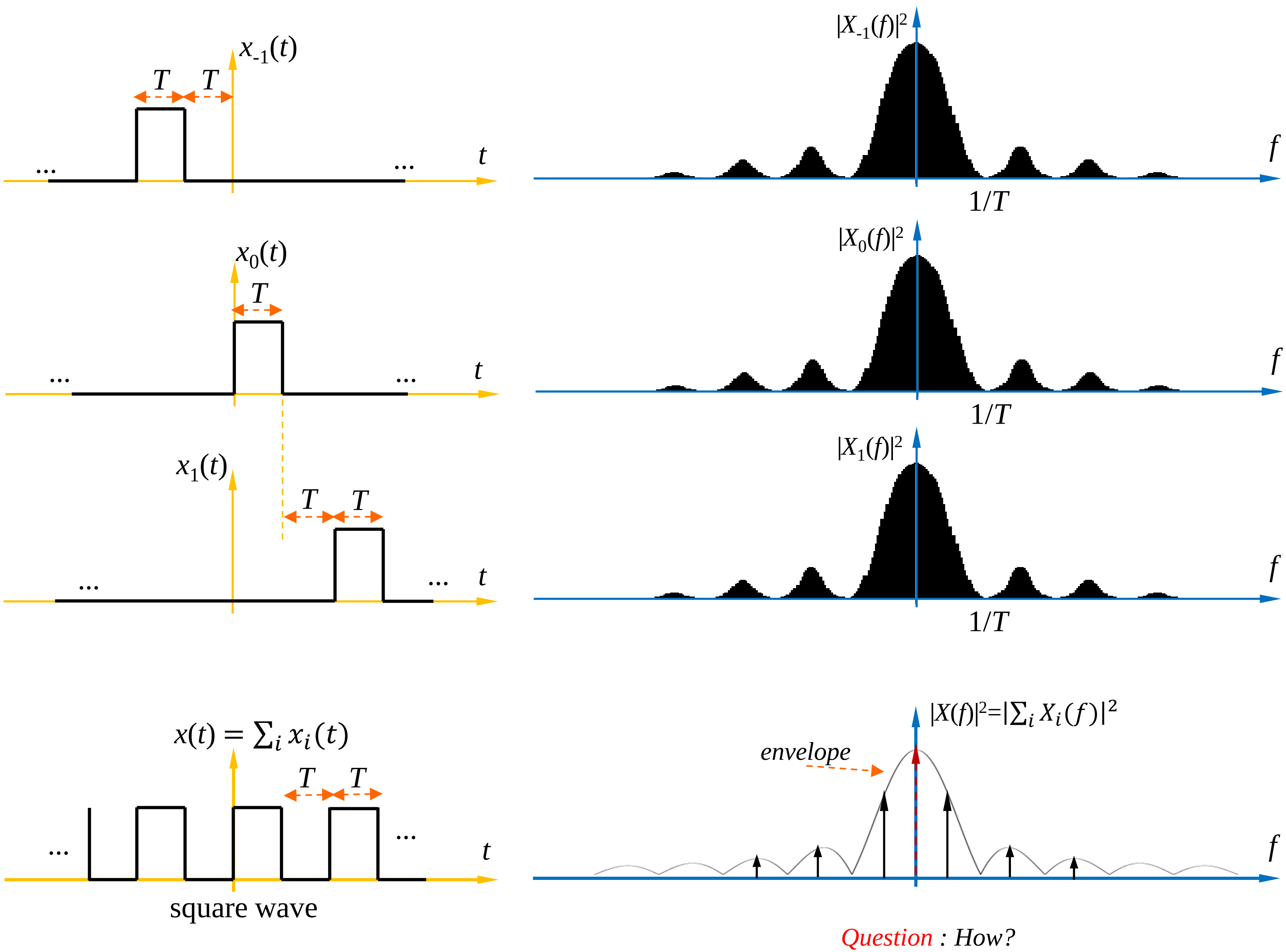
Verify



random binary pulse train
(randomly changes polarity with T intervals)



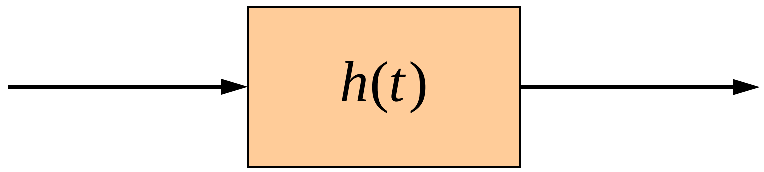
the value at zero frequency represents the average (DC) value



$x_{-1}(t)$
T
T
...
...
t
$|X_{-1}(f)|^2$
f
1/T
$x_0(t)$
T
...
...
t
$|X_0(f)|^2$
f
1/T
$x_1(t)$
T
T
...
...
t
$|X_1(f)|^2$
f
1/T
$x(t) = \sum_i x_i(t)$
T
T
...
...
t
square wave
$|X(f)|^2 = |\sum_i X_i(f)|^2$
envelope
f
Question : How?

Convolution

$x(t)$



$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

$$y(t) = x(t) * h(t)$$

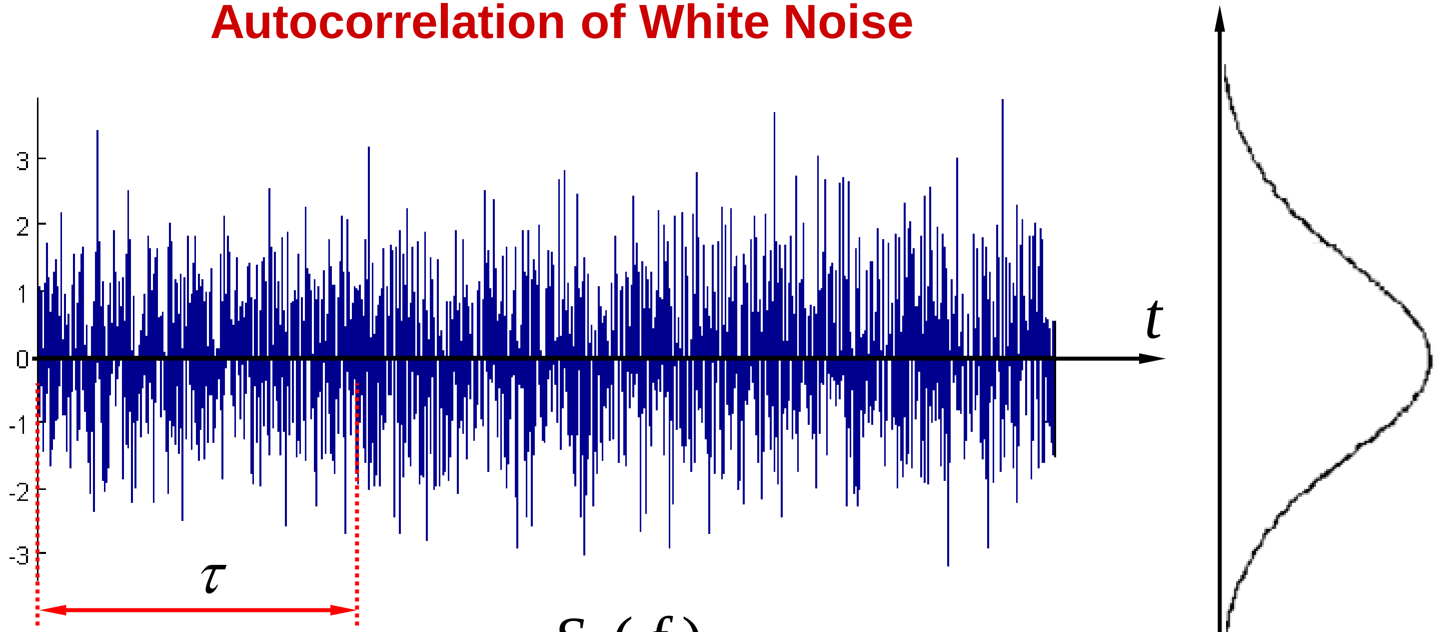
$\mathcal{F}$ourier pairs

$$X(f) \qquad H(f) \qquad Y(f) = X(f)H(f)$$

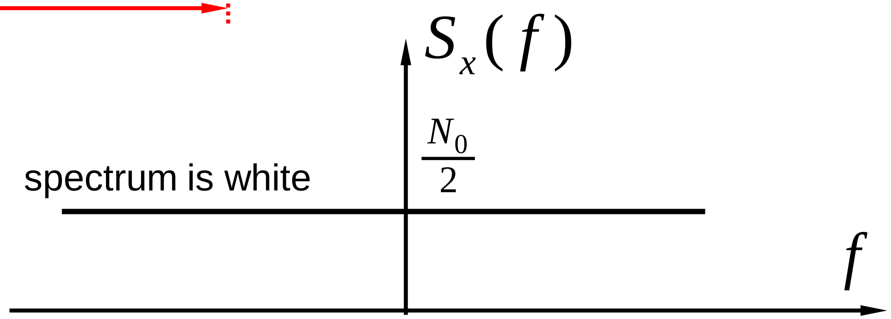
$$G_x(f) \qquad H(f) \qquad G_Y(f) = G_x(f)\left|H(f)\right|^2$$

relation of autocorrelation functions

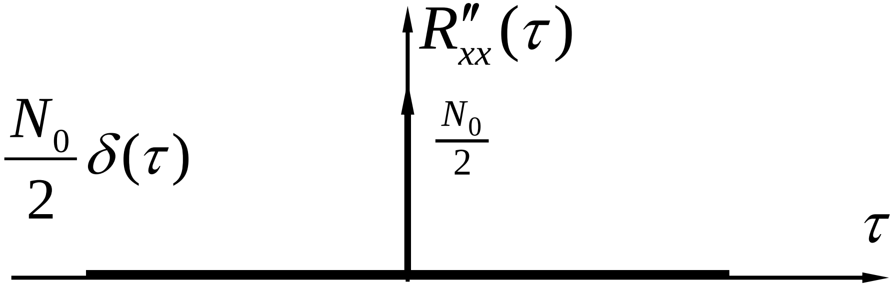
Autocorrelation of White Noise



pdf might be Gaussian



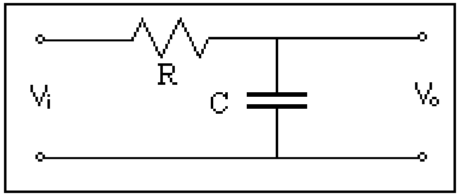
$$R''_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)x^*(t+\tau)dt = \frac{N_0}{2}\delta(\tau)$$



it is uncorrelated

Example

White noise with spectral density $G_n(f) = \frac{N_0}{2}$ is input to the filter shown



Find the power spectral density $G_Y(f)$

Find the autocorrelation function $R_Y(f)$

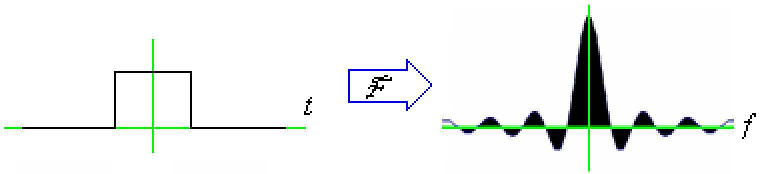
$$G_Y(f) = G_n(f)\left|H(f)\right|^2 \qquad G_Y(f) = \frac{N_0}{2}\frac{1}{1+(2\pi fRC)^2}$$

$$R_Y(\tau) = \mathcal{F}^{-1}\left\{G_Y(f)\right\} \qquad R_Y(\tau) = \frac{N_0}{4RC}e^{-\frac{|\tau|}{RC}}$$

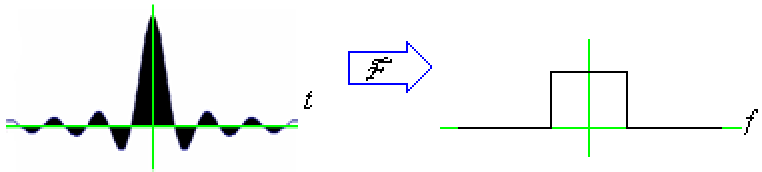
Output noise is not white

Output noise is not completely uncorrelated

The Bandwidth Dilemma



A limited duration signal has infinite bandwidth



A limited bandwidth signal has infinite duration

So, we just can not define *bandwidth* as formulated by the highest frequency component of the signal, because such a signal may not be real.

Half-power bandwidth : Defines the frequency which the signal power drops to half of the peak value (or 3dB below the peak value).

Noise equivalent bandwidth : The bandwidth of an ideal filter which passes the same amount of noise power as its real counterpart

Null-to-Null bandwidth :

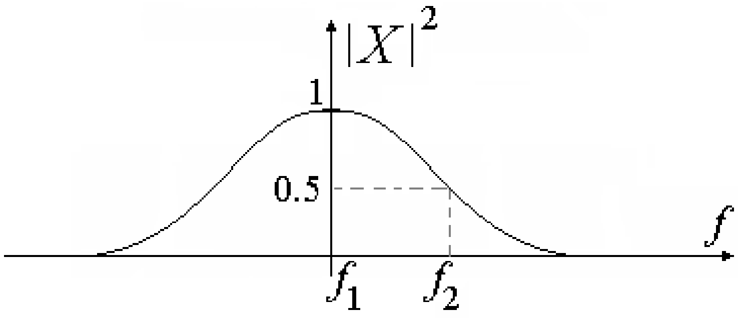
Fractional power containment bandwidth :

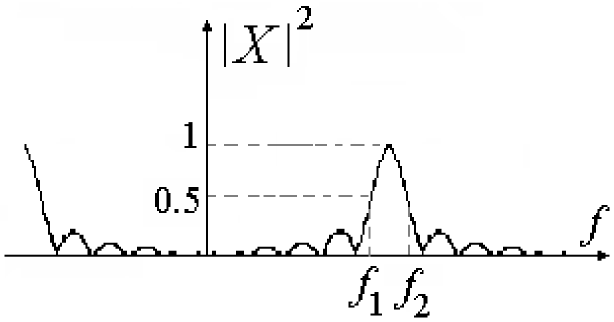
Bounded psd bandwidth :

Absolute bandwidth :

Homework : Find, read and learn about them

Half-Power Bandwidth



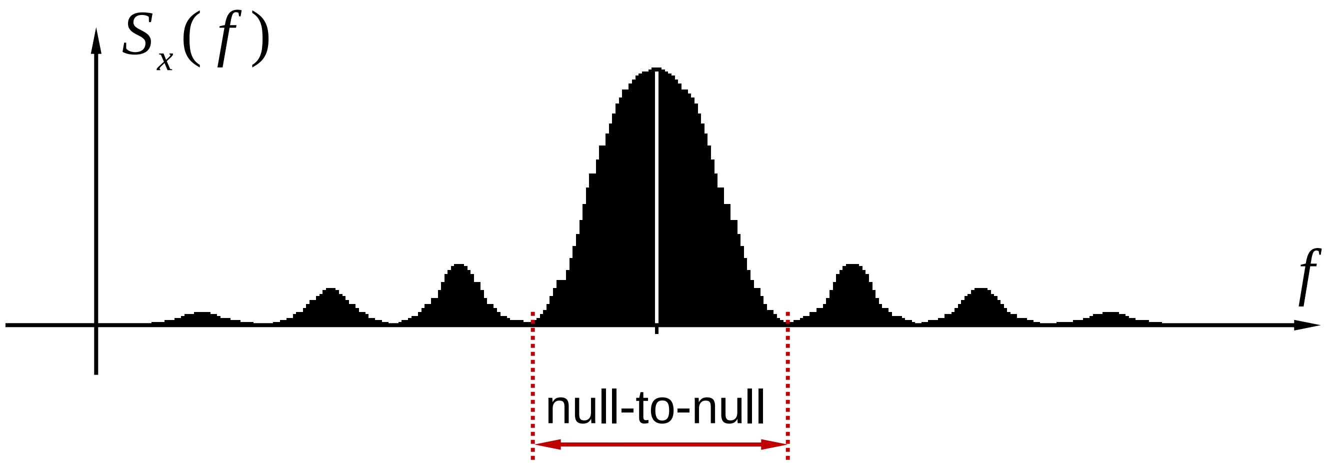


$$10\log_{10}(0.5) \cong -3 \text{ dB}$$

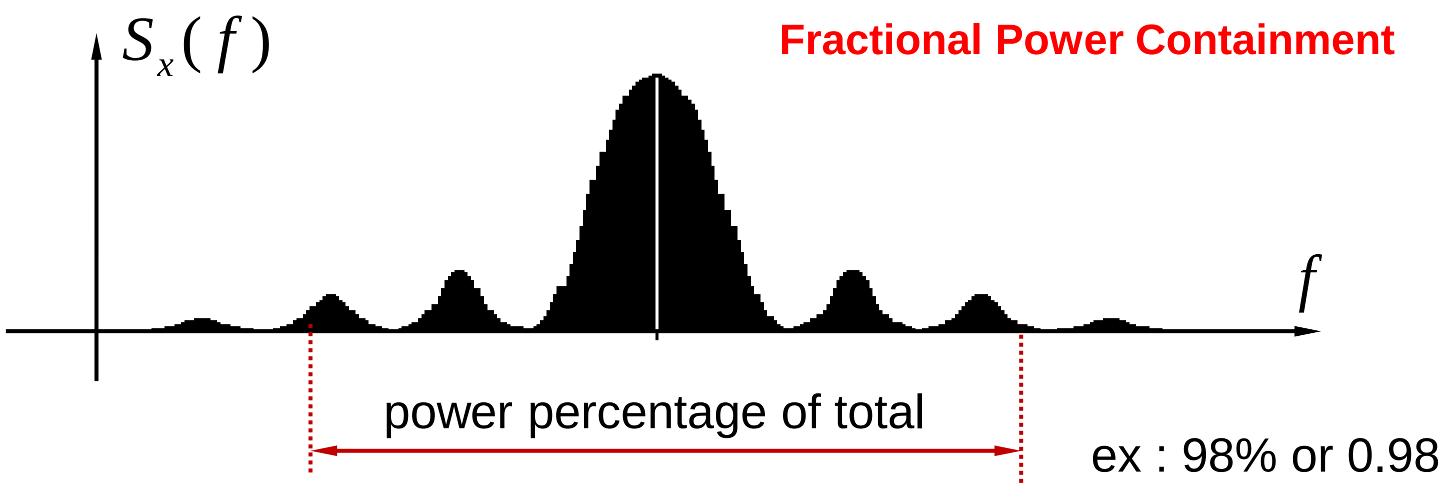
Therefore, it is sometimes called 3 dB bandwidth

Homework : Find the 3 dB bandwidth of an RC filter

Null-to-Null Bandwidth

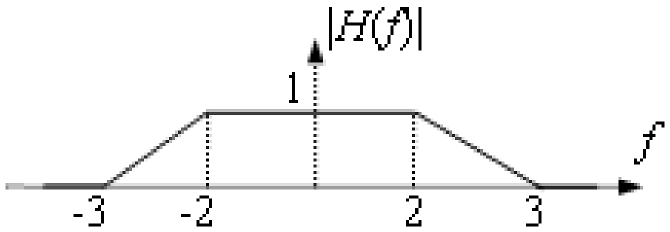


Fractional Power Containment

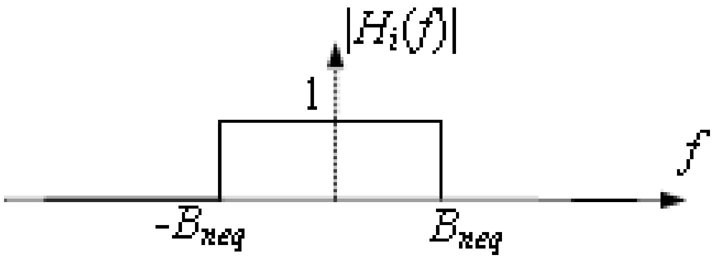


Example

$$|H(f)| = \begin{cases} 1 & , \quad |f| < 2 \\ 3 - |f| & , \quad 2 \le |f| < 3 \\ 0 & , \quad |f| \ge 3 \end{cases}$$



is given. What is the *noise equivalent bandwidth*?



$$P_o = \int_{-\infty}^{\infty} S_o(f)df$$

$$P_o = 2\int_0^2 |H(f)|^2 df + 2\int_2^3 |H(f)|^2 df$$

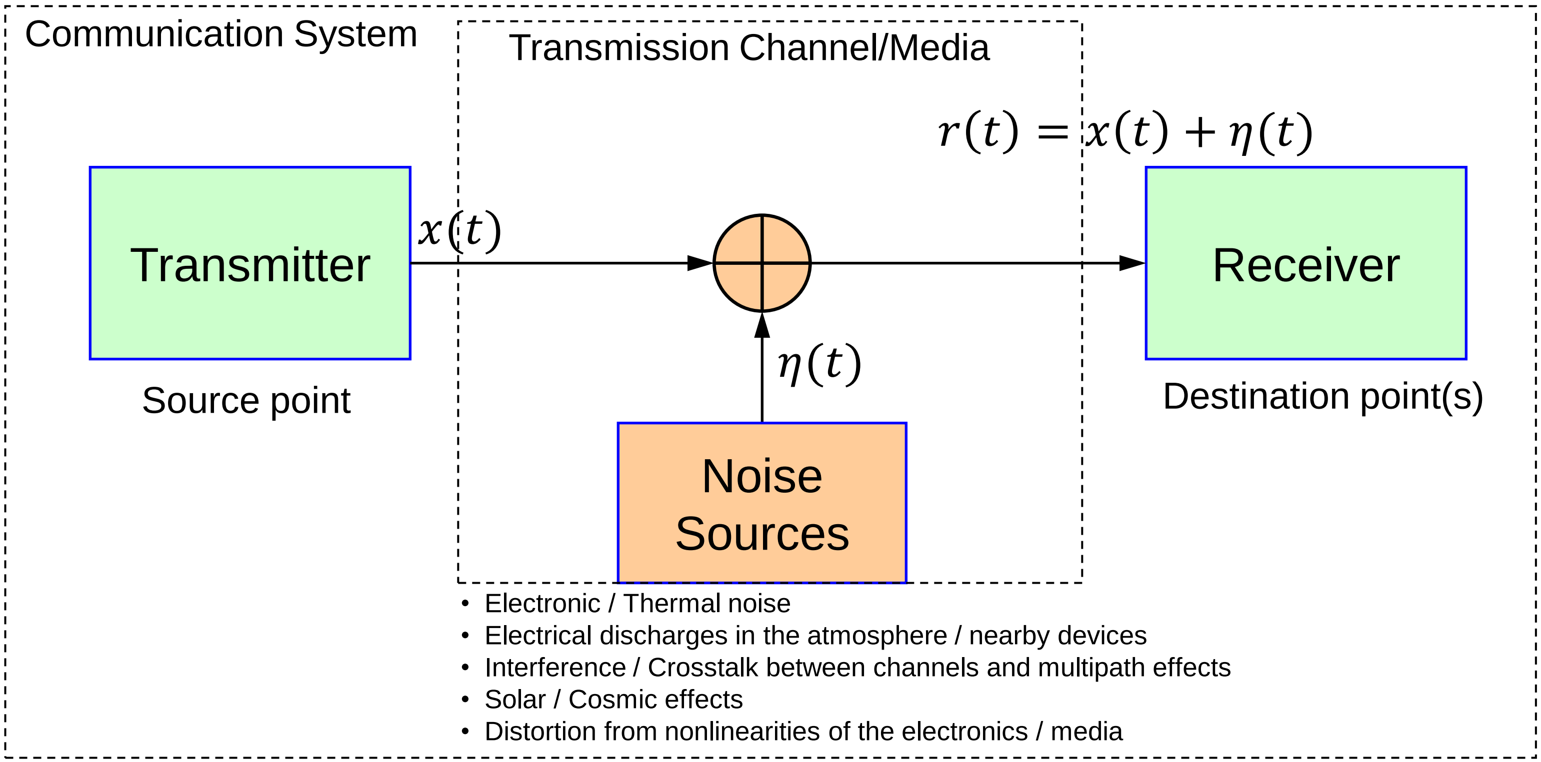
$$P_o = 2\int_0^2 df + 2\int_2^3 |3 - f|^2 df \quad = 4 + \tfrac{2}{3} = 14/3$$

$$P_{neq} = 2\int_0^{\infty} S_i(f)\,|H_i(f)|^2 df = 2\int_0^{B_{neq}} df = 2B_{neq}$$

$$2B_{neq} = 14/3 \quad \Rightarrow \quad B_{neq} = 7/3$$

Homework : Find the noise equivalent bandwidth of an RC filter

Signal Quality Measure

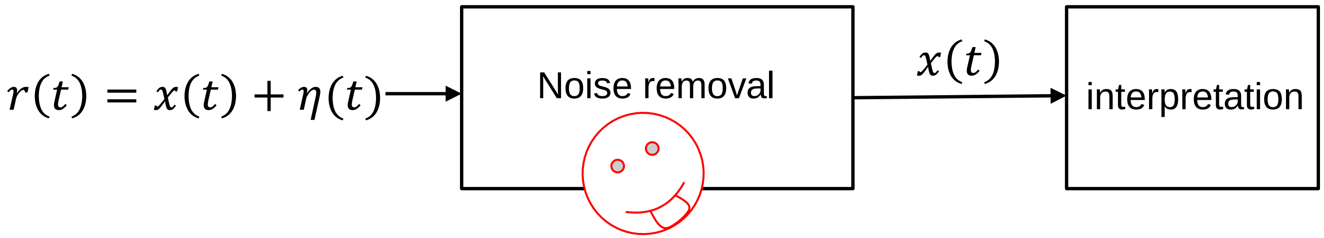


The receiver's job is to determine what is sent (the actual information) using the received noisy signal

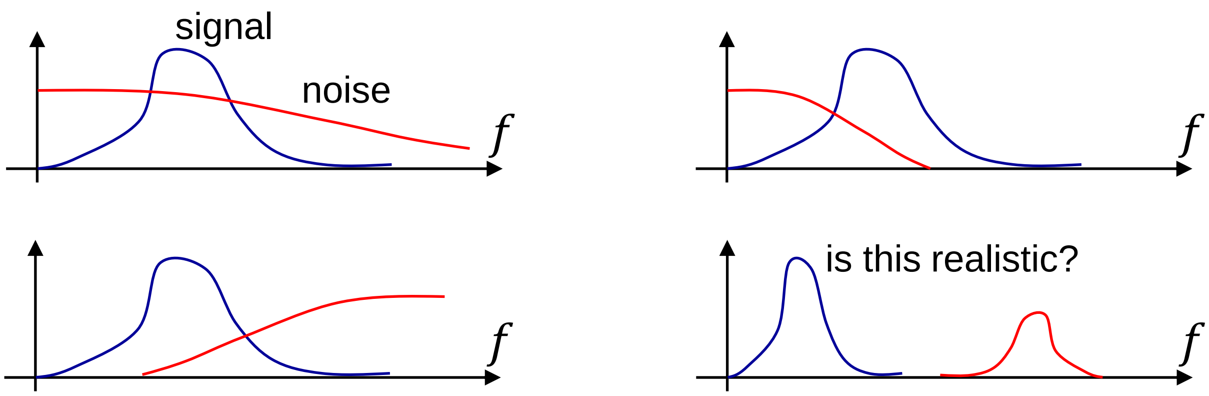
In order to compare different systems for their interpretation correctness, we need some comparison measures along with signal quality measures

Signal Quality Measure

Conceptual Receiver (the *wish*)



various scenarios



There is no such thing as “complete removal of noise”, only “gracefully accepting it”

Any signal that makes interpretation difficult, even if it is structured, is called noise.

It is not deterministic, it is stochastic.

Signal Quality Measure

The basic quality measure is Signal-to-Noise-Ratio $SNR = \frac{P_{signal}}{P_{noise}}$

Mostly, the *log* version is used $SNR_{dB} = 10log_{10}\left(\frac{P_{signal}}{P_{noise}}\right)$ deciBell

Frequency domain equivalent can also be used $SNR = \frac{\int |X(f)|^2 df}{\int S_n(f) df}$

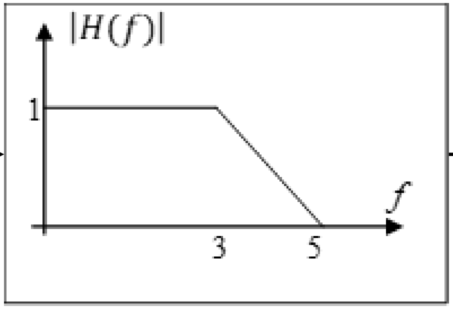
Why *log* version : dynamic range compression

Because noise is always effective, even when it is very small.

Signal Quality Measure

Example: Signal : $x(t) = cos(8\pi t) + 2sin(4\pi t + \pi/8)$

Noise : AWGN with N_o=1 $\mu W/Hz$

$x(t) + \eta(t)$ →  → $y(t) + \bar{\eta}(t)$

Output SNR ?

$$y(t) = 0.5cos(8\pi t + \varphi_1) + 2sin(4\pi t + \varphi_2)$$

phases will change, but we are not interested

Signal power, using $P_s = \int_0^T |x(t)|^2 dt$ or $P_s = V_{pp}^2/2$

$$P_{so} = P_{s1} + P_{s2} = 0.125 + 2 = 2.125 \ \ W$$

$$P_{\eta o} = \int_0^3 N_o df + N_o \int_3^5 |H(f)|^2 df = 3N_o + N_o \int_3^5 |f/2|^2 df = 11/3 \ \ \mu W$$

$$SNR = 2.125 \ W/\tfrac{11}{3}\mu W \cong 579545 \cong 57.6 \ dB$$

Signal Quality Measure

Bit Error Rate (BER) : Ratio of #bits received/detected in error and #bits transmitted

Example:

$z = \{0.36, 0.18, 0.17, 0.16, 0.13\}$ and $A = \{00, 01, 10, 110, 111\}$

1 out of 1000 symbols is received in error. What is the BER?

Average Bits per Symbol $$L_{avg} = \sum_{i=0}^{3} p_i l_i = 2.29\ bpsym$$

Assume worst case => symbol error means all bits in symbol are in error

$$BER = \frac{BitErrorsPerSymbol \times P_{symbolerror}}{L_{avg}} = 10^{-3}$$ same as SER for worst case

Best case ?

Hmw: Assume BER is known, calculate the SER.

Homework Problems

These problems are in the textbook "Digital Communications – Fundamentals and Applications, 2nd Ed." by B. Sklar

1.1. Classify the following signals as energy signals or power signals. Find the normalized energy or normalized power of each.

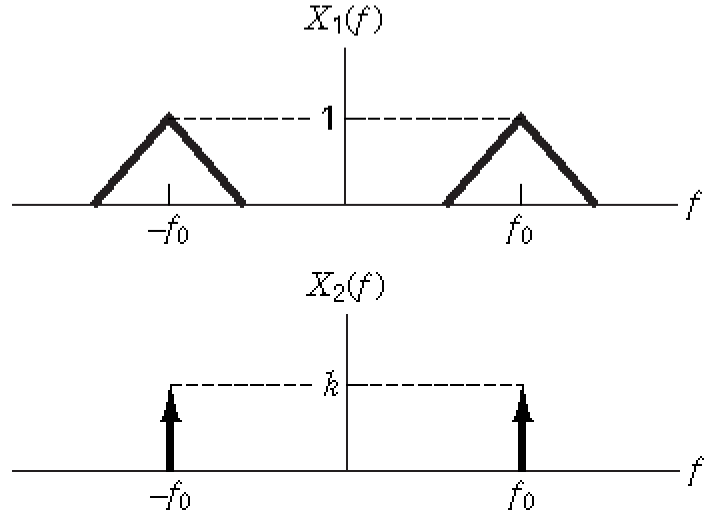
(a) $x(t) = A \cos 2\pi f_0 t \qquad \text{for } -\infty < t < \infty$

(b) $x(t) = \begin{cases} A \cos 2\pi f_0 t & \text{for } -T_0/2 \le t \le T_0/2, \text{ where } T_0 = 1/f_0 \\ 0 & \text{elsewhere} \end{cases}$

(c) $x(t) = \begin{cases} A \exp(-at) & \text{for } t > 0, a > 0 \\ 0 & \text{elsewhere} \end{cases}$

(d) $x(t) = \cos t + 5 \cos 2t \qquad \text{for } -\infty < t < \infty$

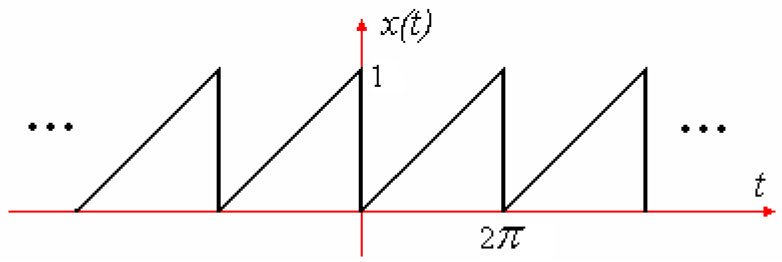
1.14. Find $X_1(f) * X_2(f)$ for the spectra shown in figures

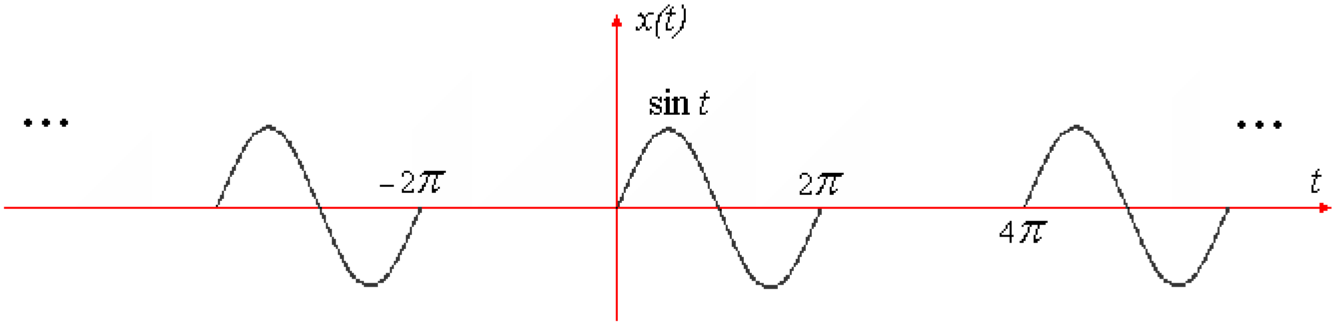


Homework Problems

These problems are in the textbook "Modern Digital and Analog Communication Systems" by B.P. Lathi

2.1- 3. Find Fourier Series representation (trigonometric or complex exponential) of the following.





2.8- 1. Energies of signals $g_1(t)$ and $g_2(t)$ are E_1 and E_2 respectively.

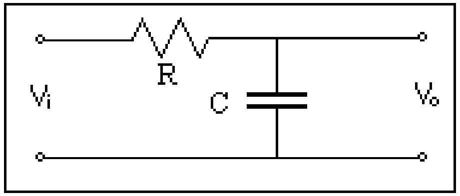
a) Show that, in general, the energy of the signal $g_1(t)+g_2(t)$ is not E_1+E_2 .

b) Under what condition is the energy of $g_1(t)+g_2(t)$ equal to E_1+E_2 .

c) Can the energy of signal $g_1(t)+g_2(t)$ be zero? If so under what condition(s)?

Example

The following LPF is fed with the signal $x(t) = 2\cos(2\pi f_1 t) + 2\sin(2\pi f_2 t)$



where $f_1 = \frac{1}{2\pi RC}$ and $f_2 = \frac{1}{\pi RC}$.

Draw the output psd ?

Given $|H(f)|^2 = \frac{1}{1+(2\pi fRC)^2}$

We just insert given frequencies and see the output powers for unit inputs.

$$G_y(f_1) = \frac{1}{1+1} = 0.5 \qquad G_y(f_2) = \frac{1}{1+4} = 0.2$$

This is, of course, for unit input powers at given frequencies.

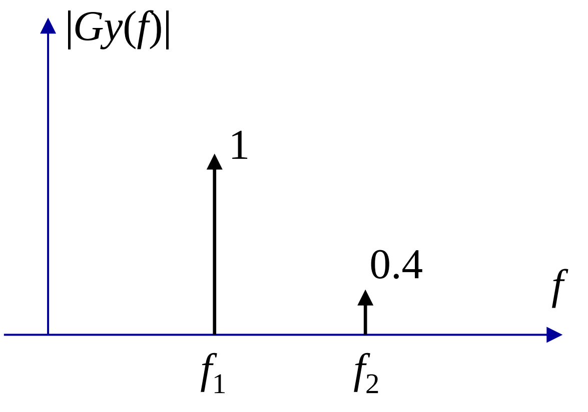
We have power in only two frequencies at the output (assuming single sided spectrum).

You may need to recalculate power of a sinusoidal using $(\frac{1}{2}A^2)$

$$G_x(f_1) = \frac{1}{T}\int_0^T |2\cos(2\pi f_1 t)|^2 dt = 2 \quad \text{and} \quad G_x(f_2) = 2$$

in case you did not memorize it already.

So, we need to multiply output values with these to get output psd graph.



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