SIGNALS AND SYSTEMS

Lecture 2

Basic Signals

CONTINUOUS-TIME

COMPLEX EXPONENTIAL SIGNALS

SINUSOIDAL SIGNALS

Continuous-time Complex Exponential Signals

The continuous-time complex exponential signal is of the form

$$x(t) = c\,e^{at}$$

where both c and a are complex numbers.

Depending upon the values of these parameters, the complex exponential can show different characteristics.

Special cases:

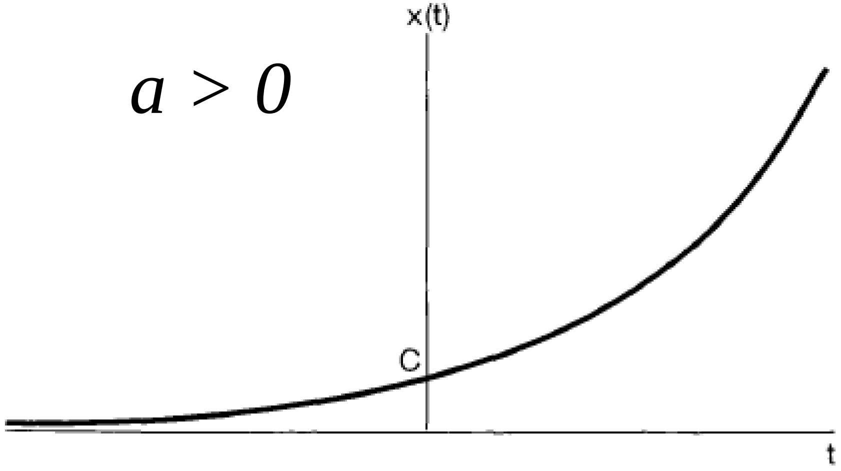
1) Both C and a are real: **Real Exponential Signals**

2) a is purely imaginary: **Periodic Complex Exponential Signals**

Real Exponential Signals: Growing Exponential

$$x(t) = c\,e^{at} \qquad \text{both } c \text{ and } a \text{ are real}$$

$a > 0$



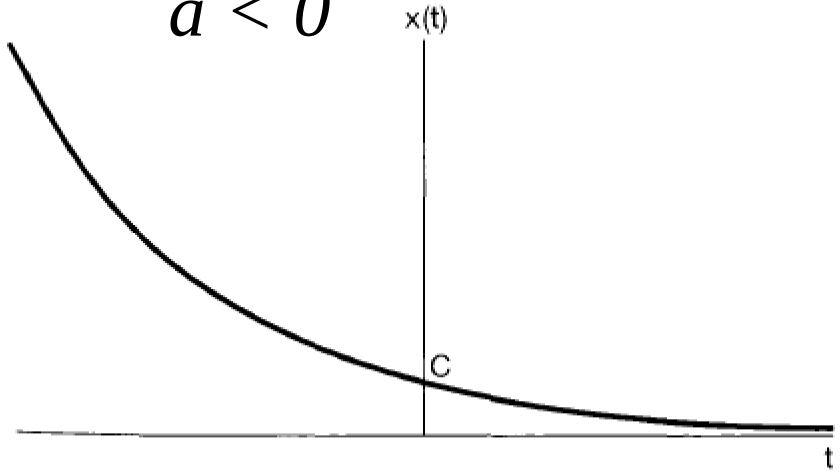
A form that is used in describing many different physical processes:

- chain reactions in atomic explosions
- complex chemical reactions

Real Exponential Signals: Decaying Exponential

$$x(t) = c\,e^{at} \qquad \text{both } c \text{ and } a \text{ are real}$$

$a < 0$



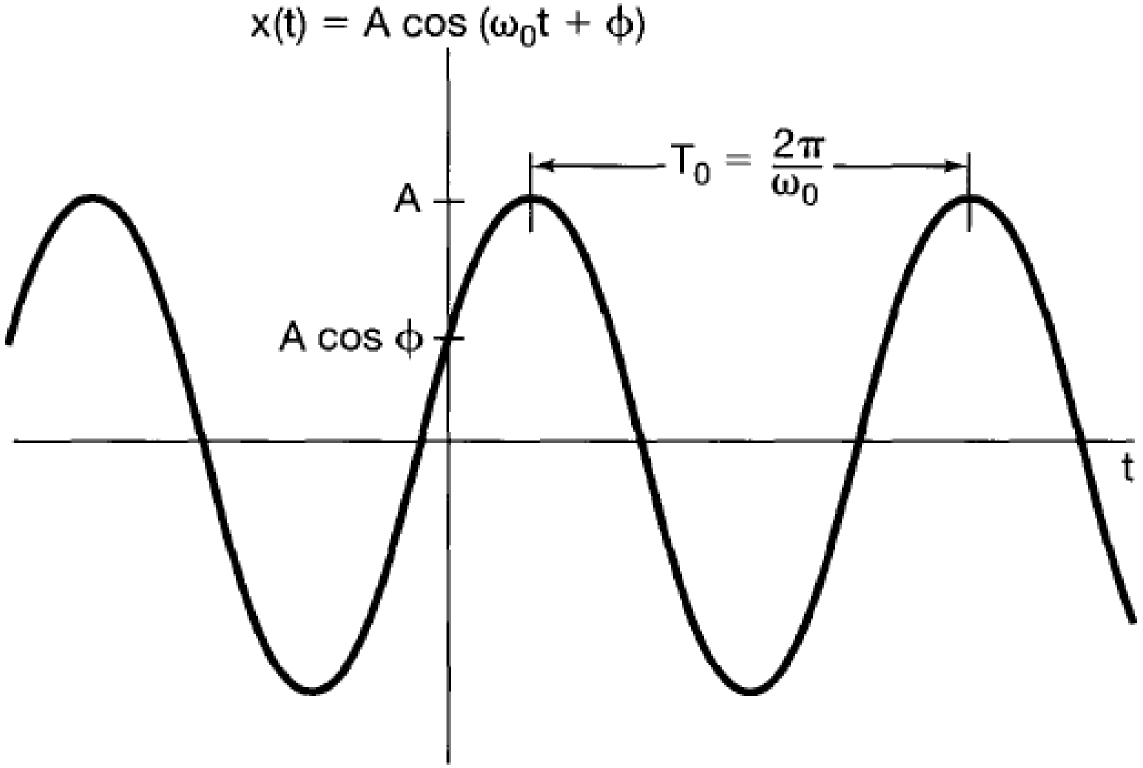
Used to describe:

- The process of radioactive decay and
- The responses of RC circuits and damped mechanical systems

Sinusoidal Signals

$$x(t) = A\cos(\omega_0 t + \phi),$$

$$\omega_0 = 2\pi f_0,$$



Units:
t (time): seconds
ω_0 :radians per seconds
f_0 : cycles per second or hertz (Hz)

Like the complex exponential signal, the sinusoidal signal is periodic with fundamental period T_0

$$T_0 = \frac{2\pi}{|\omega_0|}$$

Sinusoidal Signals

$$x(t) = A\cos(\omega_0 t + \phi)$$

$$T_0 = \frac{2\pi}{|\omega_0|}$$

ω_0 is referred to as the *fundamental frequency.* The fundamental period T_0 of a continuous-time sinusoidal signal is inversely proportional to ω_0 .

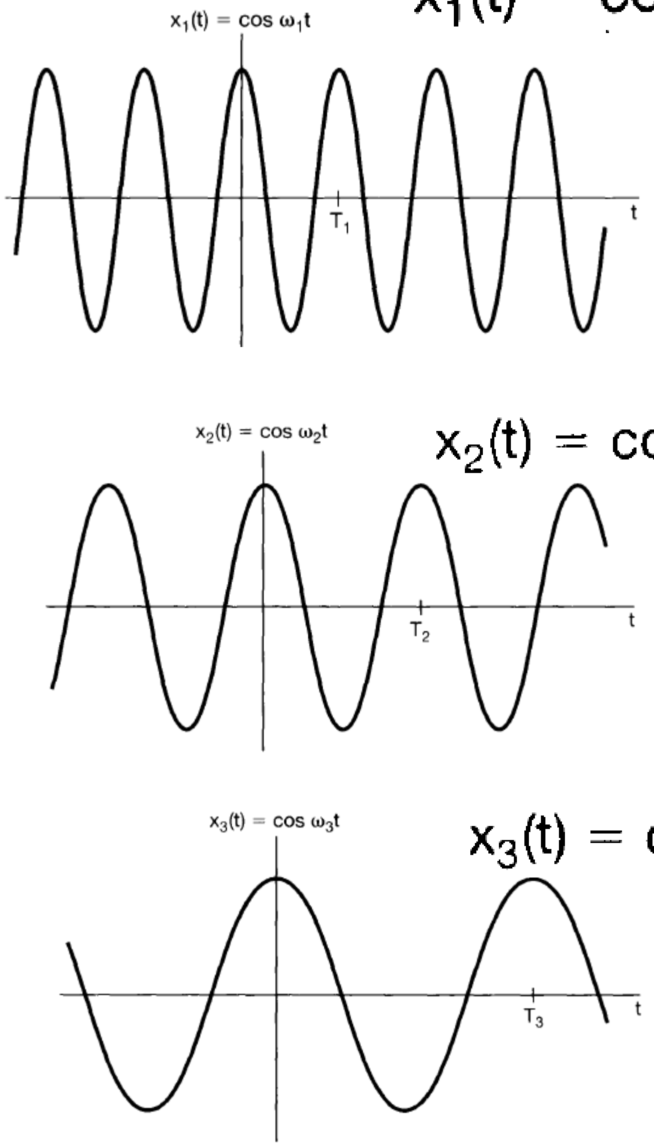
If we decrease the magnitude of ω_0 , we slow down the rate of oscillation and therefore increase the period.

Exactly the opposite effects occur if we increase the magnitude of ω_0 .

$x_1(t) = \cos \omega_1 t$

$x_2(t) = \cos \omega_2 t$

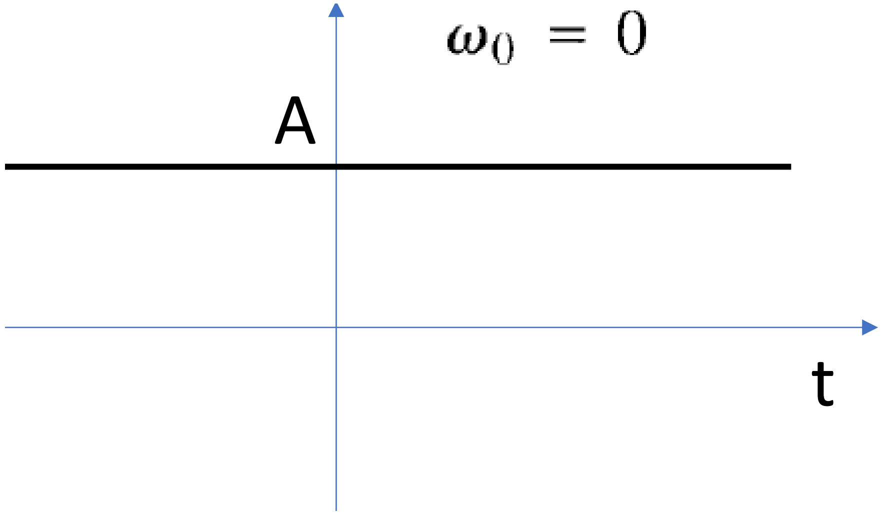
$x_3(t) = \cos \omega_3 t$



$$\omega_1 > \omega_2 > \omega_3$$

Sinusoidal Signals

$$x(t) = A\cos(\omega_0 t),$$



Consider now the case $\omega_0 = 0$.

In this case, *x(t)* is constant and therefore is periodic with period *T* for any positive value of *T.* Thus, the fundamental period of a constant signal is undefined.

On the other hand, there is no ambiguity in defining the fundamental frequency of a constant signal to be zero.

A constant signal has a zero rate of oscillation.

Complex Numbers

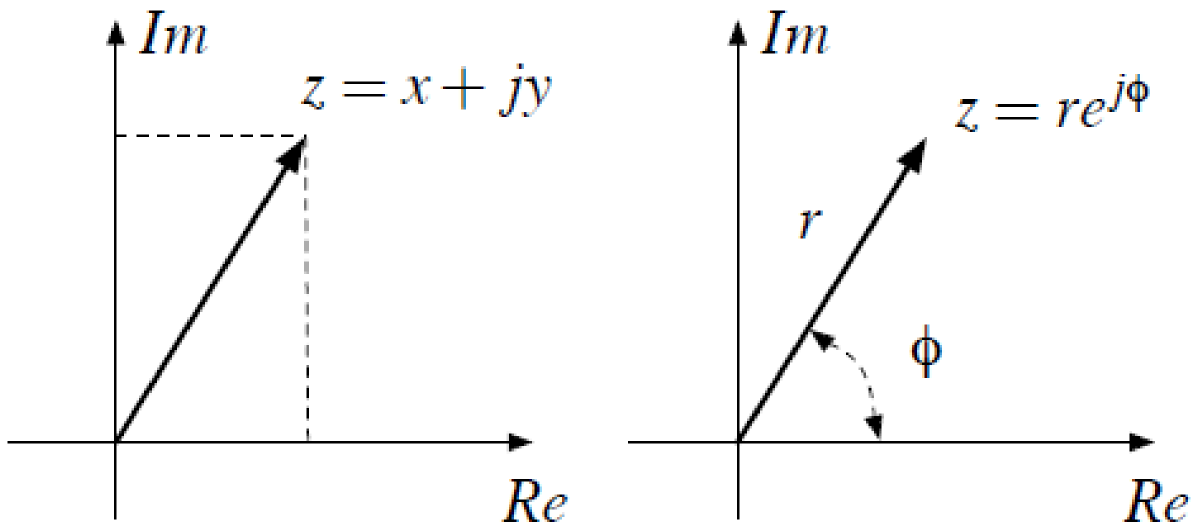
Complex number in Cartesian form: $z = x + jy$

- $x = \Re z$, the *real part* of z
- $y = \Im z$, the *imaginary part* of z
- x and y are also often called the *in-phase* and *quadrature* components of z.
- $j = \sqrt{-1}$ (engineering notation)
- $i = \sqrt{-1}$ (physics, chemistry, mathematics)

Complex Numbers

Complex number in polar form: $z = re^{j\phi}$

- r is the *modulus* or *magnitude* of z
- ϕ is the *angle* or *phase* of z
- $\exp(j\phi) = \cos\phi + j\sin\phi$



- complex exponential of $z = x + jy$:

$$e^z = e^{x+jy} = e^x e^{jy} = e^x(\cos y + j\sin y)$$

Complex Signals

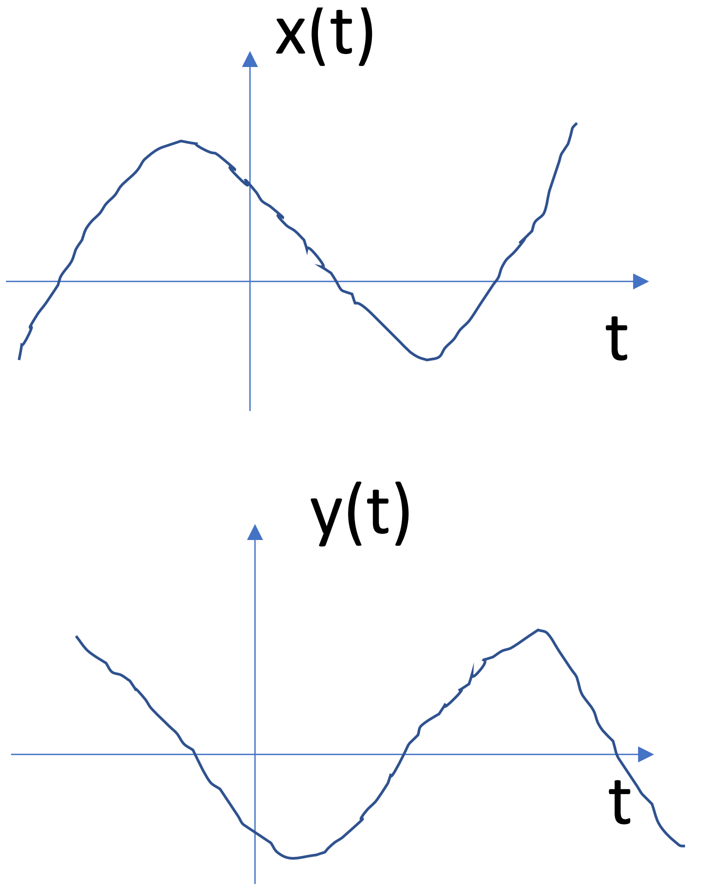
- Signals can also be complex

$$z(t) = x(t) + jy(t)$$

where $x(t)$ and $y(t)$ are each real valued signals, and $j = \sqrt{-1}$.

- Arises naturally in many problems
  - Convenient representation for sinusoids
  - Communications
  - Radar, sonar, ultrasound

$$z(t) = x(t) + jy(t)$$



Continuous-time Complex Exponential Signals

The continuous-time complex exponential signal is of the form

$$x(t) = c\,e^{at}$$

where both c and a are complex numbers.

Depending upon the values of these parameters, the complex exponential can show different characteristics.

Special cases:

1) Both C and a are real: **Real Exponential Signals**

2) a is purely imaginary: **Periodic Complex Exponential Signals**

Periodic Complex Exponential and Sinusoidal Signals

- Special case of complex exponential signals where a is purely imaginary.

$$x(t) = e^{j\omega_0 t}.$$

- This signal is periodic.
- To find the period T, we write the periodicity condition:

$$e^{j\omega_0 t} = e^{j\omega_0 (t+T)}$$

$$e^{j\omega_0 (t+T)} = e^{j\omega_0 t} e^{j\omega_0 T}$$

Then, we must have: $e^{j\omega_0 T} = 1$

Periodic Complex Exponential and Sinusoidal Signals

$$x(t) = e^{j\omega_0 t}.$$

For periodicity, we must have

$$e^{j\omega_0 T} = 1$$

The smallest positive T satisfying this equation is

$$T_0 = \frac{2\pi}{|\omega_0|}.$$

Fundamental period
of the signal $e^{j\omega_0 t}$

Periodic Complex Exponential and Sinusoidal Signals

By using Euler's relation the complex exponential can be written in terms of sinusoidal signals with the same fundamental period:

$$e^{j\omega_0 t} = \cos\omega_0 t + j\sin\omega_0 t.$$

Similarly, the sinusoidal signal can be written in terms of periodic complex exponentials, again with the same fundamental period:

$$A\cos(\omega_0 t + \phi) = \frac{A}{2}e^{j\phi}e^{j\omega_0 t} + \frac{A}{2}e^{-j\phi}e^{-j\omega_0 t}$$

Periodic Complex Exponential and Sinusoidal Signals

Alternatively, we can express a sinusoid in terms of a complex exponential signal as

$$A\cos(\omega_0 t + \phi) = A\,\mathcal{R}e\{e^{j(\omega_0 t + \phi)}\},$$

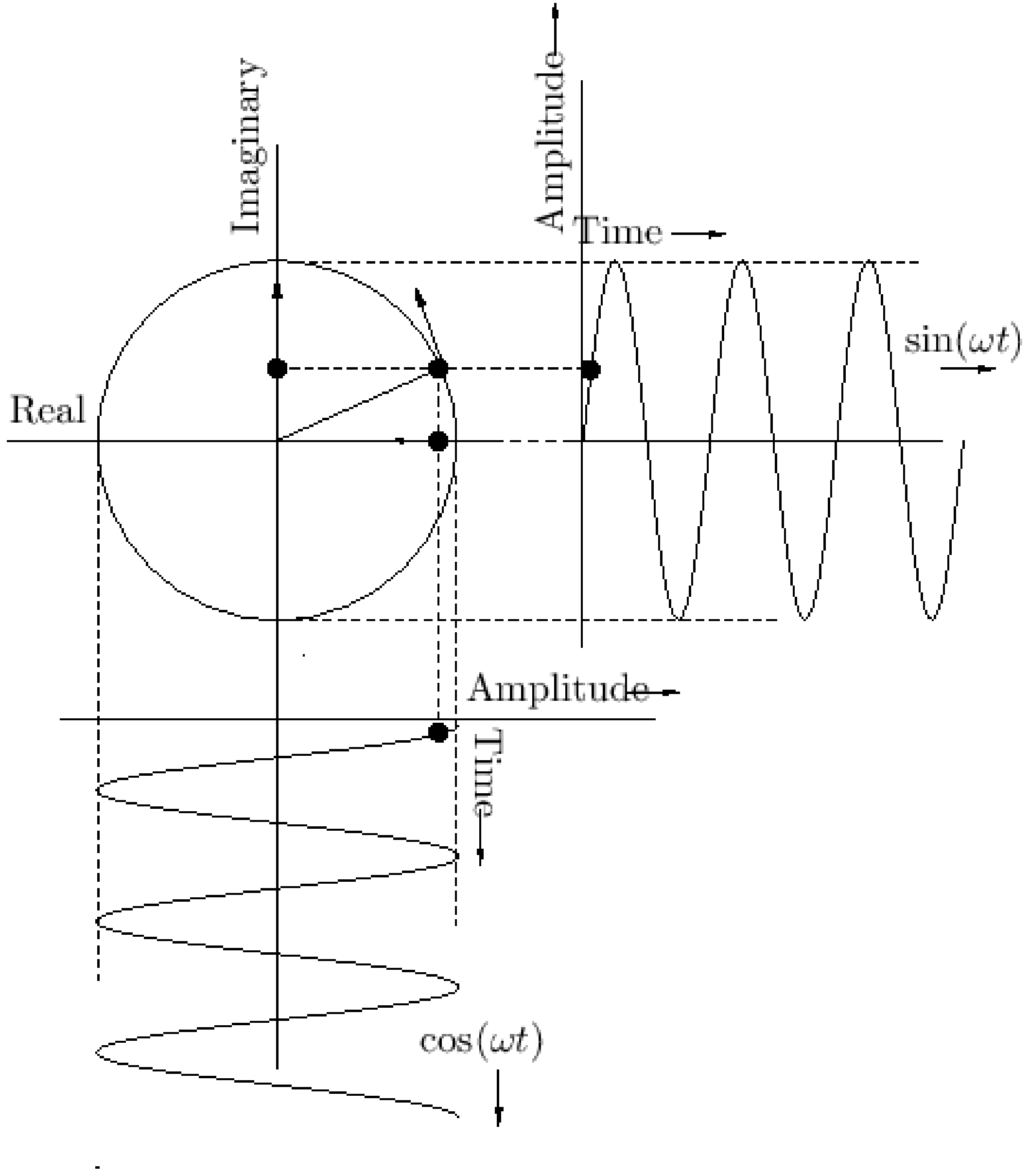
where, if *c* is a complex number, $\mathcal{R}e\{c\}$ denotes its real part.

Also;

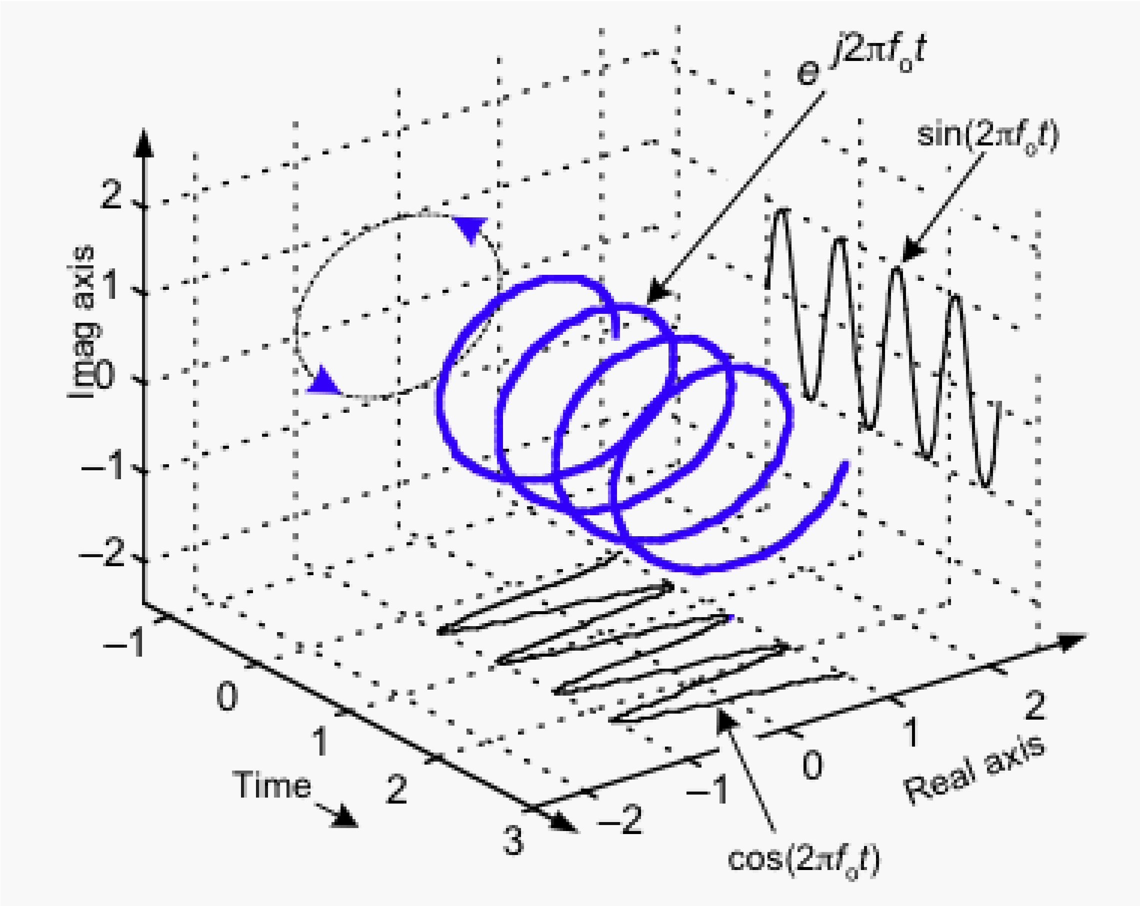
$$A\sin(\omega_0 t + \phi) = A\,\mathcal{I}m\{e^{j(\omega_0 t + \phi)}\},$$

where, if *c* is a complex number, $\mathcal{I}m\{c\}$ denotes its imaginary part.

Periodic Complex Exponential and Sinusoidal Signals



Periodic Complex Exponential and Sinusoidal Signals



Power and Energy of Periodic Complex Exponential and Sinusoidal Signals

Complex periodic exponential signals and sinusoidal signals have infinite total energy but finite average power (They are power signals).

Consider the periodic exponential signal

$$x(t) = e^{j\omega_0 t}$$

Suppose that we calculate the total energy and average power in this signal over one period:

$$E_{\text{period}} = \int_0^{T_0} \left|e^{j\omega_0 t}\right|^2 dt = \int_0^{T_0} 1 \cdot dt = T_0$$

$$P_{\text{period}} = \frac{1}{T_0} E_{\text{period}} = 1$$

Power and Energy of Periodic Complex Exponential and Sinusoidal Signals

$$x(t) = e^{j\omega_0 t}$$

The total energy and average power in this signal over one period:

$$E_{\text{period}} = T_0 \qquad\qquad P_{\text{period}} = 1$$

Since there are an infinite number of periods, the total energy integrated over all time is infinite.
However, each period of the signal looks exactly the same.
Since the average power of the signal equals 1 over each period, averaging over multiple periods always yields an average power of 1.
That is, the complex periodic exponential signal has finite average power equal to

$$P_{\infty} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \left| e^{j\omega_0 t} \right|^2 dt = 1$$

Harmonically related Complex Exponentials

Periodic complex exponentials play a central role in analysis and manipulation of signals and systems.
They are used as extremely useful building blocks for many other signals.

To do this, we will often find it useful to consider sets of harmonically related complex exponentials.

Harmonically related complex exponentials are sets of periodic exponentials, all of which are periodic with a common period T_0 .

Harmonically related Complex Exponentials

We are looking for complex exponentials which are periodic with a common period T_0

A necessary condition for a complex exponential $e^{j\omega t}$ to be periodic with period T_0 is that

$$e^{j\omega T_0} = 1$$

which implies that ωT_0 is a multiple of 2π :

$$\omega T_0 = 2\pi k, \qquad k = 0, \pm 1, \pm 2, \ldots .$$

Define: $$\omega_0 = \frac{2\pi}{T_0}$$

For the complex exponential to be periodic with T_0 ,

ω must be an integer multiple of ω_0 .

Harmonically related Complex Exponentials

Harmonically related set of complex exponentials is a set of periodic exponentials with fundamental frequencies that are all multiples of a single positive frequency ω_0 :

$$\phi_k(t) = e^{jk\omega_0 t}, \qquad k = 0, \pm 1, \pm 2, \ldots$$

For $k = 0$, $\phi_k(t)$ is a constant, while for any other value of k, $\phi_k(t)$ is periodic with fundamental frequency $|k|\omega_0$ and fundamental period $\frac{T_0}{|k|}$.

The k th harmonic $\phi_k(t)$ is still periodic with period T_0 as well, as it goes through exactly $|k|$ of its fundamental periods during any time interval of length T_0 .

General Complex Exponential Signals

The continuous-time complex exponential signal is of the form

$$x(t) = c\,e^{at}$$

where both c and a are complex numbers.

Special cases:

1) Both C and *a* are real: **Real Exponential Signals**

2) a is purely imaginary: **Periodic Complex Exponential Signals**

Genaral case:

The most general case of a complex exponential can be expressed and interpreted in terms of the two cases we have examined so far: the real exponential and the periodic complex exponential.

General Complex Exponential Signals

The most general case of a complex exponential can be expressed and interpreted in terms of the two cases we have examined so far: the real exponential and the periodic complex exponential.

$$x(t) = Ce^{at}$$

Both *C* and *a* are complex numbers.
We will express *C* in polar form and *a* in rectangular form:

$$C = |C|e^{j\theta} \qquad a = r + j\omega_0$$

$$Ce^{at} = |C|e^{j\theta}e^{(r+j\omega_0)t} = |C|e^{rt}e^{j(\omega_0 t+\theta)}$$

General Complex Exponential Signals

$$C = |C|e^{j\theta} \qquad a = r + j\omega_0$$

$$Ce^{at} = |C|e^{j\theta}e^{(r+j\omega_0)t} = |C|e^{rt}e^{j(\omega_0 t+\theta)}$$

Using Euler's relation:

$$x(t) = Ce^{at} = |C|e^{rt}\cos(\omega_0 t + \theta) + j|C|e^{rt}\sin(\omega_0 t + \theta)$$

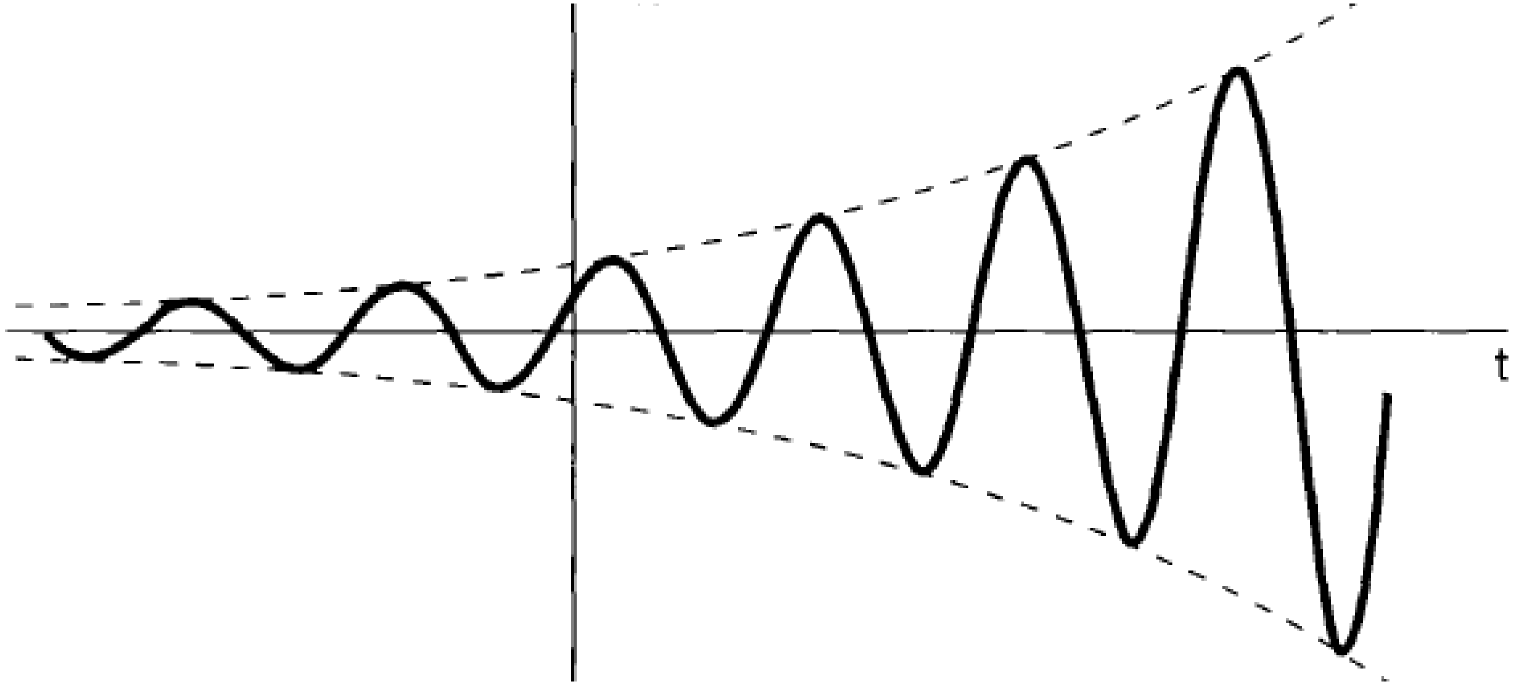
- For r = 0, the real and imaginary parts of a complex exponential are sinusoidal.
- For *r* > 0 they correspond to sinusoidal signals multiplied by a growing exponential.
- For r < 0 they correspond to sinusoidal signals multiplied by a decaying exponential.

General Complex Exponential Signals

$$x(t) = Ce^{at} = |C|e^{rt}\cos(\omega_0 t + \theta) + j|C|e^{rt}\sin(\omega_0 t + \theta)$$

- For r > 0 we have sinusoidal signals multiplied by a growing exponential.

Real part of x(t) = $|C|e^{rt}\cos(\omega_0 t + \theta)$



General Complex Exponential Signals

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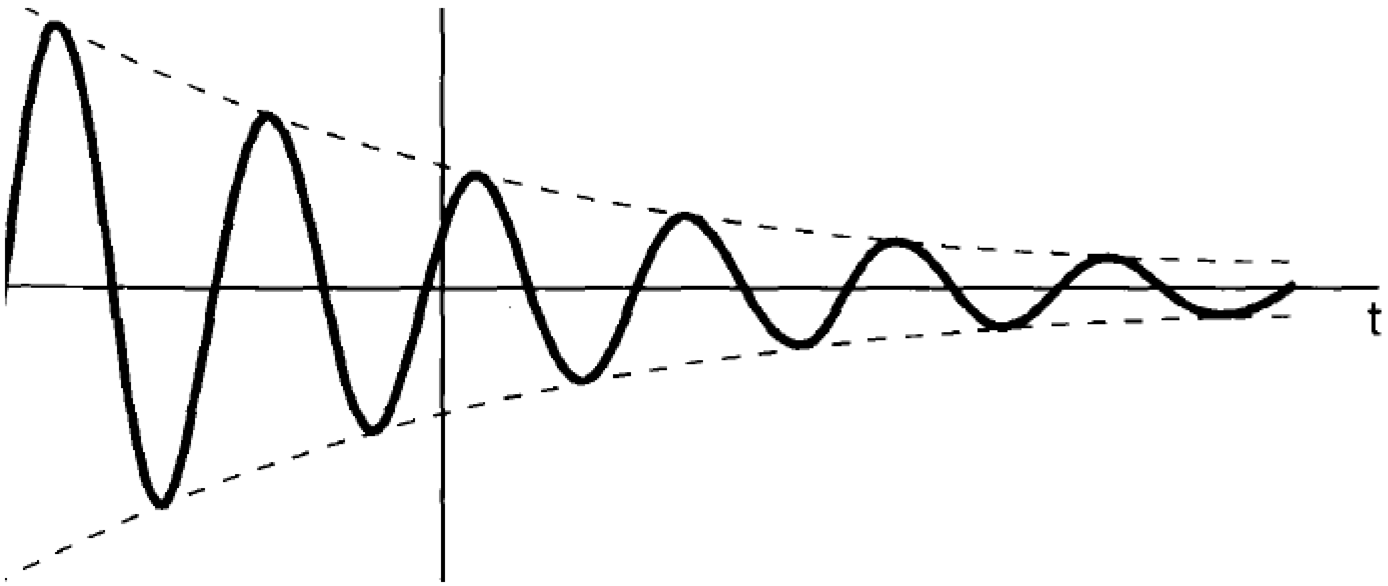
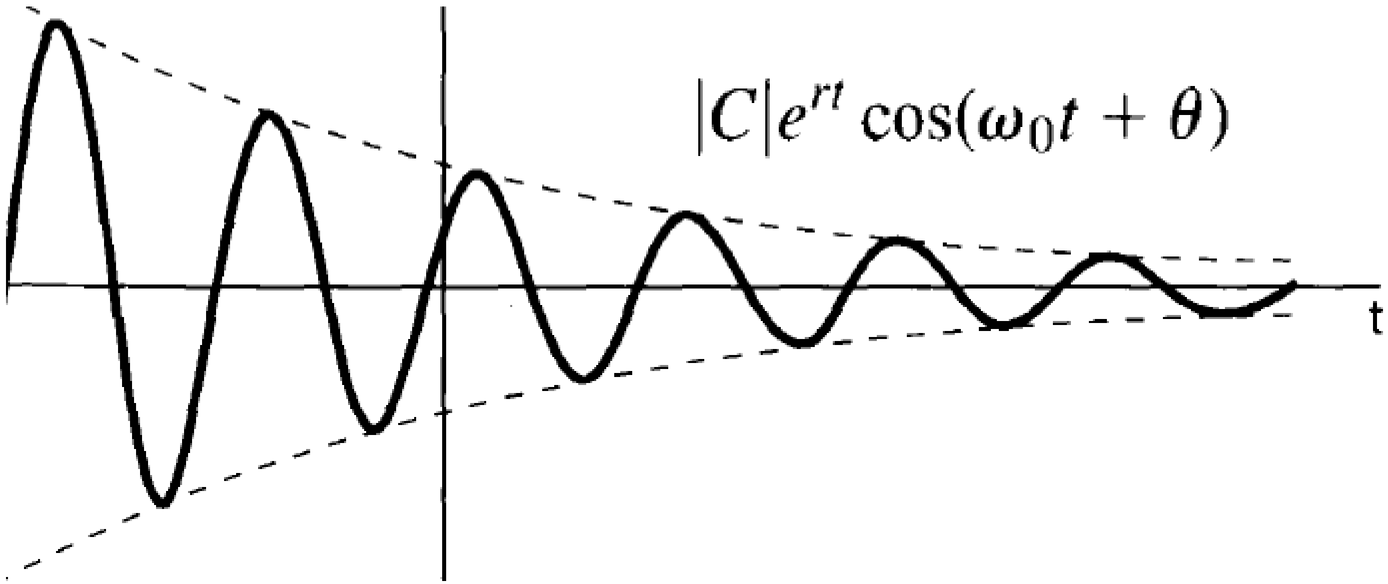
- For $r < 0$ we have sinusoidal signals multiplied by a decaying exponential.

Real part of x(t) = $|C|e^{rt}\cos(\omega_0 t + \theta)$





Sinusoidal signals multiplied by decaying exponentials are commonly referred to as *damped sinusoids.*

Examples of damped sinusoids arise in the response of *RLC* circuits
and in mechanical systems containing both damping and restoring forces, such as automotive suspension systems.

These kinds of systems have mechanisms that dissipate energy (resistors, damping forces such as friction) with oscillations that decay in time.

DISCRETE-TIME

COMPLEX EXPONENTIAL SIGNALS

SINUSOIDAL SIGNALS

Discrete-time Complex Exponential Signals

A discrete-time complex exponential signal or sequence is defined by

$$x[n] = C\alpha^n$$

C and α are, in general, complex numbers.

This could alternatively be expressed in the form

$$x[n] = Ce^{\beta n}$$

$$\alpha = e^{\beta}$$

Real Exponential Signals

$$x[n] = C\alpha^n$$

Special case: *C* and α are real numbers.

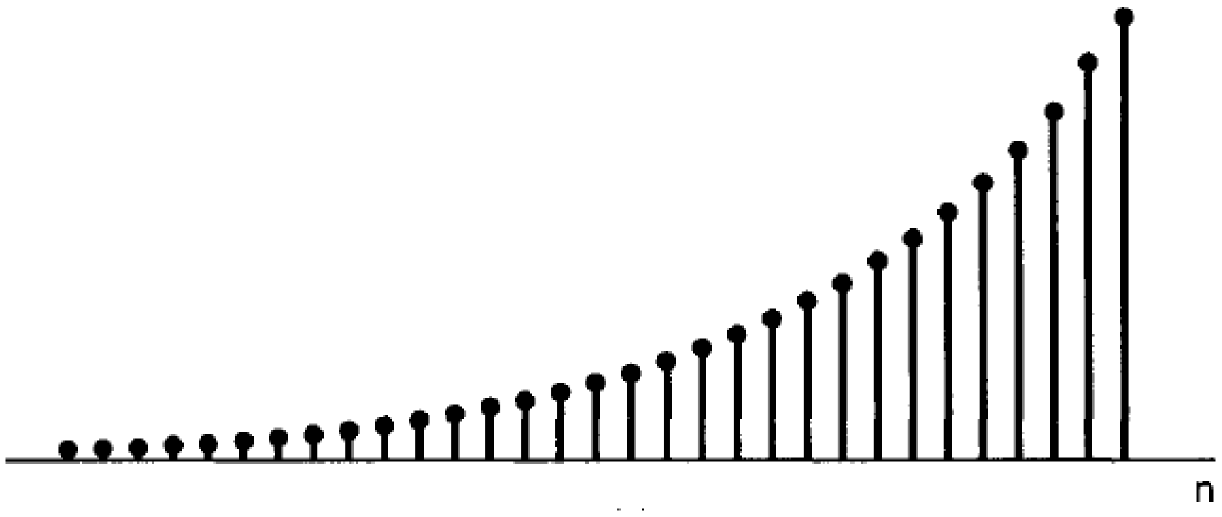
- If $|\alpha| > 1$ the magnitude of the signal grows exponentially with *n.*
- If $|\alpha| < 1$, we have a decaying exponential.
- If α is positive, all the values of $x[n]$ are of the same sign.
- If α is negative, then the sign of $x[n]$ alternates.
- If $\alpha = 1$ then $x[n]$ is a constant.
- If $\alpha = -1$, $x[n]$ alternates in value between $+C$ and $-C$.

Real Exponential Signals

$$x[n] = C\alpha^n$$

C and α are real numbers.

$\alpha > 1$

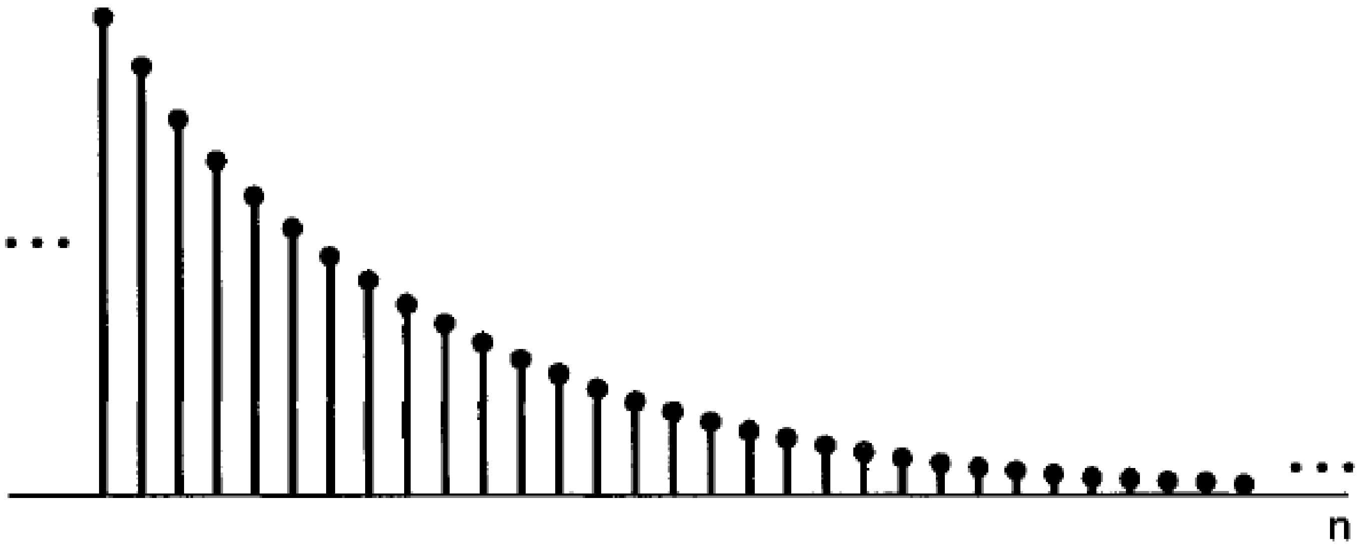


Real Exponential Signals

$$x[n] = C\alpha^n$$

C and α are real numbers.

$0 < \alpha < 1$

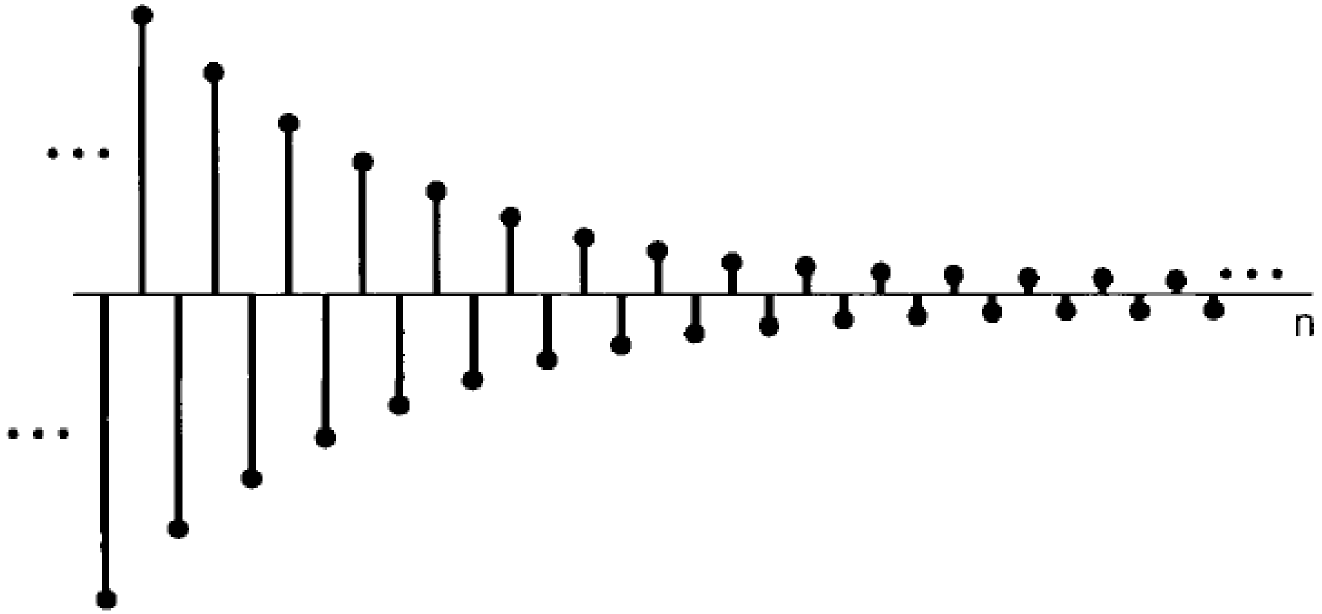


Real Exponential Signals

$$x[n] = C\alpha^n$$

C and α are real numbers.

$-1 < \alpha < 0$

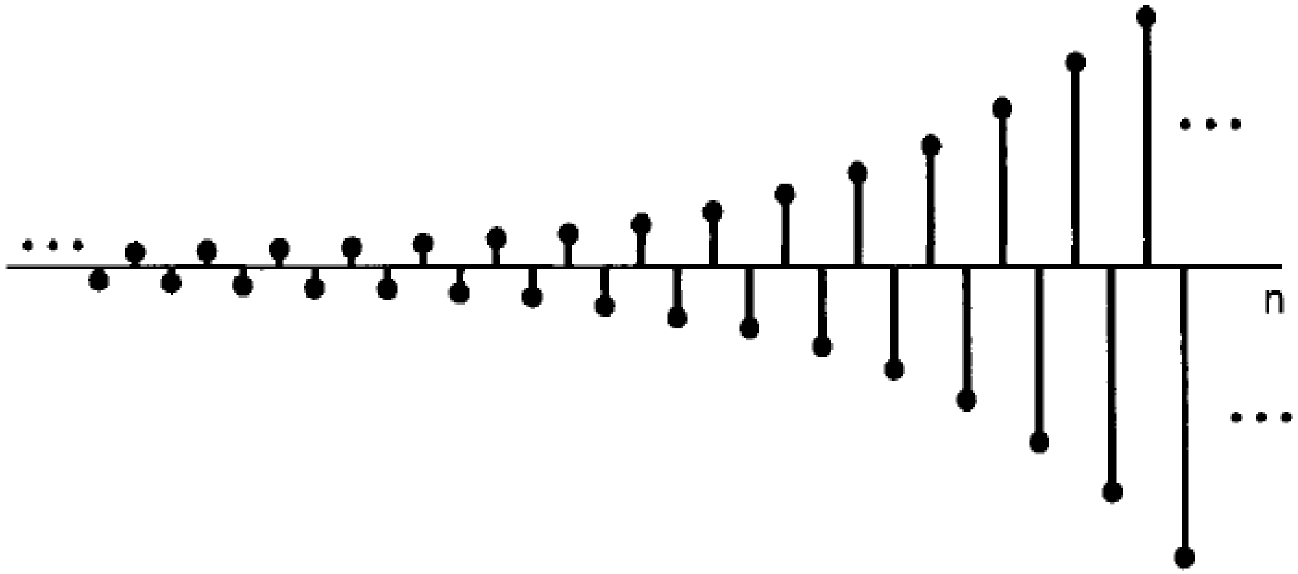


Real Exponential Signals

$$x[n] = C\alpha^n$$

C and α are real numbers.

$\alpha < -1$



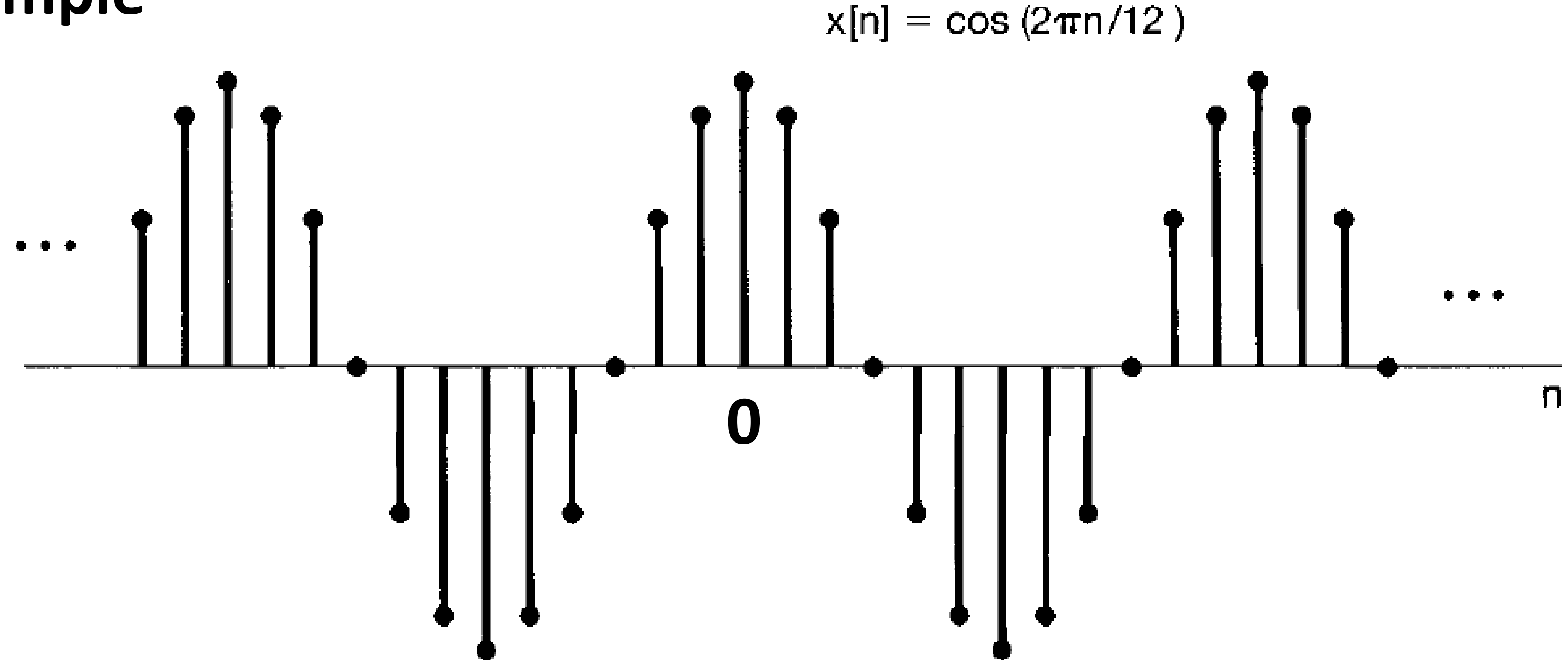
Discrete-time sinusoidal signals

A discrete-time sinusoidal signal is of the form

$$x[n] = A\cos(\omega_0 n + \phi)$$

If we take *n* to be dimensionless, then both ω_0 and ϕ have units of radians.

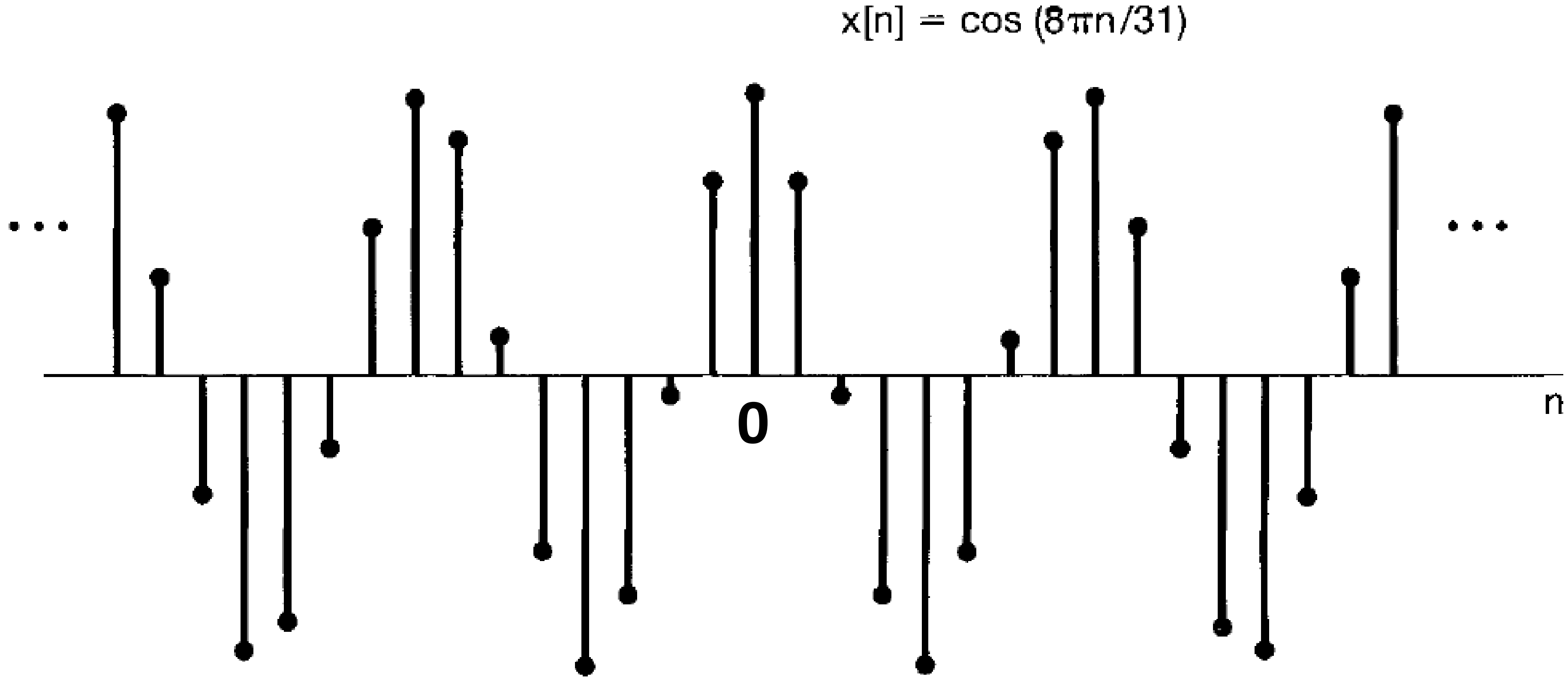
Example



Discrete-time sinusoidal signals

$$x[n] = A\cos(\omega_0 n + \phi)$$

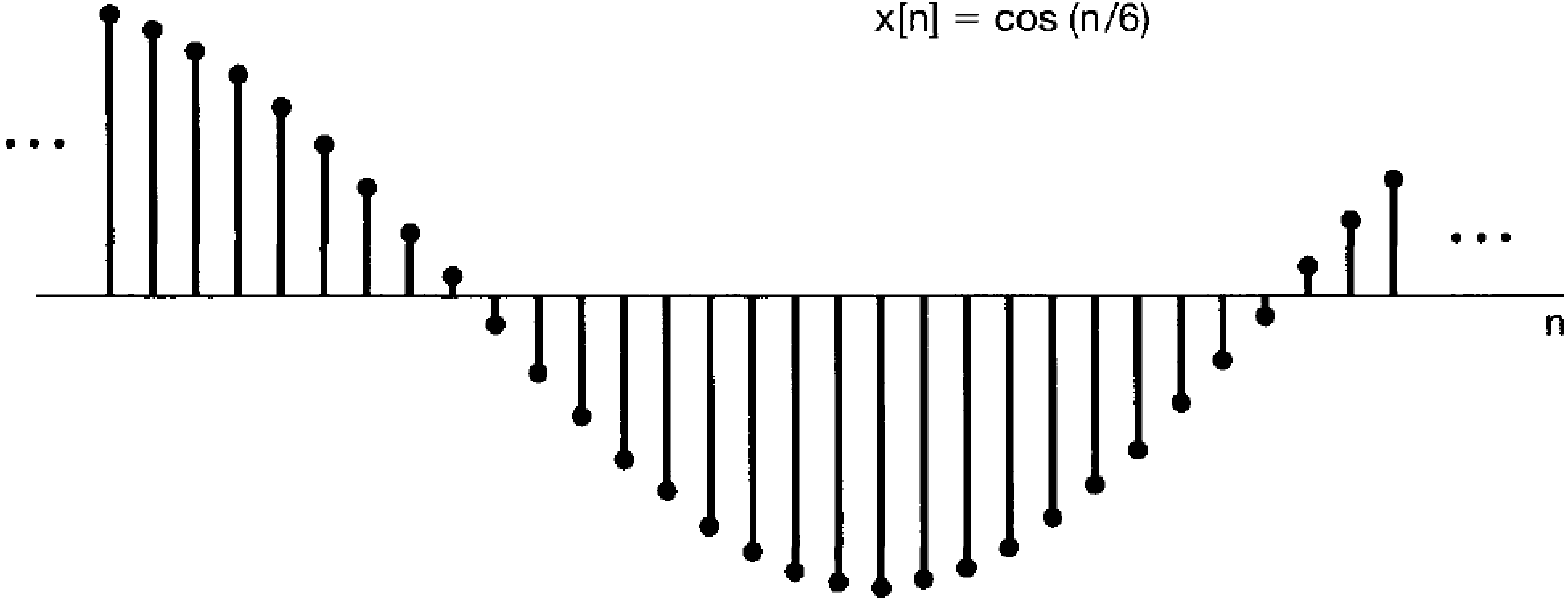
Example



Discrete-time sinusoidal signals

$$x[n] = A\cos(\omega_0 n + \phi)$$

Example



Discrete-time Complex Exponential Signals

A discrete-time complex exponential signal or sequence is defined by

$$x[n] = C\alpha^n$$

C and α are, in general, complex numbers.

This could alternatively be expressed in the form

$$x[n] = Ce^{\beta n} \qquad \alpha = e^{\beta}$$

Special case: β is purely imaginary.

$$x[n] = e^{j\omega_0 n}$$

Complex Exponential Signals

Special case: β is purely imaginary.

$$x[n] = e^{j\omega_0 n}$$

Using Euler's identity we can relate complex exponentials and sinusoids:

$$e^{j\omega_0 n} = \cos \omega_0 n + j \sin \omega_0 n$$

and

$$A\cos(\omega_0 n + \phi) = \frac{A}{2} e^{j\phi} e^{j\omega_0 n} + \frac{A}{2} e^{-j\phi} e^{-j\omega_0 n}$$

General Complex Exponential Signals

$$x[n] = C\alpha^n$$

Writing *C* and *α* in polar form:

$$C = |C|e^{j\theta} \qquad \alpha = |\alpha|e^{j\omega_0}$$

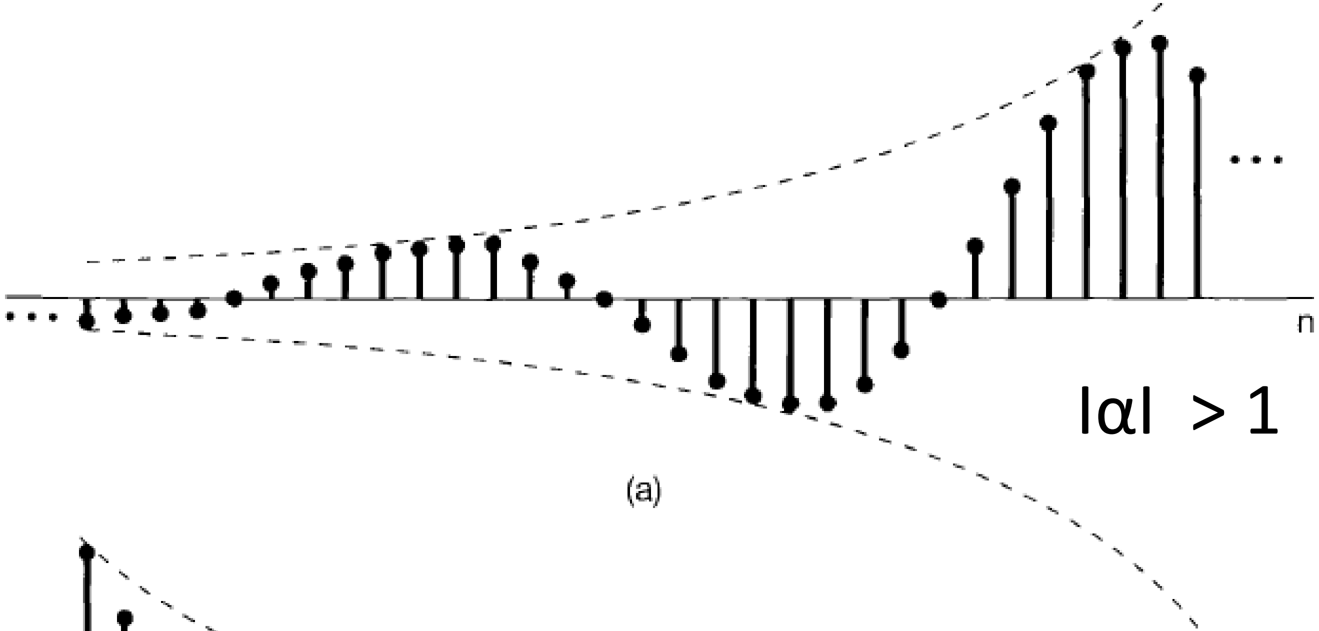
$$C\alpha^n = |C||\alpha|^n \cos(\omega_0 n + \theta) + j|C||\alpha|^n \sin(\omega_0 n + \theta)$$

For IαI = 1 , the real and imaginary parts of a complex exponential sequence are sinusoidal.

For IαI < 1, they correspond to sinusoidal sequences multiplied by a decaying exponential.

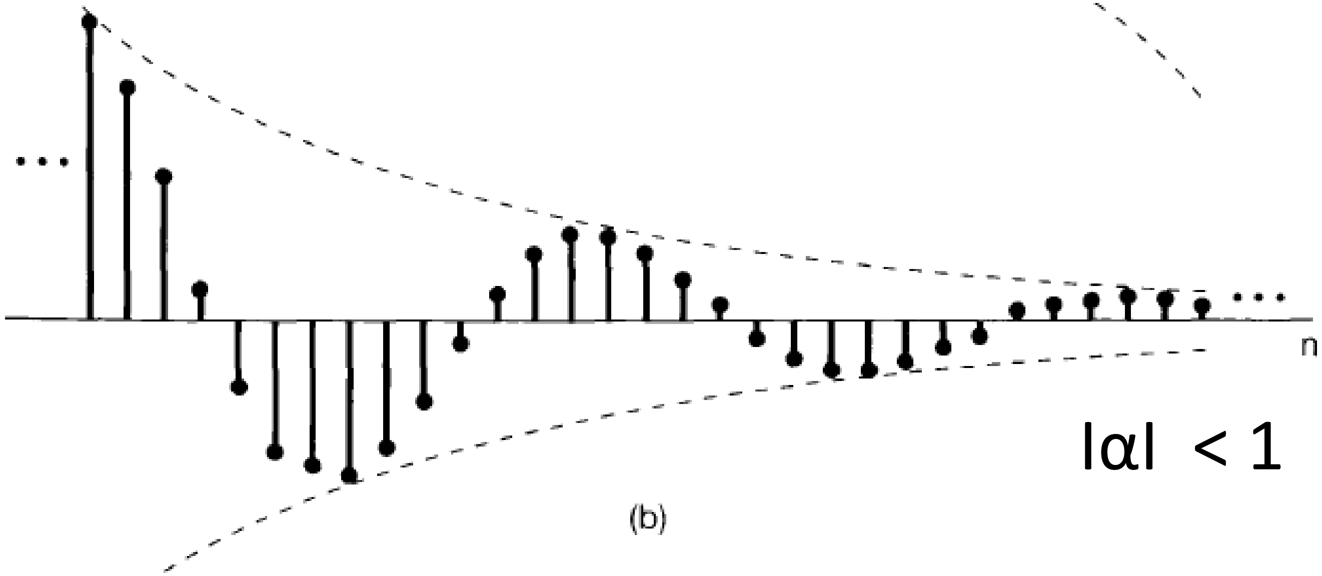
For IαI > 1, they correspond to sinusoidal sequences multiplied by a growing exponential.

$$|C||\alpha|^n \cos(\omega_0 n + \theta)$$



(a)

$|\alpha| > 1$: Growing discrete-time sinusoid



(b)

$|\alpha| < 1$: Decaying discrete-time sinusoid

Properties of Discrete-Time Complex Exponentials and Sinusoids

We get identical signals $e^{j\omega_0 n}$ for values of ω_0 separated by multiples of 2π

Specifically, consider the discrete-time complex exponential with frequency $\omega_0 + 2\pi$:

$$e^{j(\omega_0+2\pi)n} = e^{j2\pi n}e^{j\omega_0 n} = e^{j\omega_0 n}$$

We see that the exponential at frequency $\omega_0 + 2\pi$ is the same as the exponential at frequency ω_0 .

Properties of Discrete-Time Complex Exponentials and Sinusoids

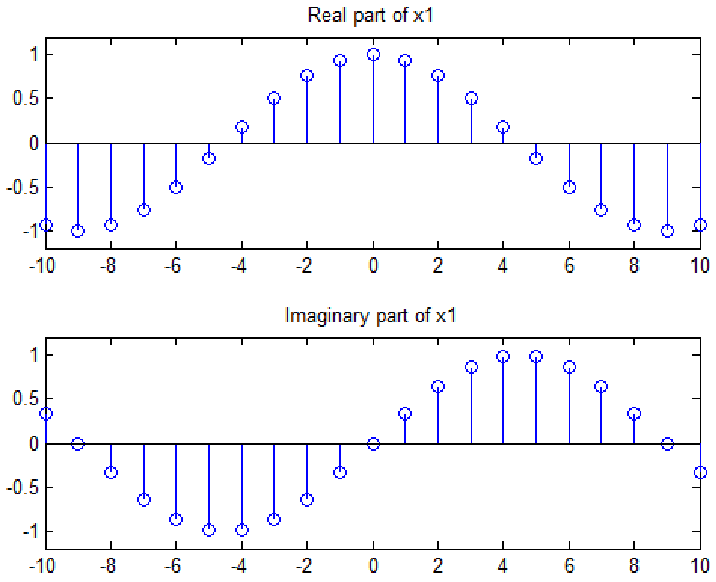
We get identical signals $e^{j\omega_0 n}$ for values of ω_0 separated by multiples of 2π

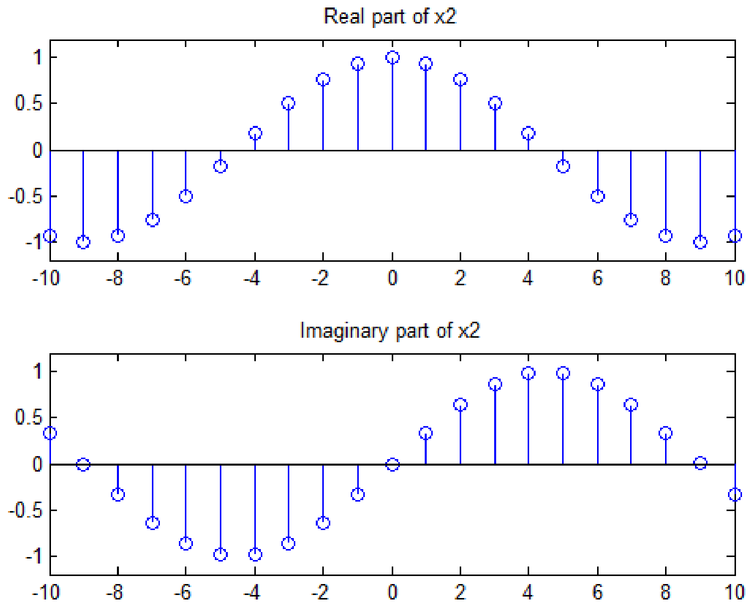
Same situation with discrete-time sinusoids

Example

$$x_1 = e^{j\frac{3}{27}\pi n}$$

$$x_2 = e^{j(\frac{3}{27}\pi + k2\pi)n}$$





Properties of Discrete-Time Complex Exponentials and Sinusoids

We get identical signals $e^{j\omega_0 n}$ for values of ω_0 separated by multiples of 2π
Same situation with discrete-time sinusoids

We have a very different situation from the continuous-time case, in which the signals $e^{j\omega_0 t}$ are all distinct for distinct values of ω_0.

In discrete time, these signals are not distinct, as the signal with frequency ω_0 is identical to the signals with frequencies $\omega_0 \pm 2\pi, \omega_0 \pm 4\pi$ and so on.

Therefore, in considering discrete-time complex exponentials, we need only consider a frequency interval of length 2π in which to choose ω_0.

We can map all ω_0 to any interval of length 2π.

For convenience, will use the interval $0 \leq \omega_0 < 2\pi$ or the interval $-\pi \leq \omega_0 < \pi$

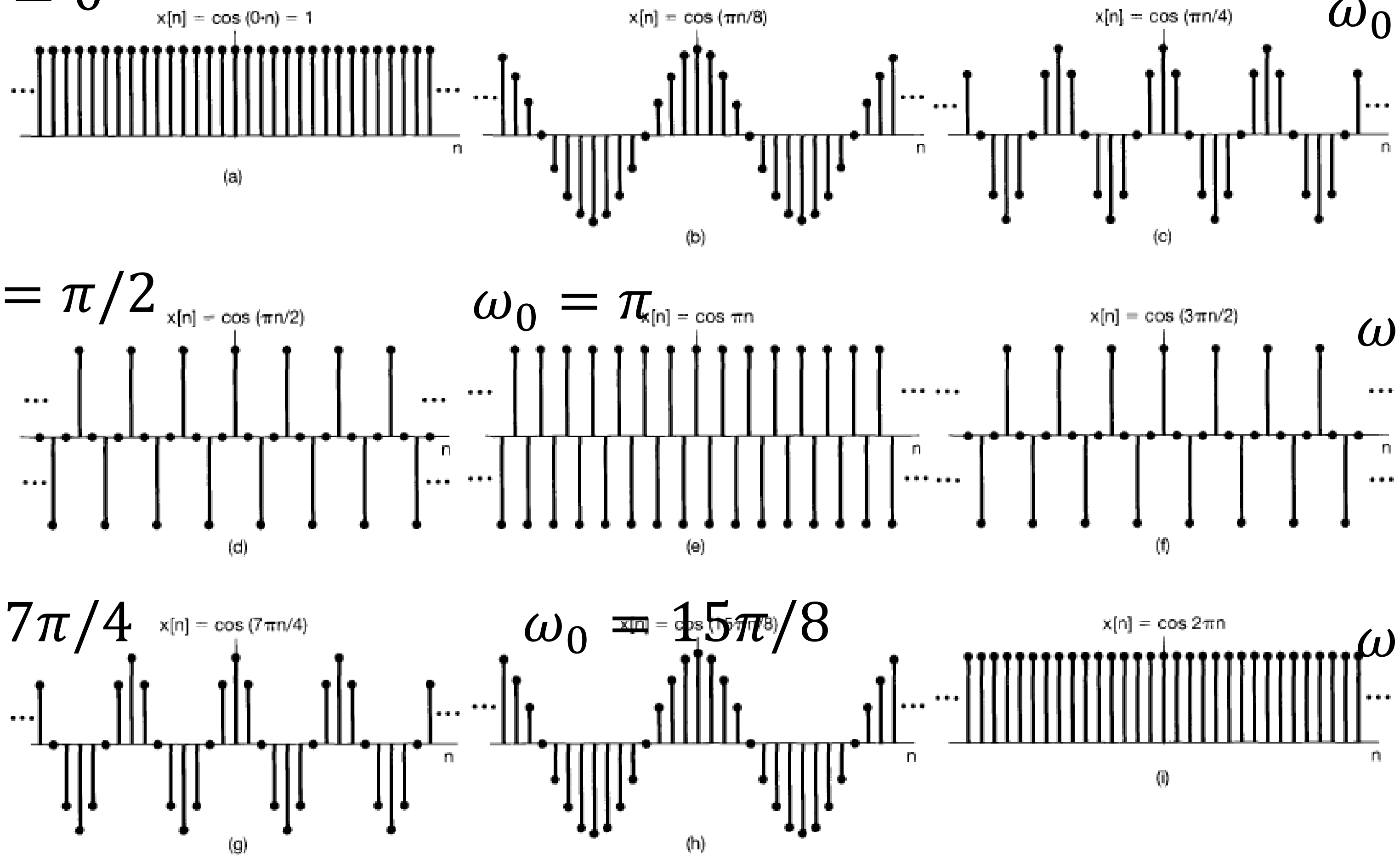
Properties of Discrete-Time Complex Exponentials and Sinusoids

$$x[n] = \cos(\omega_0 n)$$

$\omega_0 = 0$ $\omega_0 = \pi/8$ $\omega_0 = \pi/4$

$\omega_0 = \pi/2$ $\omega_0 = \pi$ $\omega_0 = 3\pi/2$

$\omega_0 = 7\pi/4$ $\omega_0 = 15\pi/8$ $\omega_0 = 2\pi$



The signal $e^{j\omega_0 n} = \cos\omega_0 n + j\sin\omega_0 n$ does not have a continually increasing rate of oscillation as ω_0 is increased in magnitude.

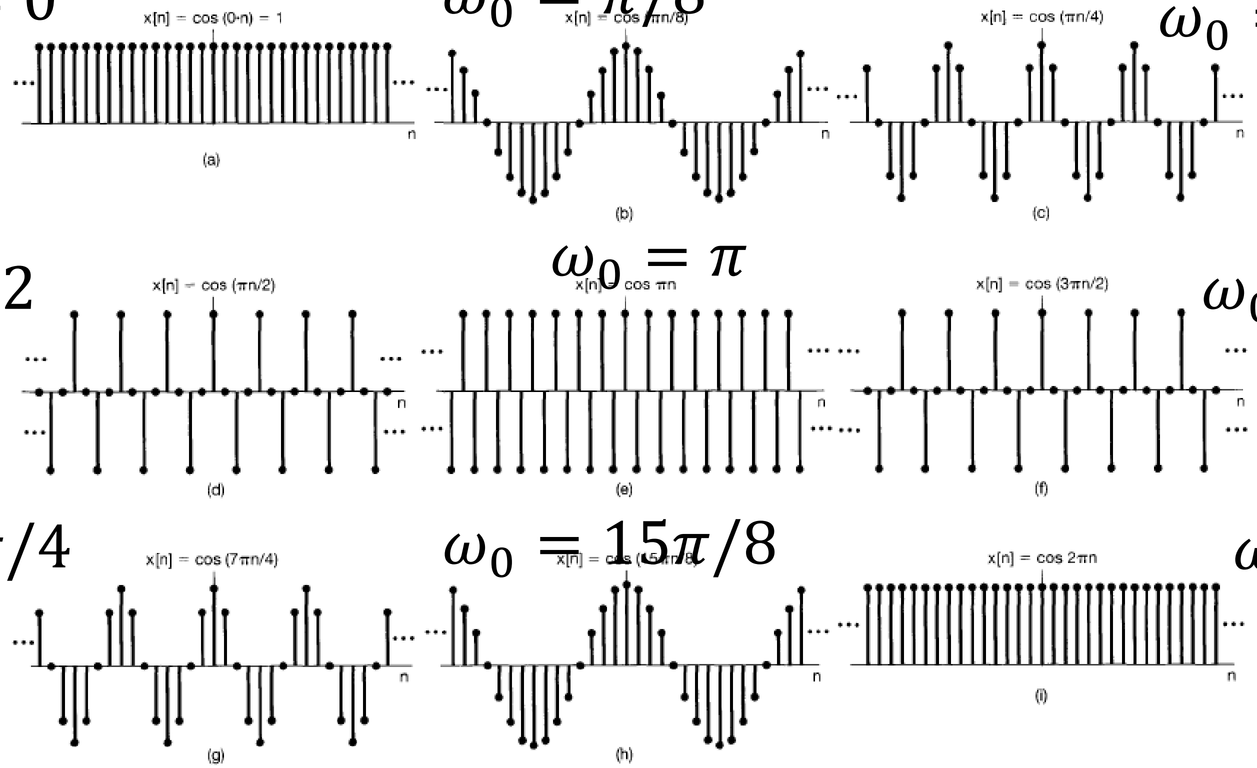
Properties of Discrete-Time Complex Exponentials and Sinusoids

$$x[n] = \cos(\omega_0 n)$$

$\omega_0 = 0$ $\omega_0 = \pi/8$ $\omega_0 = \pi/4$

$\omega_0 = \pi/2$ $\omega_0 = \pi$ $\omega_0 = 3\pi/2$

$\omega_0 = 7\pi/4$ $\omega_0 = 15\pi/8$ $\omega_0 = 2\pi$



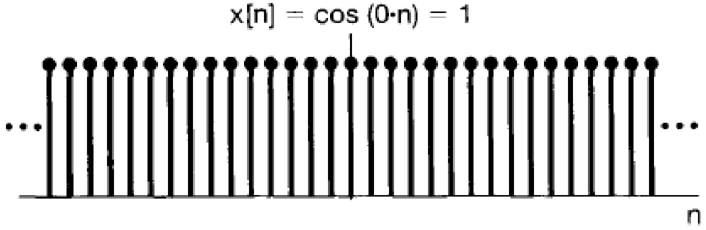
As we increase ω_0 from 0, we obtain signals that oscillate more and more rapidly until we reach $\omega_0 = \pi$.

As we continue to increase ω_0, we decrease the rate of oscillation until we reach $\omega_0 = 2\pi$, which produces the same constant sequence as $\omega_0 = 0$.

Properties of Discrete-Time Complex Exponentials and Sinusoids

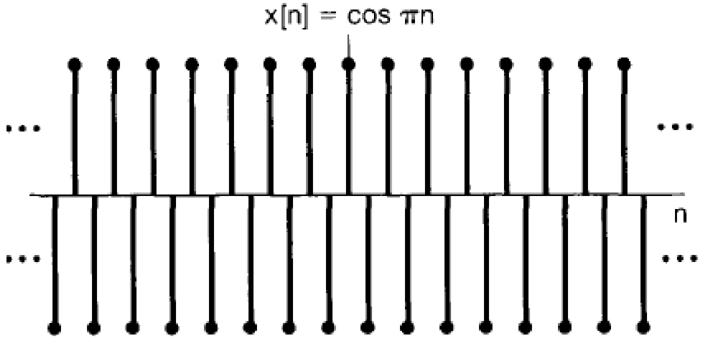
$$x[n] = \cos(\omega_0 n)$$

$\omega_0 = 0$



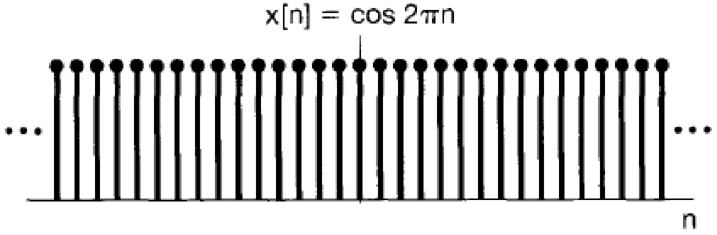
Low frequencies around
$\omega_0 = 2k\pi$

$\omega_0 = \pi$



High frequencies around
$\omega_0 = (2k+1)\pi$

$\omega_0 = 2\pi$



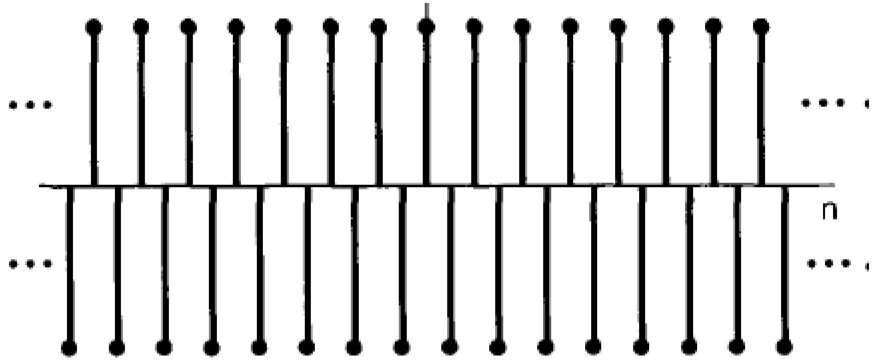
Low frequencies around
$\omega_0 = 2k\pi$

The low-frequency (that is, slowly varying) discrete-time exponentials have values of ω_0 near 0, 2π, and any other even multiple of π.

The high frequencies (corresponding to rapid variations) are located near $\omega_0 = \pm\pi$ and other odd multiples of π.

Properties of Discrete-Time Complex Exponentials and Sinusoids

$e^{j\pi n}$ $\omega_0 = \pi$



Note in particular that for $\omega_0 = \pi$ or any other odd multiple of π

$$e^{j\pi n} = (e^{j\pi})^n = (-1)^n$$

so that this signal oscillates rapidly, changing sign at each point in time.

Periodicity of Discrete-Time Complex Exponentials and Sinusoids

In order for the signal $e^{j\omega_0 n}$ to be periodic with period $N > 0$, we must have

$$e^{j\omega_0(n+N)} = e^{j\omega_0 n}$$

or equivalently

$$e^{j\omega_0 N} = 1$$

For the above equation to hold, $\omega_0 N$ must be a multiple of 2π. That is, there must be an integer *m* such that

$$\omega_0 N = 2\pi m \qquad \text{or equivalently} \qquad \frac{\omega_0}{2\pi} = \frac{m}{N}$$

Periodicity of Discrete-Time Complex Exponentials and Sinusoids

The signal $e^{j\omega_0 n}$ is periodic if $\omega_0/2\pi$ is a rational number and is not periodic otherwise.

These same observations also hold for discrete-time sinusoids.

If $e^{j\omega_0 n}$ or $\cos(\omega_0 n)$ is periodic, then their fundamental period is the smallest positive integer satisfying:

$$N = m\left(\frac{2\pi}{\omega_0}\right).$$

Periodicity of Discrete-Time Complex Exponentials and Sinusoids

Example:

For $e^{jn2\pi/3}$ $\quad w_0/(2\pi) = 1/3$ $\quad N = m\left(\frac{2\pi}{\omega_0}\right) \longrightarrow$ fundamental period 3

Rational;

The signal is periodic

Example:

$x[n] = \sin(3n/4)$ $\quad \frac{w_0}{2\pi} = \frac{3}{8\pi}$

Irrational;

The signal is not periodic.

Periodicity of Discrete-Time Complex Exponentials and Sinusoids

Example:

Is $x[n] = e^{jn2\pi/3} + e^{jn3\pi/4}$ periodic? If it is periodic, what's its fundamental period?

For $e^{jn2\pi/3}$ $\quad w_0/(2\pi) = 1/3 \quad N = m\left(\frac{2\pi}{\omega_0}\right) \longrightarrow$ fundamental period 3

Rational

For $e^{jn3\pi/4}$ $\quad w_0/(2\pi) = 3/8 \quad N = m\left(\frac{2\pi}{\omega_0}\right) \longrightarrow$ fundamental period 8.

Rational

$x[n] = e^{jn2\pi/3} + e^{jn3\pi/4}$ is periodic with fundamental period $24 = lcm(3, 8)$.

Least common multiple

Harmonically related periodic exponentials in discrete-time

Set of periodic exponentials with a common period *N*.

The periodic exponential signals which are at frequencies which are multiples of $2\pi/N$.

$$\phi_k[n] = e^{jk(2\pi/N)n}, \qquad k = 0, \pm 1, \ldots.$$

In discrete-time, we have only *N* distinct signals in the set of periodic exponentials with the common period *N*:

$$\begin{aligned}\phi_{k+N}[n] &= e^{j(k+N)(2\pi/N)n} \\ &= e^{jk(2\pi/N)n} e^{j2\pi n} = \phi_k[n].\end{aligned}$$

Harmonically related periodic exponentials in discrete-time

$$\phi_k[n] = e^{jk(2\pi/N)n}, \qquad k = 0, \pm 1, \ldots.$$

There are only *N* distinct periodic exponentials in the set.

$$\phi_0[n] = 1, \phi_1[n] = e^{j2\pi n/N}, \phi_2[n] = e^{j4\pi n/N}, \ldots, \phi_{N-1}[n] = e^{j2\pi(N-1)n/N}$$

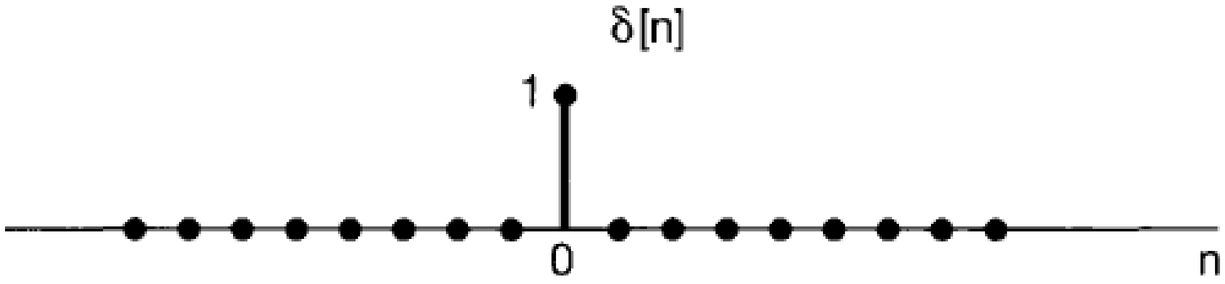
For example:

$$\phi_N[n] = \phi_0[n] \qquad\qquad \phi_{-1}[n] = \phi_{N-1}[n]$$

The Discrete-Time Unit Impulse

The discrete-time unit impulse (or unit sample) is defined as:

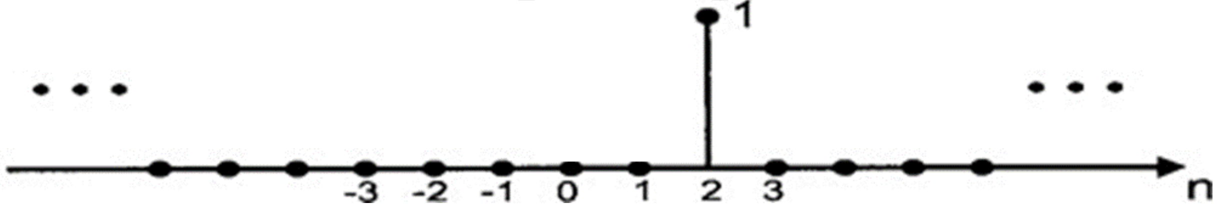
$$\delta[n] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}$$



The Discrete-Time Unit Impulse

$$\delta[n-1]$$

$$\delta[n-2]$$

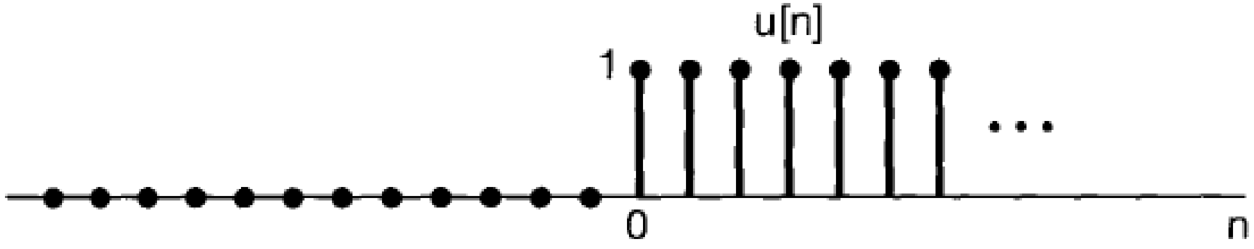


$$\delta[n+3]$$

The Discrete-Time Unit Step

The discrete-time unit step is defined as:

$$u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$



The Discrete-Time Unit Impulse and Unit Step

There is a close relationship between the discrete-time unit impulse and unit step.

The discrete-time unit impulse is the first difference of the discrete-time step:

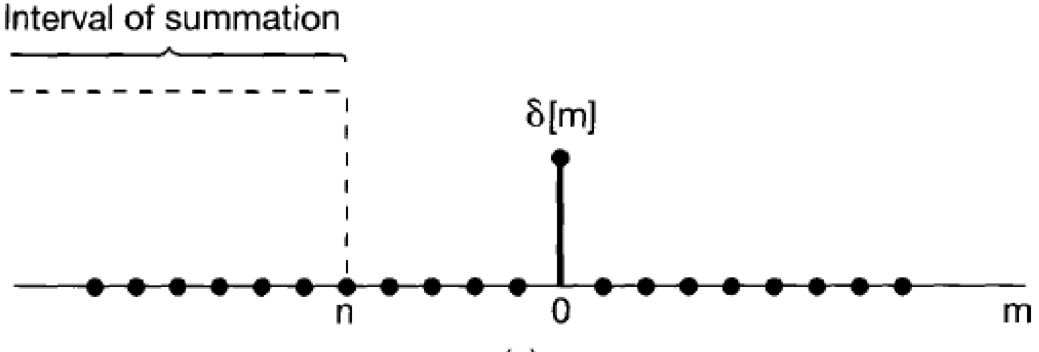
$$\delta[n] = u[n] - u[n-1].$$

Conversely, the discrete-time unit step is the running sum of the unit sample:

$$u[n] = \sum_{m=-\infty}^{n} \delta[m]$$

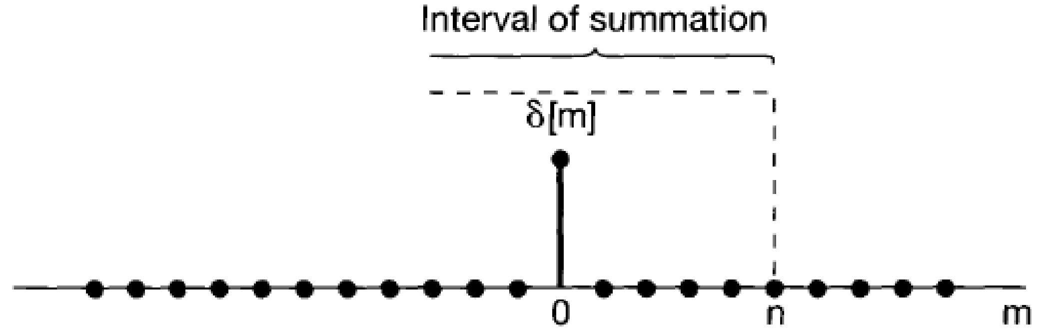
The Discrete-Time Unit Impulse and Unit Step

$$u[n] = \sum_{m=-\infty}^{n} \delta[m] = \quad u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$



(a)

The running sum is zero for $n < 0$



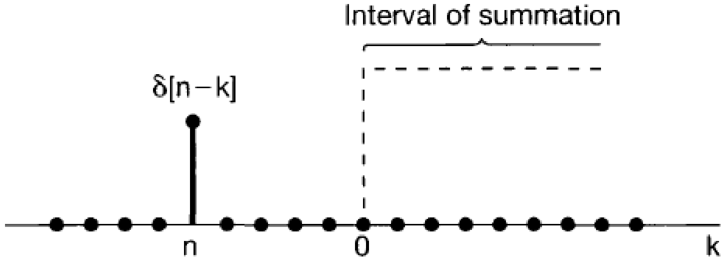
(b)

The running sum is 1 for $n \geq 0$

The Discrete-Time Unit Impulse and Unit Step

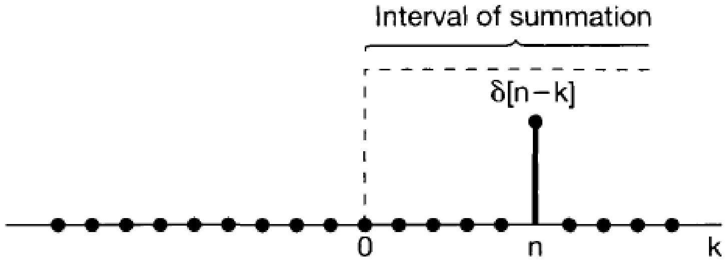
The discrete-time unit step can also be written in terms of the unit sample as

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k]$$



(a)

The sum is 0 for $n < 0$



(b)

The sum is 1 for $n \geq 0$

The Discrete-Time Unit Impulse – Sampling Property

The unit impulse sequence can be used to sample the value of a signal at $n = 0$.

In particular, since $\delta[n]$ is nonzero (and equal to 1) only for $n = 0$, it follows that

$$x[n]\delta[n] = x[0]\delta[n]$$

More generally, if we consider a unit impulse $\delta[n - n_0]$ at $n = n_0$, then

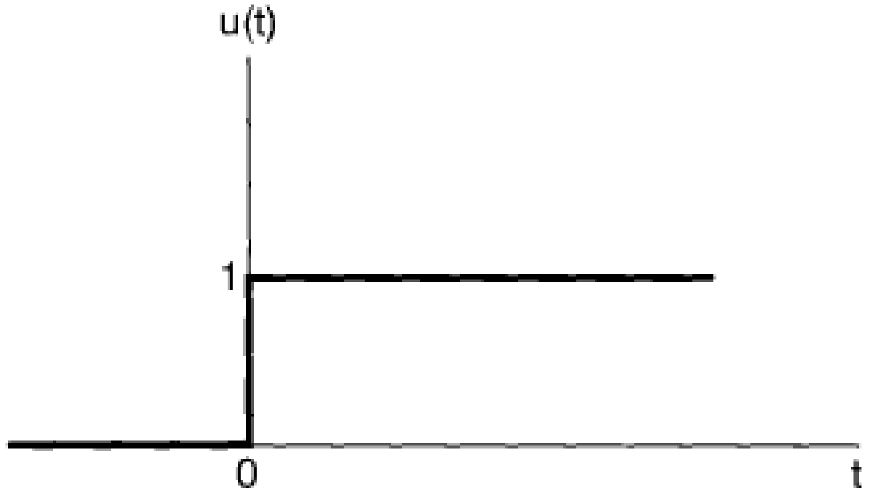
$$x[n]\delta[n - n_0] = x[n_0]\delta[n - n_0]$$

This is called the sampling property of the unit impulse

The Continuous-Time Unit Step Function

The continuous-time *unit step function* $u(t)$ is defined as:

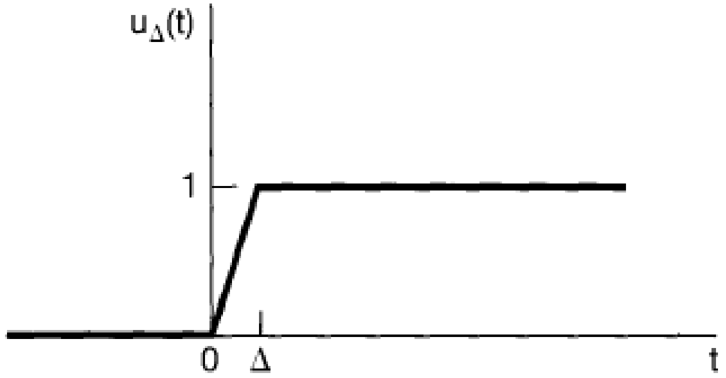
$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$



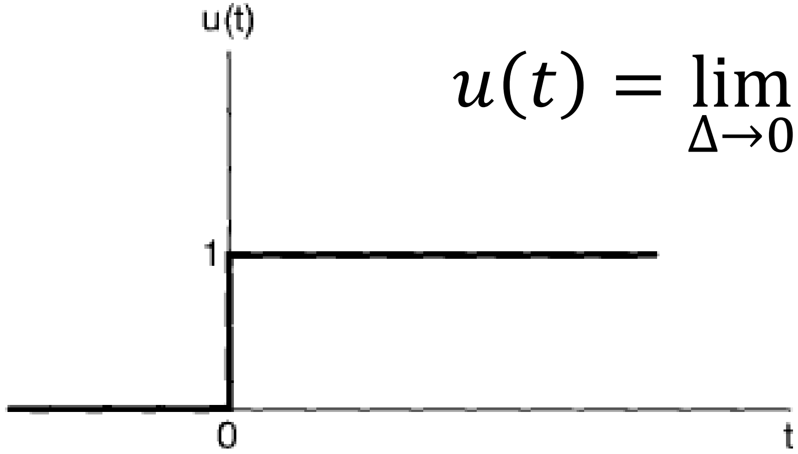
Note that the *unit step function* $u(t)$ is discontinuous at $t = 0$

The Continuous-Time Unit Step Function

We can interpret the continuous-time *unit step function* $u(t)$ as:

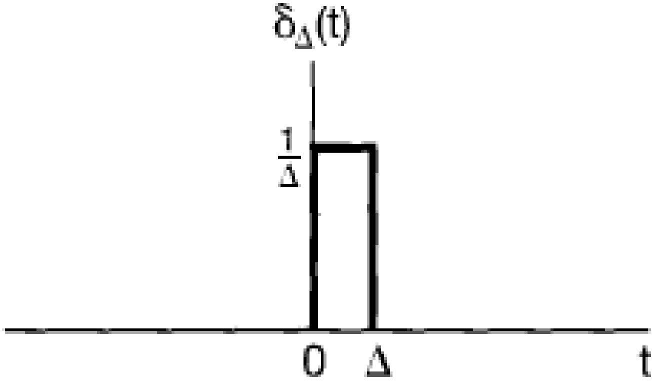


$\Delta \rightarrow 0$

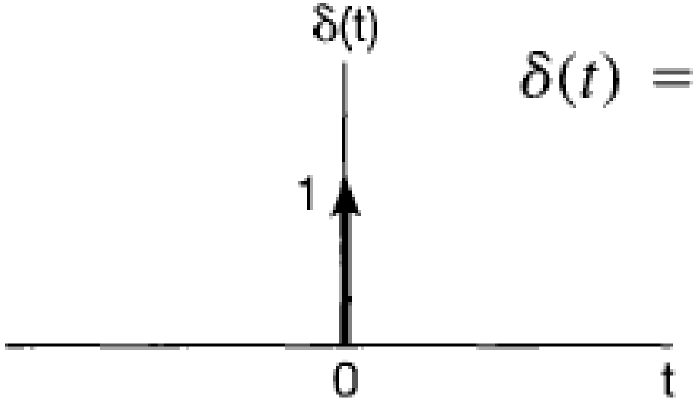


$$u(t) = \lim_{\Delta \to 0} u_\Delta(t)$$

Consider $\delta_\Delta(t) = \dfrac{du_\Delta(t)}{dt}$



$\Delta \rightarrow 0$

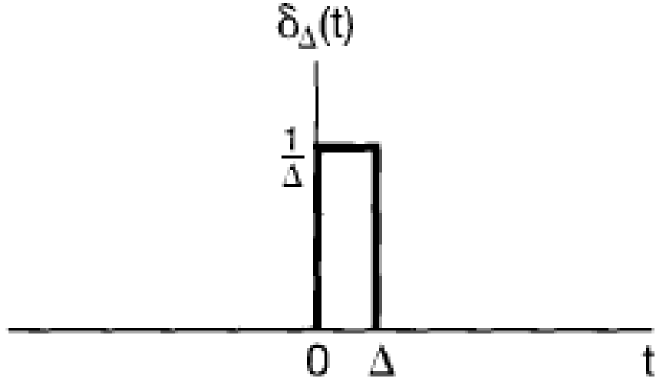


$$\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)$$

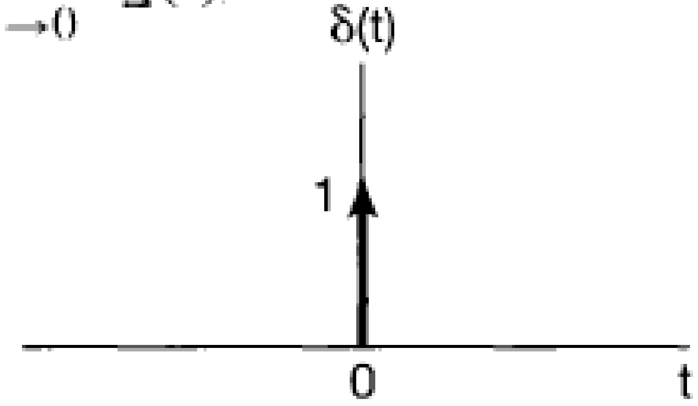
The Continuous-Time Unit Impulse Function

We can interpret the continuous-time *unit impulse function* $\delta(t)$ as:

$$\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)$$



$\Delta \rightarrow 0$



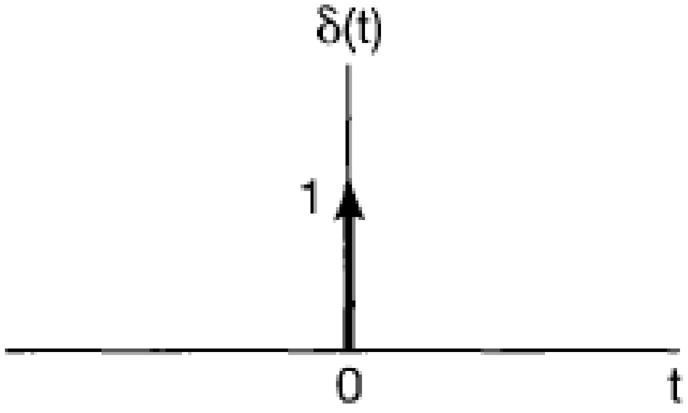
Note that $\delta_\Delta(t)$ is a short pulse, of duration Δ and with unit area for any value of Δ.

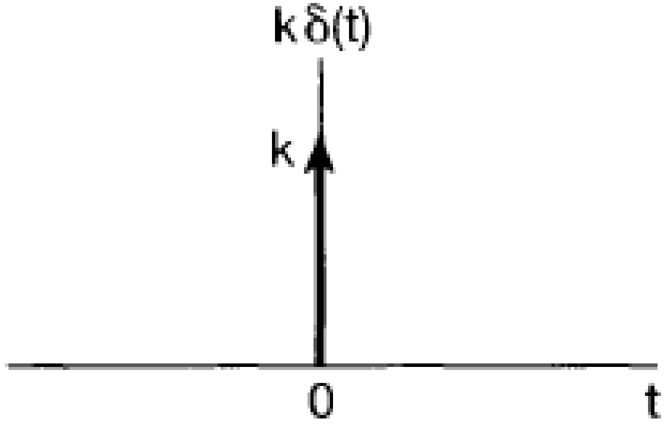
As $\Delta \rightarrow 0$, $\delta_\Delta(t)$ becomes narrower and higher, maintaining its unit area.

$\delta(t)$ can then be thought of as an idealization of the short pulse $\delta_\Delta(t)$ as the duration Δ becomes insignificant.

The Continuous-Time Unit Impulse Function

We adopt the graphical notation:





The arrow at t = 0 indicates that the area of the pulse is concentrated at t = 0 and the height of the arrow and the "1" next to the arrow are used to represent the area of the impulse.

More generally, a scaled impulse $\mathrm{k}\delta(t)$ will have an area k.

The Continuous-Time Unit Step and Unit Impulse Functions

The continuous-time unit impulse function $\delta(t)$ is related to the unit step $u(t)$.

The continuous-time unit step is the running integral of the unit impulse:

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau$$

More genarally:

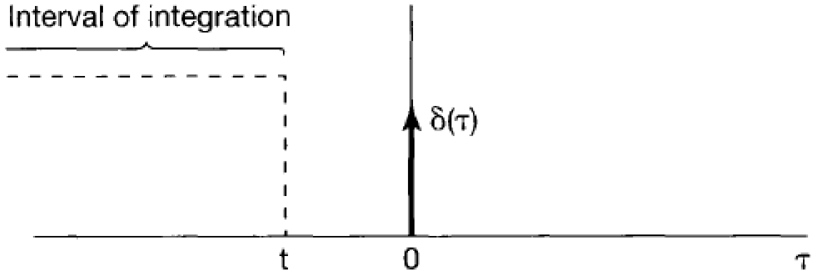
$$\int_{-\infty}^{t} k\delta(\tau)\, d\tau = ku(t)$$

Also, the continuous-time unit impulse can be thought of as the first derivative of the continuous-time unit step:

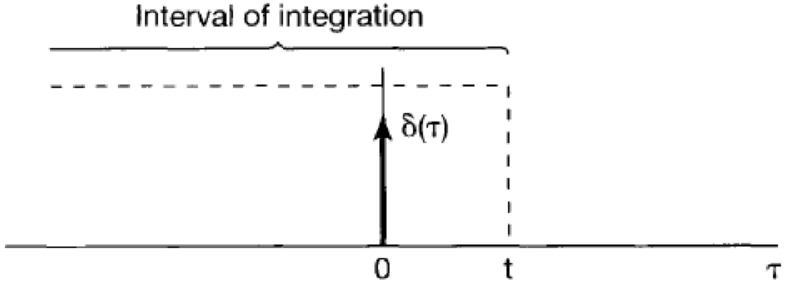
$$\delta(t) = \frac{du(t)}{dt}.$$

The Continuous-Time Unit Step and Unit Impulse Functions

The continuous-time unit step is the running integral of the unit impulse:



(a)



Since the area of the continuous-time unit impulse $\delta(t)$ is concentrated at $t = 0$,

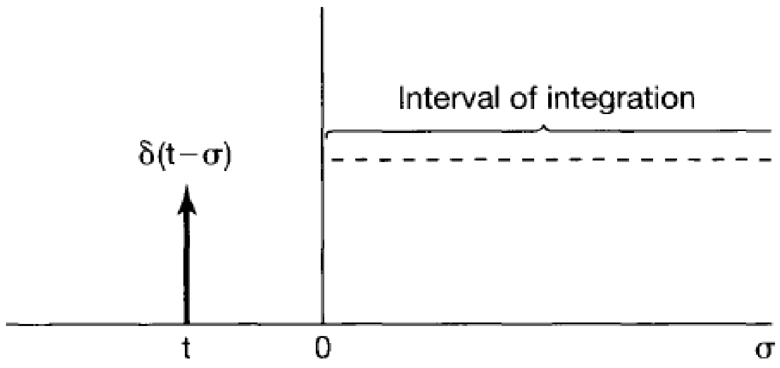
The running integral is 0 for $t < 0$

$$\int_{-\infty}^{t} \delta(\tau)\, d\tau = u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

The running integral is 1 for $t > 0$

The Continuous-Time Unit Step and Unit Impulse Functions

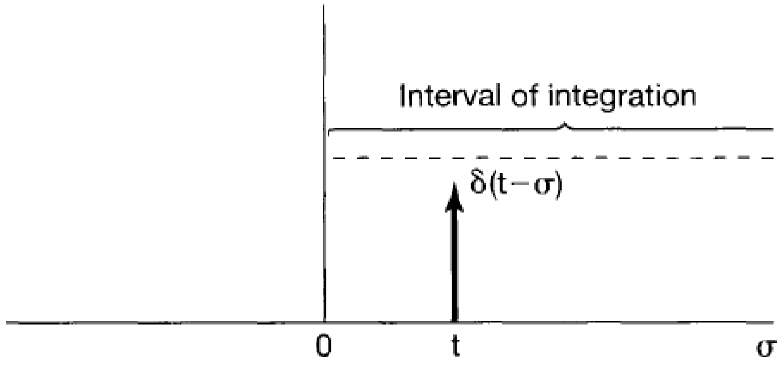
We can also write: $$u(t) = \int_0^{\infty} \delta(t-\sigma)\,d\sigma$$



(a)

The integral is 0 for $t < 0$

$$\int_0^{\infty} \delta(t-\sigma)\,d\sigma \;=\; u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$



The integral is 1 for $t > 0$

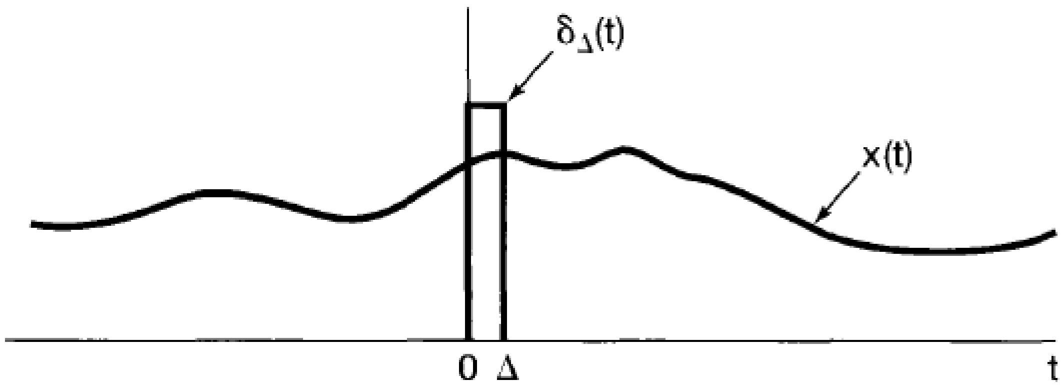
The Continuous-Time Unit Impulse – Sampling Property

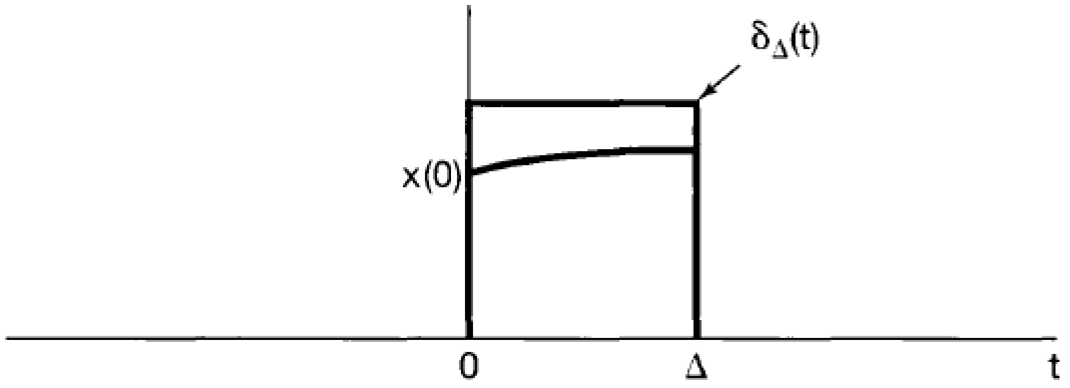
The continuous-time impulse has a very important sampling property.

We will consider the product of an impulse and more well-behaved continuous-time functions $x(t)$.

Consider:

$$x_1(t) = x(t)\delta_\Delta(t)$$



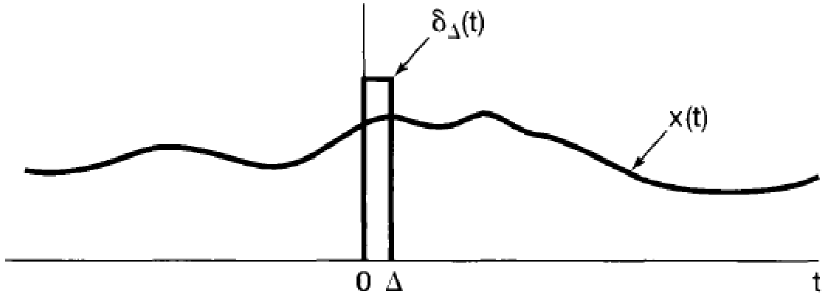


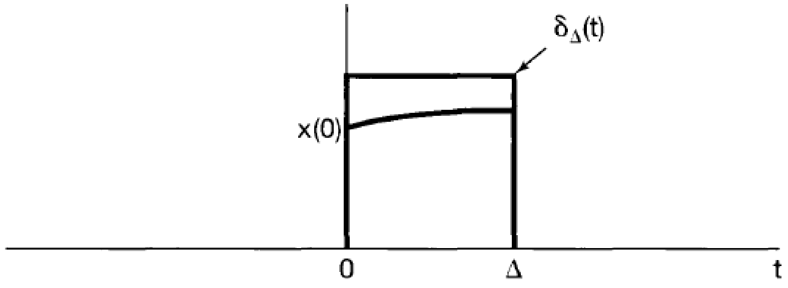
For Δ sufficiently small so that $x(t)$ is approximately constant over this interval:

$$x(t)\delta_\Delta(t) \approx x(0)\delta_\Delta(t)$$

The Continuous-Time Unit Impulse – Sampling Property

$$x_1(t) = x(t)\delta_\Delta(t)$$





For Δ sufficiently small so that $x(t)$ is approximately constant over this interval:

$$x(t)\delta_\Delta(t) \approx x(0)\delta_\Delta(t)$$

Since $\delta(t)$ is the limit as $\Delta \rightarrow 0$ of $\delta_\Delta(t)$, it follows that

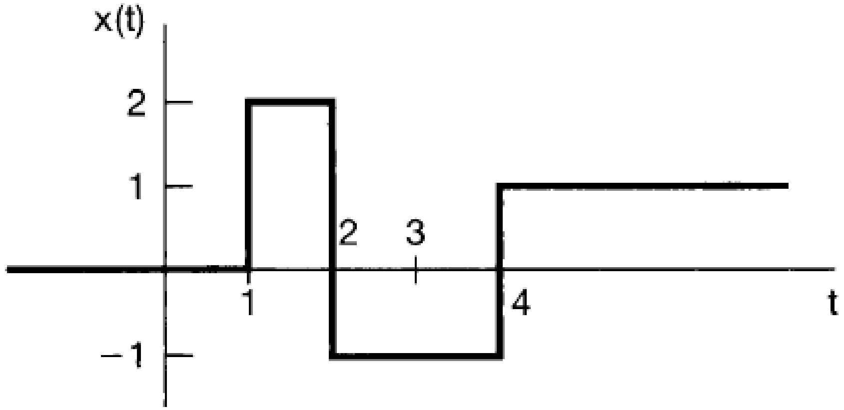
$$x(t)\delta(t) = x(0)\delta(t)$$

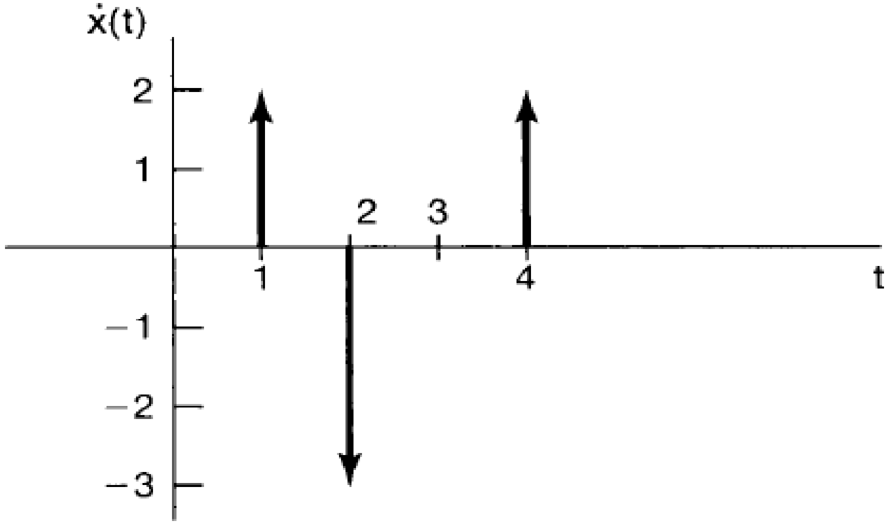
For an impulse concentrated at an arbitrary point t_0 :

$$x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0)$$

Example

Sketch the derivative of $x(t)$.





Differentiation gives rise to a unit impulse located at the point of discontinuity.

The derivative of a unit step with a discontinuity of size k gives rise to an impulse of area k at the point of discontinuity.