Signals and Systems

Lecture 3

CONTINUOUS-TIME AND DISCRETE-TIME SYSTEMS

INTRODUCTION

Systems

Physical systems in the broadest sense are an interconnection of components, devices, or subsystems.

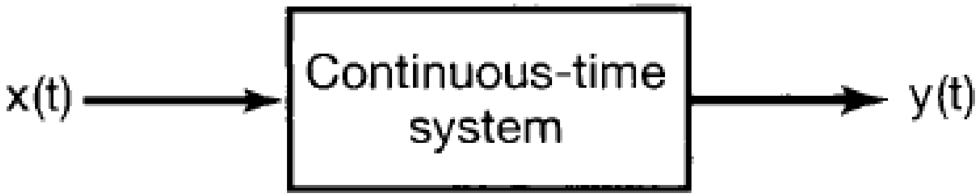
A system can be viewed as a process in which input signals are transformed by the system or cause the system to respond in some way, resulting in other signals as outputs.

For example, a high-fidelity system takes a recorded audio signal and generates a reproduction of that signal.

Continuous-time Systems

A continuous-time system is a system in which continuous-time input signals are applied and result in continuous-time output signals.

We will represent a continuous-time system pictorially as



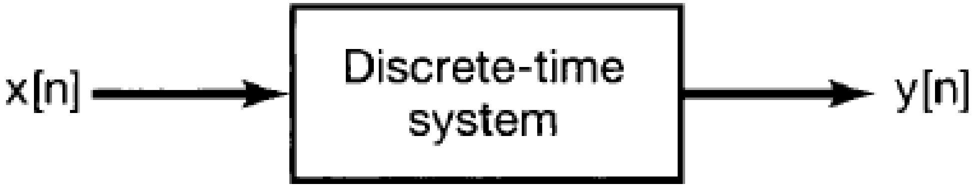
where x(t) is the input and y(t) is the output.

Alternatively, we will often represent the input-output relation of a continuous-time system by the notation

$$x(t) \rightarrow y(t)$$

Discrete-time Systems

A discrete-time system-that is, a system that transforms discrete-time inputs into discrete-time outputs-will be depicted as

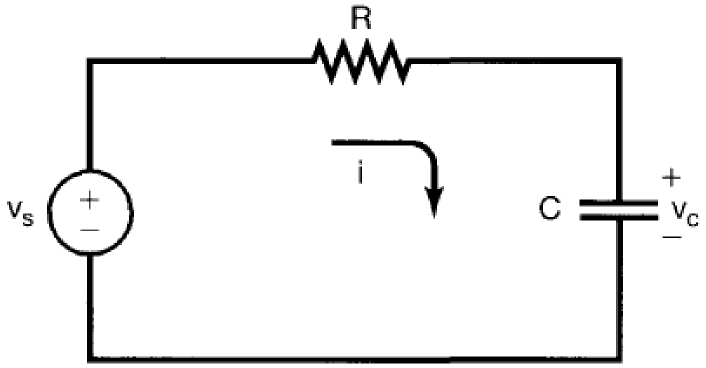


where x[n] is the input and y[n] is the output.

Alternatively, we will often represent the input-output relation of a discrete-time system by the notation

$$x[n] \rightarrow y[n].$$

Example



A continuous-time system:

Input signal: $v_s(t)$

Output signal: $v_c(t)$

We can use simple circuit analysis to derive an equation describing the relationship between the input and output.

The current through the resistor and the capacitor:

$$i(t) = \frac{v_s(t) - v_c(t)}{R}. \qquad\qquad i(t) = C\frac{dv_c(t)}{dt}$$

Equating the right-hand sides of equations, we obtain a differential equation describing the relationship between the input $v_s(t)$ and the output $v_c(t)$:

$$\frac{dv_c(t)}{dt} + \frac{1}{RC}v_c(t) = \frac{1}{RC}v_s(t).$$

Example

As a simple example of a discrete-time system, consider a simple model for the balance in a bank account from month to month.

Specifically, let $y[n]$ denote the balance at the end of the n th month, and suppose that $y[n]$ evolves from month to month according to the equation:

$$y[n] = 1.01y[n-1] + x[n]$$

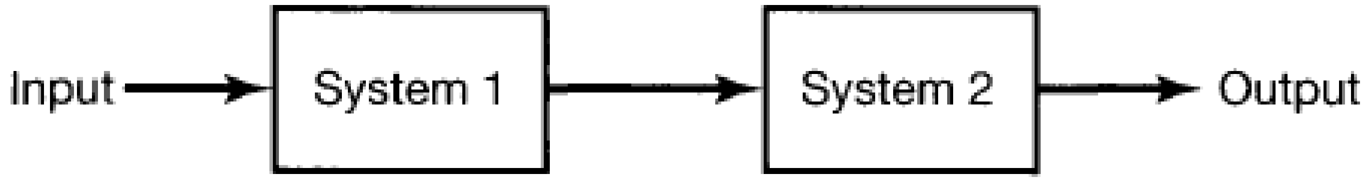
$$y[n] - 1.01y[n-1] = x[n]$$

$x[n]$ represents the net deposit (i.e., deposits minus withdrawals) during the n th month

The term $1.01\, y[n]$ models the fact that we get 1% interest each month.

Interconnections of Systems

A series or cascade interconnection of two systems:



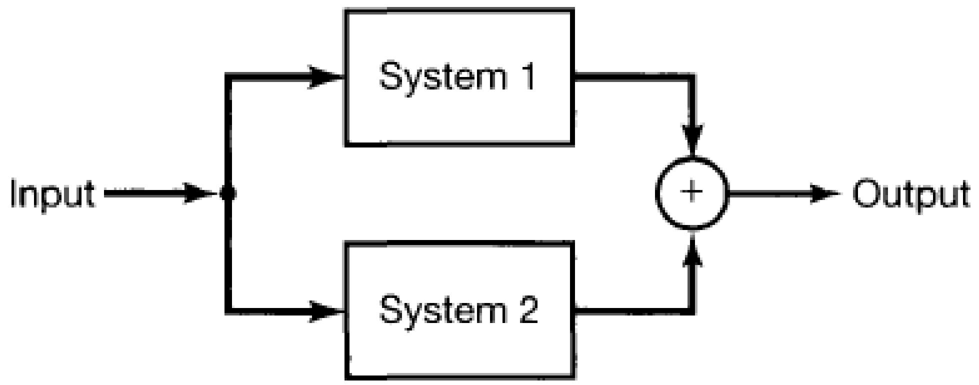
The output of System 1 is the input to System 2, and the overall system transforms an input by processing it first by System 1 and then by System 2.

An example of a series interconnection is a radio receiver followed by an amplifier.

One can define a series interconnection of three or more systems.

Interconnections of Systems

A parallel interconnection of two systems



The same input signal is applied to Systems 1 and 2. The symbol $\oplus$ in the figure denotes addition, so that the output of the parallel interconnection is the sum of the outputs of Systems 1 and 2.

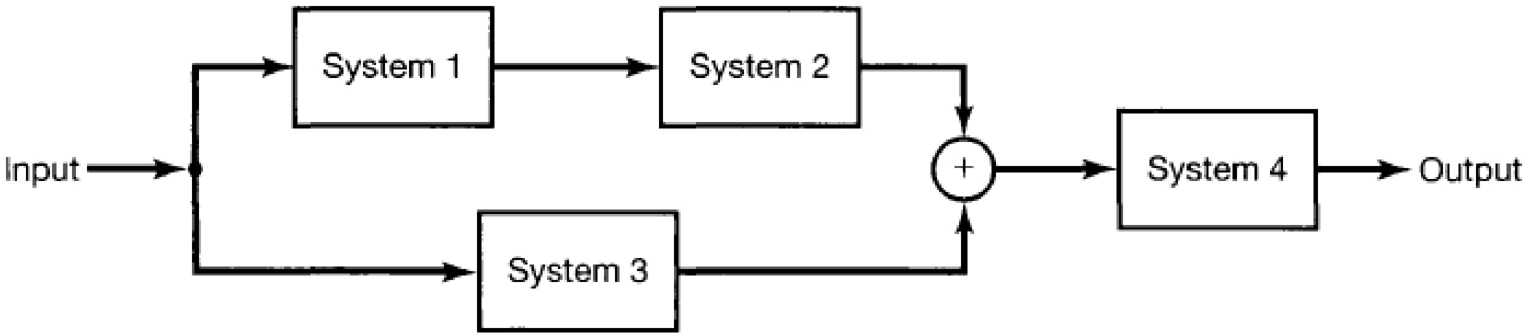
An example of a parallel interconnection is a simple audio system with several microphones feeding into a single amplifier and speaker system.

We can define parallel interconnections of more than two systems.

Interconnections of Systems

We can combine both cascade and parallel interconnections to obtain more complicated interconnections.

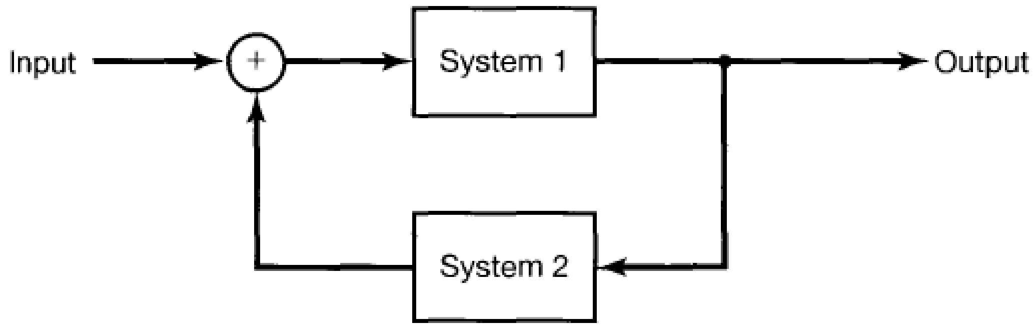
Example:



By describing a system in terms of an interconnection of simpler subsystems, we may model complex systems using simpler, basic building blocks.

Interconnections of Systems

Feedback interconnection



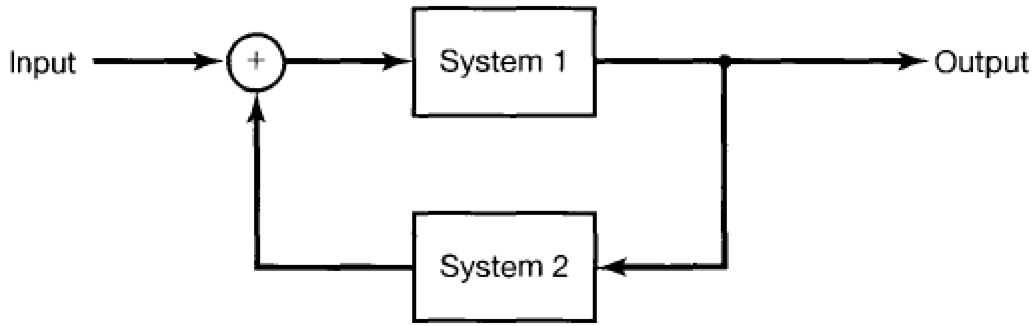
The output of System 1 is the input to System 2, while the output of System 2 is fed back and added to the external input to produce the actual input to System 1.

Feedback systems arise in a wide variety of applications.

For example, a cruise control system on an automobile senses the vehicle's velocity and adjusts the fuel flow in order to keep the speed at the desired level.

Interconnections of Systems

Feedback interconnection



A digitally controlled aircraft is a feedback system in which differences between actual and desired speed, heading, or altitude are fed back through the autopilot in order to correct these discrepancies.

BASIC SYSTEM PROPERTIES

Systems with and without Memory

A system is said to be memoryless if its output for each value of the independent variable at a given time is dependent only on the input at that same time.

Example

The system specified by the relationship

$$y[n] = (2x[n] - x^2[n])^2$$

is memoryless, as the value of $y[n]$ at any particular time n_0 depends only on the value of $x[n]$ at that time.

Systems with and without Memory

A system is said to be memoryless if its output for each value of the independent variable at a given time is dependent only on the input at that same time.

Example

A resistor is a memoryless system; with the input $x(t)$ taken as the current and with the voltage taken as the output $y(t)$, the input-output relationship of a resistor is

$$y(t) = Rx(t)$$

where R is the resistance.

Systems with and without Memory

Systems with memory: A system corresponds with a mechanism that retains or stores information about input values at times other than the current time.

Example

An example of a discrete-time system with memory is an accumulator or summer:

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

The accumulator must "remember" or store information about past inputs.

The accumulator computes the running sum of all inputs up to the current time.

It adds the current input to the preceding value of the running sum (which is exactly the presceding output).

$$y[n] = \sum_{k=-\infty}^{n-1} x[k] + x[n], \qquad y[n] = y[n-1] + x[n]$$

Systems with and without Memory

Systems with memory: A system corresponds with a mechanism that retains or stores information about input values at times other than the current time.

Example

Another example of a discrete-time system with memory is the delay element:

$$y[n] = x[n-1]$$

The delay must retain or store the preceding value of the input.

Example

A capacitor is an example of a continuous-time system with memory, since if the input is taken to be the current and the output is the voltage, then

$$y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)\,d\tau$$

The capacitor stores energy by accumulating electrical charge, represented as the integral of the current.

Systems with and without Memory

If the output depends on a past or future value of the input (i.e. a value of the input other than the current time), the system is with memory.

Example

Consider the discrete time system:

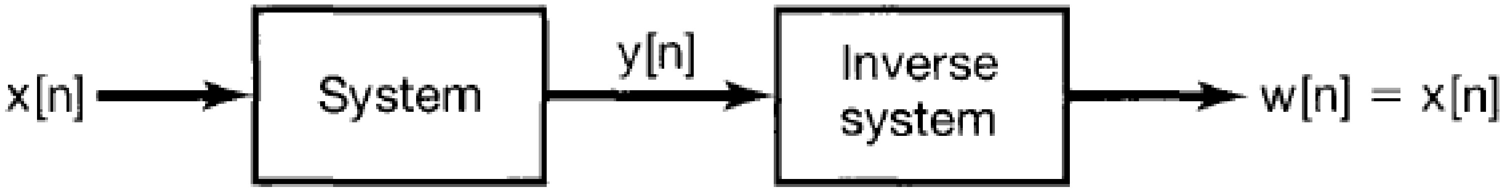
$$y[n] = x[n] + 0.2x[n + 2]$$

The output of the system is affected by a future value of the input. This system is with memory.

Invertibility and Inverse Systems

A system is said to be invertible if distinct inputs lead to distinct outputs.

If a system is invertible, then an inverse system exists.



When cascaded with the original system, the inverse system yields an output $w[n]$ equal to the input $x[n]$ of the first system.

Thus, the series interconnection has an overall input-output relationship which is the same as that for the identity system.

Invertibility and Inverse Systems

A system is said to be invertible if distinct inputs lead to distinct outputs.

If a system is invertible, then an inverse system exists.

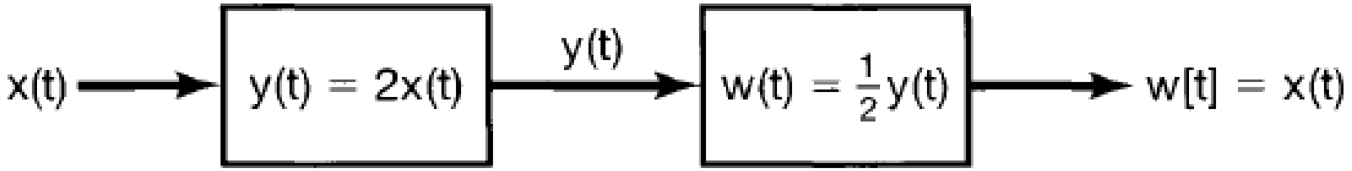
Example

An example of an invertible continuous-time system is

$$y(t) = 2x(t)$$

for which the inverse system is

$$w(t) = \frac{1}{2}y(t)$$



Invertibility and Inverse Systems

A system is said to be invertible if distinct inputs lead to distinct outputs.

If a system is invertible, then an inverse system exists.

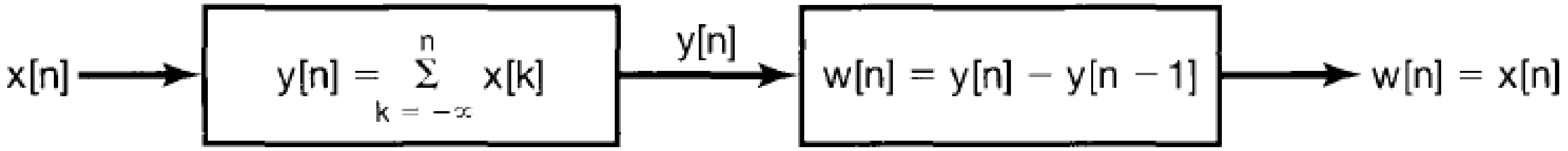
Example

Another example of an invertible system is the accumulator

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

for which the inverse system is

$$w[n] = y[n] - y[n-1]$$



Invertibility and Inverse Systems

A system is said to be invertible if distinct inputs lead to distinct outputs.

If a system is invertible, then an inverse system exists.

Examples of noninvertible systems

$$y[n] = 0$$

The system that produces the zero output sequence for any input sequence. There is no inverse system to recover the input.

Examples of noninvertible systems

The squarer

$$y(t) = x^2(t)$$

We cannot determine the sign of the input from knowledge of the output.

Invertibility and Inverse Systems

The concept of invertibility is important in many contexts.

One example: In communications applications, a signal that we wish to transmit is first applied as the input to a system known as an encoder. The encoder should be invertible so that the input to the encoder must be exactly recoverable from the output.

Similarly, for lossless coding, the input to the encoder must be exactly recoverable from the output.

Causality

A system is causal if the output at any time depends only on values of the input at the present time and in the past.

Such a system is often referred to as being nonanticipative, as the system output does not anticipate future values of the input.

Consequently, if two inputs to a causal system are identical up to some point in time t_0 or n_0 , the corresponding outputs must also be equal up to this same time.

Example

A capacitor is an example of a causal continuous-time system, since the capacitor voltage responds only to the present and past values of the source voltage

$$y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)\,d\tau$$

Causality

A system is causal if the output at any time depends only on values of the input at the present time and in the past.

Consequently, if two inputs to a causal system are identical up to some point in time t_0 or n_0 , the corresponding outputs must also be equal up to this same time.

Example

The following two systems are causal. The output does not depend on a future value of the input.

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

$$y[n] = x[n-1]$$

Causality

A system is causal if the output at any time depends only on values of the input at the present time and in the past.

Consequently, if two inputs to a causal system are identical up to some point in time t_0 or n_0 , the corresponding outputs must also be equal up to this same time.

Example

The following two systems are not causal. The output is affected by a future value of the input.

$$y[n] = x[n] - x[n+1]$$

$$y(t) = x(t+1)$$

Causality and Memory

All memoryless systems are causal.

If a system is not causal, it is with memory.

If a system is with memory, it might or might not be causal.

Causality

For some applications, such as image processing, causality is not an essential constraint.

While processing data that have been recorded previously, as often happens with speech, geophysical, or meteorological signals, we are not constrained to causal processing.

As another example, in many applications, including historical stock market analysis and demographic studies, we may be interested in determining a slowly varying trend in data that also contain high-frequency fluctuations about that trend. In this case, a commonly used approach is to average data over an interval in order to smooth out the fluctuations and keep only the trend.

An example of a noncausal averaging system is $y[n] = \frac{1}{2M+1} \sum_{k=-M}^{+M} x[n-k].$

Causality

Example

Determine whether the following system is causal.

$$y[n] = x[-n].$$

Note that the output $y[n_0]$ at a positive time n_0 depends only on the value of the input signal $x[-n_0]$ at time $(-n_0)$, which is negative and therefore in the past of n_0.

However, we should always be careful to check the input-output relation for all times.

In particular, for $n < 0$, e.g. $n = -4$, we see that $y[-4] = x[4]$, so that the output at this time depends on a future value of the input.

Hence, the system is not causal.

Causality

Example

Determine whether the following system is causal.

$$y(t) = x(t)\cos(t+1)$$

In this system, the output at any time t equals the input at that same time multiplied by a number that varies with time.

Specifically, we can rewrite the equation as

$$y(t) = x(t)g(t)$$

where $g(t)$ is a time-varying function, namely $g(t) = \cos(t + 1)$. Thus, only the current value of the input $x(t)$ influences the current value of the output $y(t)$.

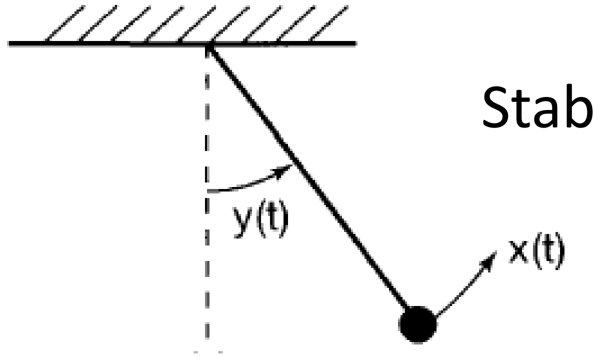
We conclude that this system is causal (and, in fact, memoryless).

Stability

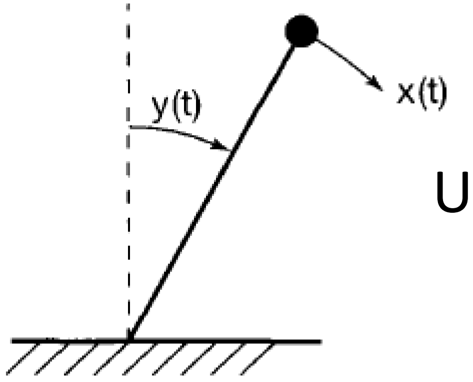
Informally, a stable system is one in which small inputs lead to responses that do not diverge.

Example

Consider the two pendulums given below. The input is the applied force $x(t)$ and the output is the angular deviation $y(t)$ from the vertical.



Stable system



Unstable system

If a small force $x(t)$ is applied, the resulting deflection from vertical will also be small.

If a small force $x(t)$ is applied, the resulting deflection from vertical will be «too large».

Stability

Formally, if the input to a stable system is bounded (i.e., if its magnitude does not grow without bound), then the output must also be bounded and therefore cannot diverge.

This is the definition of stability that we will use in this course.

Example

Consider an automobile where you apply a constant force $f(t) = F$, with the vehicle initially at rest. In this case the velocity of the car will increase, but not without bound, since the retarding frictional force also increases with velocity. In fact, the velocity will continue to increase until the frictional force exactly balances the applied force.

Stability

Formally, if the input to a stable system is bounded (i.e., if its magnitude does not grow without bound), then the output must also be bounded and therefore cannot diverge.

We will consider Bounded Input - Bounded Output (BIBO) Stability

We say that a system is BIBO stable if an input $x(t)$ that is bounded (finite) for all time produces an output $y(t)$ that is also bounded or finite for all time.

Mathematically, we write if $|x(t)| < B1 \rightarrow |y(t)| < B2$, where $B1$ and $B2$ are finite constants and y(t) is the output.

Stability

Formally, if the input to a stable system is bounded (i.e., if its magnitude does not grow without bound), then the output must also be bounded and therefore cannot diverge.

Example

The system

$$y[n] = \frac{1}{2M+1} \sum_{k=-M}^{+M} x[n-k].$$

is stable. Suppose that the input $x[n]$ is bounded in magnitude by some number, say, B, for all values of n. Then the largest possible magnitude for $y[n]$ is also B, because $y[n]$ is the average of a finite set of values of the input. Therefore, $y[n]$ is bounded and the system is stable.

Stability

Formally, if the input to a stable system is bounded (i.e., if its magnitude does not grow without bound), then the output must also be bounded and therefore cannot diverge.

Example

The system

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

is not stable. This system sums all of the past values of the input rather than just a finite set of values, and the system is unstable, since the sum can grow continually even if $x[n]$ is bounded. For example, if we take $x[n] = u[n]$. The output will be

$$y[n] = \sum_{k=-\infty}^{n} u[k] = (n+1)u[n].$$

That is, $y[O] = 1, y[1] = 2, y[2] = 3$, and so on, and y[n] grows without bound.

Check for Stability

To show that a system is unstable, one example for the input signal is enough.

To show that a system is stable, do not use examples of input signals. Since the stability should hold for all bounded input signals.

If you suspect that the system is unstable, find a bounded input for which the output is unbounded.

If you suspect that the system is stable, verify that all bounded inputs, in general, result in bounded outputs.

Check for Stability

Example

Determine whether the following system is stable.

$$y(t) = t x(t)$$

In order to disprove stability, we might try simple bounded inputs such as a constant or a unit step.

For this system, a constant input $x(t) = 1$ yields $y(t) = t$, which is unbounded, since no matter what finite constant we pick, $|y(t)|$ will exceed that constant for some t. We conclude that the system is unstable.

Check for Stability

Example

Determine whether the following system is stable.

$$y(t) = e^{x(t)}$$

For this system, we would be unable to find a bounded input that results in an unbounded output.

We proceed to verify that all bounded inputs result in bounded outputs.

Let B be an arbitrary positive number, and let $x(t)$ be an arbitrary signal bounded by B; that is, we are making no assumption about $x(t)$, except that

$$|x(t)| < B \qquad -B < x(t) < B$$

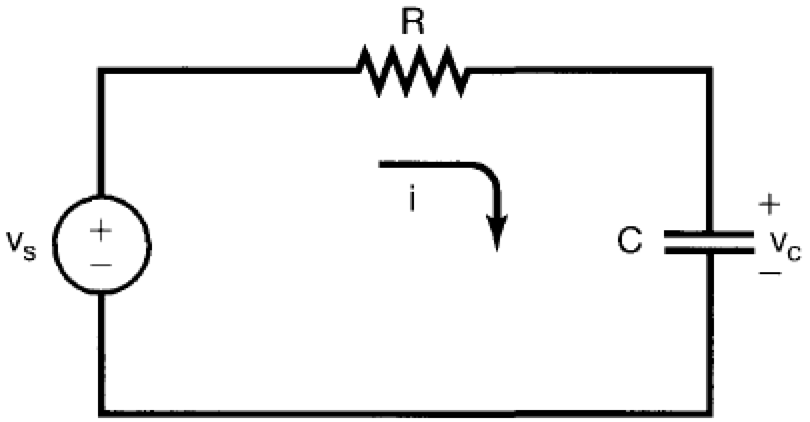
Then $y(t)$ must satisfy

$$e^{-B} < |y(t)| < e^{B}.$$

We conclude that if any input is bounded by an arbitrary positive number B, the corresponding output is guaranteed to be bounded by e^{B}. Thus, the system is stable.

Time-invariance

A system is time invariant if the behavior and characteristics of the system are fixed over time.



For example, an RC system is time invariant if the resistance and capacitance values R and C are constant over time: We would expect to get the same results from an experiment with this circuit today as we would if we ran the identical experiment tomorrow.

On the other hand, if the values of R and C are changed or fluctuate over time, then we would expect the results of our experiment to depend on the time at which we run it.

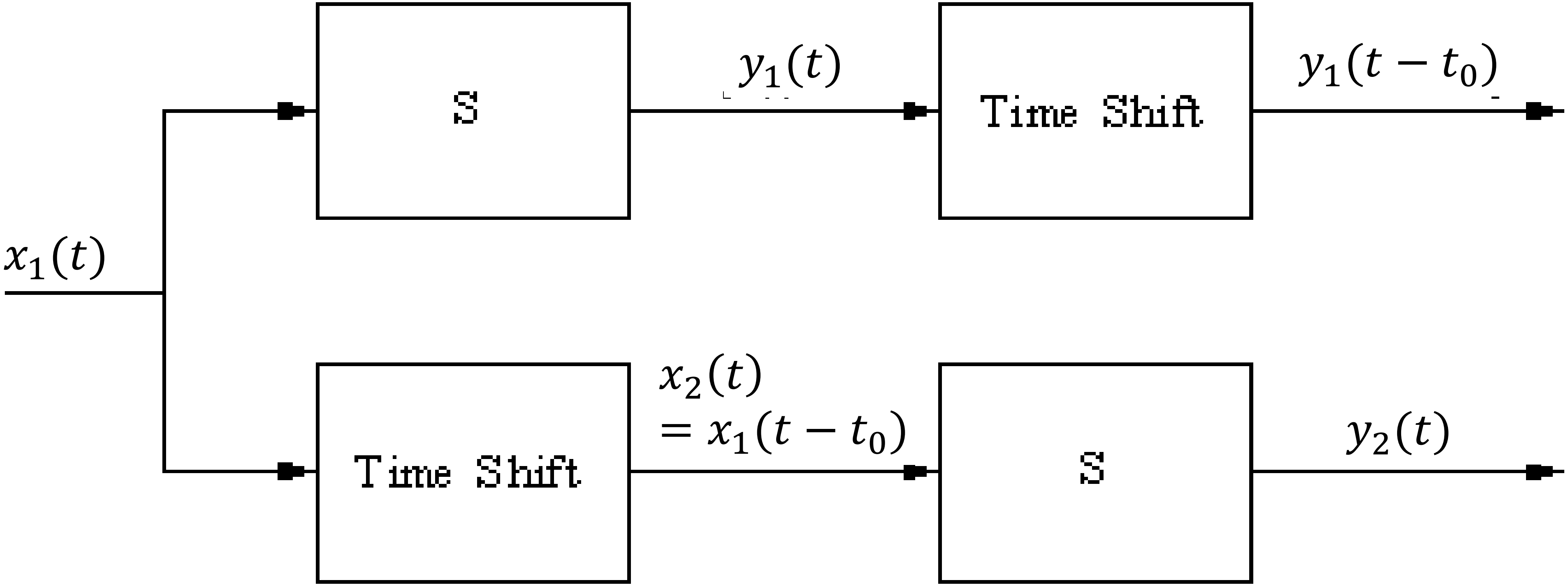
Time-invariance

A system is time invariant if a time shift in the input signal results in an identical time shift in the output signal.

That is, if $y[n]$ is the output of a discrete-time, time-invariant system when $x[n]$ is the input, then $y[n- \ n_0]$ is the output when $x[n-n_0]$ is applied.

In continuous time with $y(t)$ the output corresponding to the input $x(t)$, a time-invariant system will have $y(t-t_0)$ as the output when $x(t-t_0)$ is the input.

Time-invariance



If $y_2(t) = y_1(t-t_0)$ for any $x_1(t)$, the system is time invariant.

Time-invariance

Example

Consider the continuous-time system defined by

$$y(t) = \sin\left[x(t)\right]$$

To check that this system is time invariant, we must determine whether the time-invariance property holds for *any* input and *any* time shift t_0.

Let $x_1(t)$ be an arbitrary input to this system, and let the corresponding output be

$$y_1(t) = \sin\left[x_1(t)\right]$$

Then consider a second input obtained by shifting $x_1(t)$ by t_0

$$x_2(t) = x_1(t - t_0)$$

Time-invariance

Example - continued

$x_1(t) \longrightarrow y_1(t) = \sin[x_1(t)]$

$x_2(t) = x_1(t - t_0) \longrightarrow y_2(t) \stackrel{?}{=} y_1(t - t_0)$

The output corresponding to $x_2(t)$ is

$$y_2(t) = \sin[x_2(t)] = \sin[x_1(t - t_0)]$$

Check if $y_2(t)$ is equal to $y_1(t - t_0)$ by substituting t with $(t - t_0)$ in

$$y_1(t) = \sin[x_1(t)]$$

$$y_1(t - t_0) = \sin[x_1(t - t_0)]$$

We see that $y_2(t) = y_1(t - t_0)$ and therefore, this system is time invariant.

Time-invariance

Example

Consider the discrete-time system defined by

$$y[n] = nx[n].$$

Let $x_1[n]$ be an arbitrary input to this system, and let the corresponding output be

$$y_1[n] = \mathrm{n}x_1[n]$$

Then consider a second input obtained by shifting $x_1[n]$ by n_0.

$$x_2[n] = x_1[n - n_0]$$

The output corresponding to $x_2[n]$ is

$$y_2[n] = \mathrm{n}x_2[n] = nx_1[n - n_0]$$

Check if $y_2[n]$ is equal to $y_1[n - n_0]$ by substituting n with $(n - n_0)$ in

$$y_1[n] = \mathrm{n}x_1[n]$$

$$y_1[n - n_0] = (n - n_0)x_1[n - n_0]$$

We see that $y_2[n]$ is not equal to $y_1[n - n_0]$. This system is not time-invariant.

Linearity

A *linear system,* in continuous time or discrete time, is a system that possesses the important property of **superposition**:

If an input consists of the weighted sum of several signals, then the output is the superposition - that is, the weighted sum- of the responses of the system to each of those signals.

More precisely, let $y_1(t)$ be the response of a continuous-time system to an input $x_1(t)$, and let $y_2(t)$ be the output corresponding to the input $x_2(t)$.

Then the system is linear if:

1. The response to $x_1(t) + x_2(t)$ is $y_1(t) + y_2(t)$.
   (additivity property)
2. The response to $ax_1(t)$ is $ay_1(t)$, where a is any complex constant.
   (scaling or homogeneity property)

The same definition holds in discrete time.

Linearity

1. The response to $x_1(t) + x_2(t)$ is $y_1(t) + y_2(t)$.
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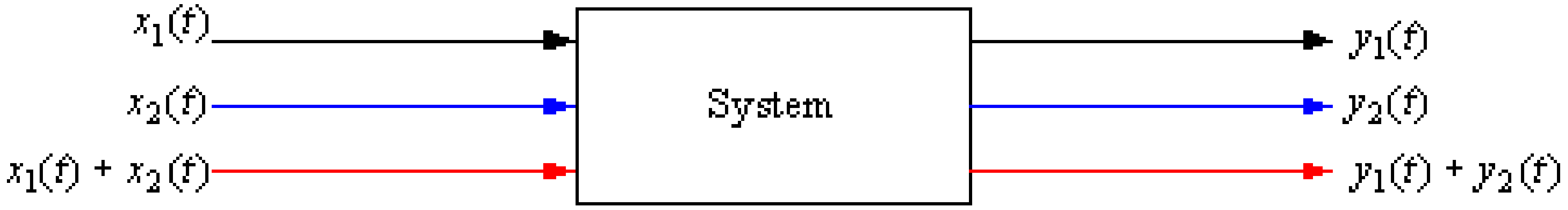

2. The response to $ax_1(t)$ is $ay_1(t)$, where a is any complex constant.
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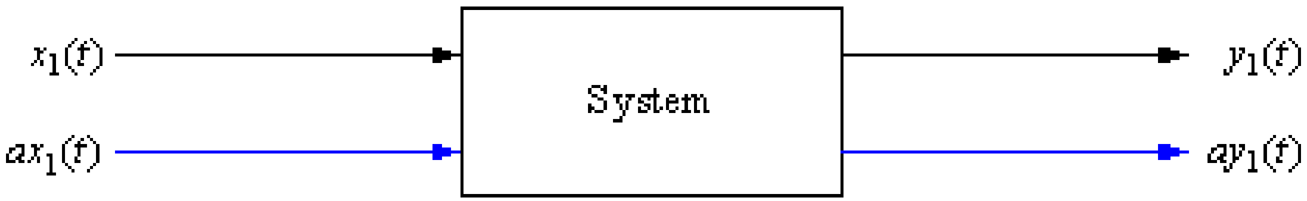

The same definition holds in discrete time.

Linearity

The two properties defining a linear system can be combined into a single statement:

$$\text{continuous time: } ax_1(t) + bx_2(t) \rightarrow ay_1(t) + by_2(t),$$

$$\text{discrete time: } ax_1[n] + bx_2[n] \rightarrow ay_1[n] + by_2[n].$$

Here, a and b are any complex constants.

Furthermore, if $x_k[n],\ k = 1, 2, 3, \ldots$, are a set of inputs to a linear system with corresponding outputs $y_k[n],\ k = 1, 2, 3, \ldots$, then the response to a linear combination of these inputs given by

$$x[n] = \sum_k a_k x_k[n] = a_1 x_1[n] + a_2 x_2[n] + a_3 x_3[n] + \ldots$$

is

$$y[n] = \sum_k a_k y_k[n] = a_1 y_1[n] + a_2 y_2[n] + a_3 y_3[n] + \ldots.$$

Linearity

Furthermore, if $x_k[n],\ k = 1, 2, 3, \dots$, are a set of inputs to a linear system with corresponding outputs $y_k[n],\ k = 1, 2, 3, \dots$, then the response to a linear combination of these inputs given by

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is

$$y[n] = \sum_k a_k y_k[n] = a_1 y_1[n] + a_2 y_2[n] + a_3 y_3[n] + \dots.$$

This very important fact is known as the *superposition property,* which holds for linear systems in both continuous and discrete time.

A direct consequence of the superposition property is that, for linear systems, an input which is zero for all time results in an output which is zero for all time. For example, if $x[n] \rightarrow y[n]$, then the homogeneity property tells us that

$$0 = 0 \cdot x[n] \rightarrow 0 \cdot y[n] = 0.$$

Linearity

Example

Consider a system S whose input $x(t)$ and output $y(t)$ are related by

$$y(t) = tx(t)$$

To determine whether or not S linear, we consider two arbitrary inputs $x_1(t)$ and $x_2(t)$

$$x_1(t) \to y_1(t) = tx_1(t)$$

$$x_2(t) \to y_2(t) = tx_2(t)$$

Let $x_3(t)$ be a linear combination of $x_1(t)$ and $x_2(t)$. That is,

$$x_3(t) = ax_1(t) + bx_2(t)$$

where a and b are arbitrary scalars. If $x_3(t)$ is the input to S, then the output is

$$\begin{aligned} y_3(t) &= tx_3(t) \\ &= t(ax_1(t) + bx_2(t)) \\ &= atx_1(t) + btx_2(t) \\ &= ay_1(t) + by_2(t) \end{aligned}$$

System S is linear.

Linearity

Example

Consider a system *S* whose input $x(t)$ and output $y(t)$ are related by

$$y(t) = x^2(t)$$

To determine whether or not *S* linear, we consider two arbitrary inputs $x_1(t)$ and $x_2(t)$

$$x_1(t) \rightarrow y_1(t) = x_1^2(t)$$

$$x_2(t) \rightarrow y_2(t) = x_2^2(t)$$

Let $x_3(t)$ be a linear combination of $x_1(t)$ and $x_2(t)$. That is,

$$x_3(t) = ax_1(t) + bx_2(t)$$

where a and b are arbitrary scalars. If $x_3(t)$ is the input to *S*, then the output is

$$\begin{aligned} x_3(t) \rightarrow y_3(t) &= x_3^2(t) \\ &= (ax_1(t) + bx_2(t))^2 \\ &= a^2x_1^2(t) + b^2x_2^2(t) + 2abx_1(t)x_2(t) \\ &= a^2y_1(t) + b^2y_2(t) + 2abx_1(t)x_2(t) \end{aligned}$$

$$y_3(t) \neq ay_1(t) + by_2(t)$$

System *S* is not linear.

Linearity

Example

Consider the system:

$$y[n] = 2x[n] + 3$$

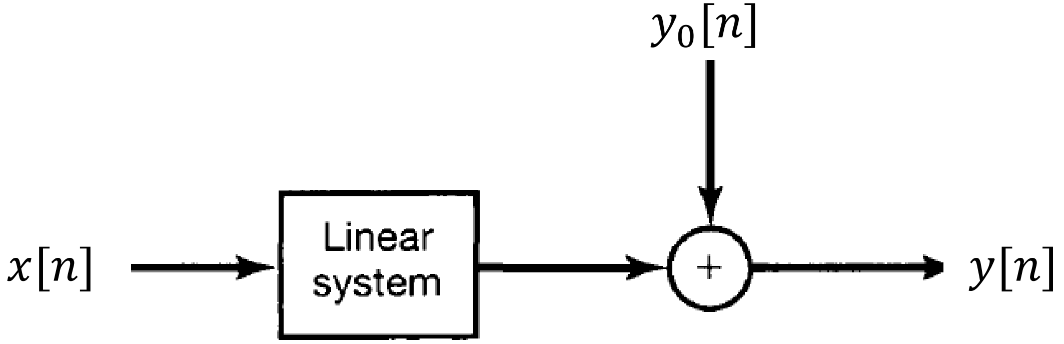
$$x_1[n] \to y_1[n] = 2x_1[n] + 3$$
$$x_2[n] \to y_2[n] = 2x_2[n] + 3$$

$$y_3[n] = 2[x_1[n] + x_2[n]] + 3 \neq y_1[n] + y_2[n]$$

This system is not linear, but the output of this system can be represented as the sum of the output of a linear system and another signal equal to the *zero-input response* of the system.

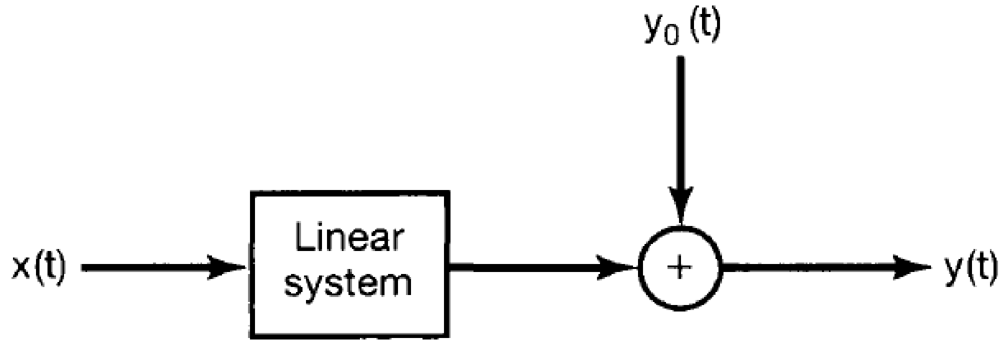
$x[n] \to 2x[n]$. Linear system

$y_0[n] = 3$. Zero-input response of the system



Incrementally linear system

Linearity



Incrementally linear systems are systems in continuous or discrete time that respond linearly to changes in the input.
In other words, the difference between the responses to any two inputs to an incrementally linear system is a linear (i.e., additive and homogeneous) function of the difference between the two inputs.

Further examples for properties of systems

System	Linear	Time-invariant	Causal
$y(t) = 3x(t)\cos(t)$	yes	no	yes
$y(t) = \sqrt{x^2(t)}$	no	yes	yes
$y[n] = \begin{cases} +1, & x[n] \geq 0 \\ -1, & x[n] < 0 \end{cases}$	no	yes	yes

Further examples for properties of systems

System	Linear	Time-invariant	Causal
$y(t) = \int_{t}^{t+1} x(\lambda) d\lambda$	yes	yes	no
$y[n] = 2(x[n+1]u[n] - x[n]) + 1$	no	no	no