Signals and Systems

Lecture 4

LTI Systems

Impulse Response and Convolution

Linear Time-Invariant (LTI) Systems

- Many physical processes have the properties of linearity and time-invariance and thus can be modeled as linear time-invariant (LTI) systems.
- Superposition property -> If we can represent the input to an LTI system in terms of a linear combination of a set of basic signals, we can then use superposition to compute the output of the system in terms of its responses to these basic signals.
- Both in discrete time and in continuous time, very general signals can be represented as linear combinations of delayed impulses.
- These allow us to develop a complete characterization of any LTI system in terms of its response to a unit impulse.

DISCRETE-TIME LTI SYSTEMS:

THE CONVOLUTION SUM

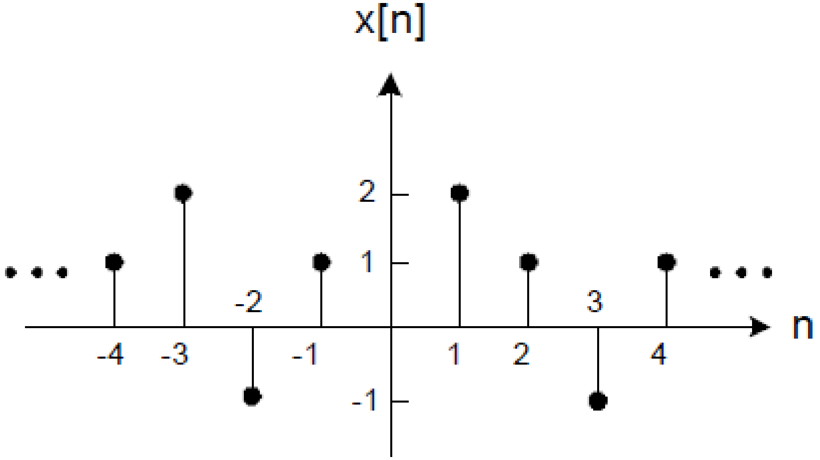
The Representation of Discrete-Time Signals in Terms of Impulses

A discrete-time signal can be decomposed into a sequence of individual impulses:

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]$$

Sifting property of the discrete-time unit impulse.

Example 1:



This signal can be expressed as a weighted sum of shifted impulses:

$$x[n] = \ldots + x[-3]\delta[n+3] + x[-2]\delta[n+2] + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + \ldots$$

The Discrete-Time Unit Impulse Response and the Convolution Sum Representation of LTI Systems

Let $h_k[n]$ be the response of a linear system to the shifted unit impulse $\delta[n-k]$.

Then from the superposition property for a linear system, the response of the linear system to the input $x[n]$ is the weighted linear combination of these basic responses:

$$\delta[n-k] \longrightarrow h_k[n]$$

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k] \longrightarrow y[n] = \sum_{k=-\infty}^{+\infty} x[k]h_k[n]$$

If the linear system is time invariant, then the responses to time-shifted unit impulses are all time-shifted versions of the same impulse response:

$$h_k[n] = h[n-k]$$

The Discrete-Time Unit Impulse Response of an LTI system

$$\delta[n] \longrightarrow h[n]$$

$$\delta[n-k] \longrightarrow h_k[n]$$

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k] \longrightarrow y[n] = \sum_{k=-\infty}^{+\infty} x[k]h_k[n]$$

If the linear system is time invariant:

$$h_k[n] = h[n-k]$$

$h[n]$ is called the unit impulse (sample) response of the LTI systems.
That is, $h[n]$ is the output of the LTI system when $\delta[n]$ is the input.

The impulse response $h[n]$ of an LTI system characterizes the system completely.

Convolution-Sum Representation of LTI Systems

For an LTI system:

$$\boxed{y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k].}$$

This sum is referred to as the convolution sum or superposition sum.

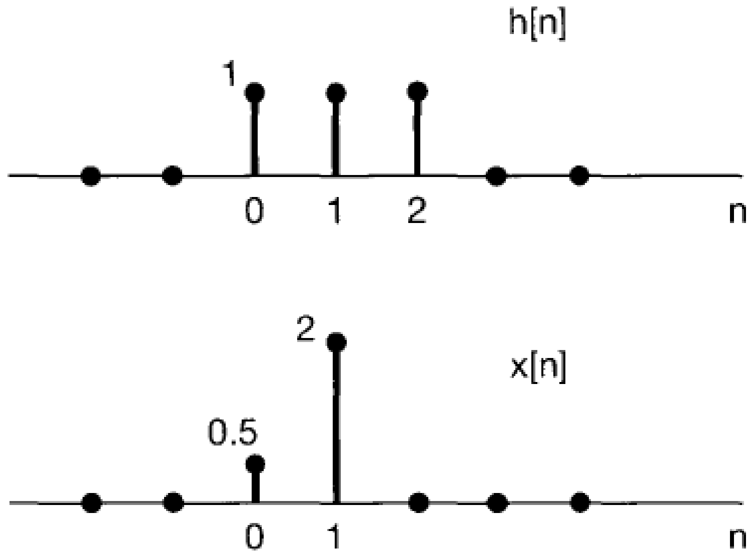
The operation on the right-hand side of the equation is known as the convolution of the sequences $x[n]$ and $h[n]$.

We will represent the operation of convolution symbolically as

$$y[n] = x[n] * h[n]$$

Example 2

Consider an LTI system with impulse response $h[n]$ and input $x[n]$, as illustrated below



$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

simplifies to

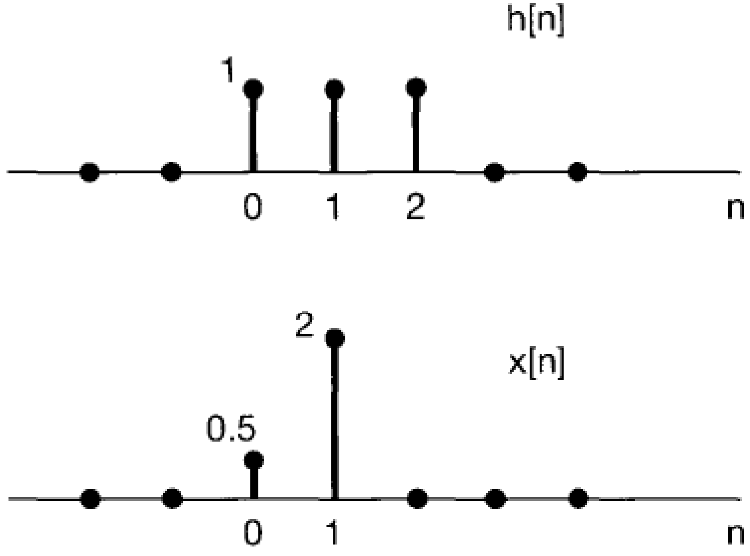
$$y[n] = x[0]h[n-0] + x[1]h[n-1] = 0.5h[n] + 2h[n-1]$$

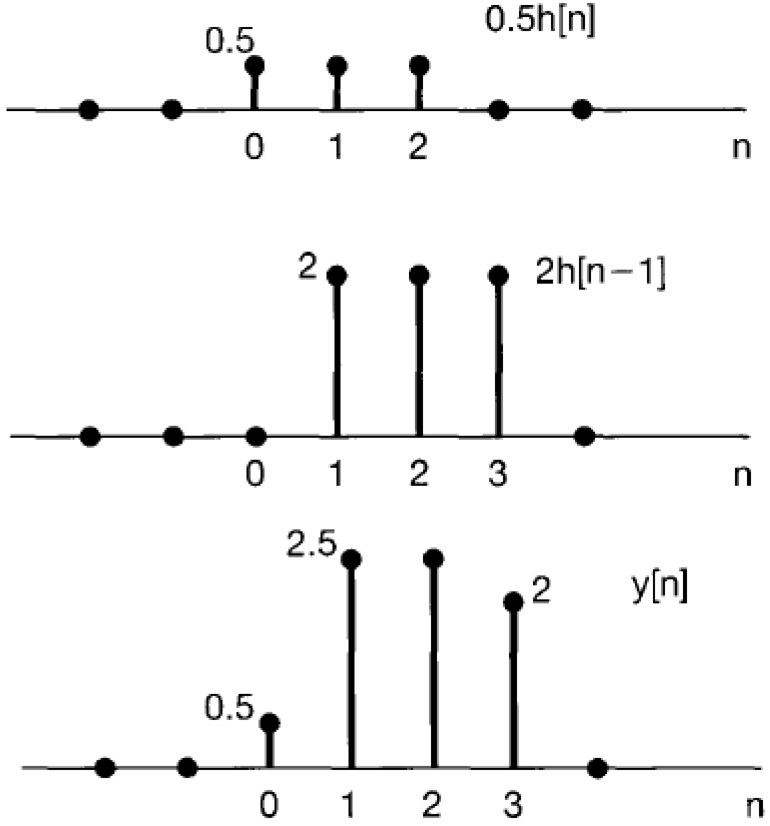
The sequences $0.5h[n]$ and $2h[n-1]$ are the two echoes of the impulse response needed for the superposition involved in generating $y[n]$.

Example 2 - continued

By summing the two echoes for each value of n, we obtain $y[n]$.

$$y[n] = 0.5h[n] + 2h[n-1]$$





Convolution Sum

Another way to visualize the convolution sum is

for a fixed n

- draw the signals $x[k]$ and $h[n-k]$ as functions of k
- multiply them to form the signal g[k]

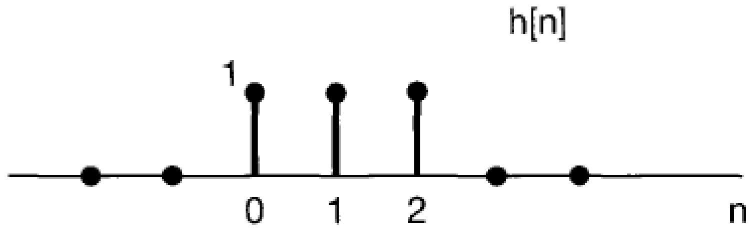
$$g[k] = x[k]h[n-k]$$

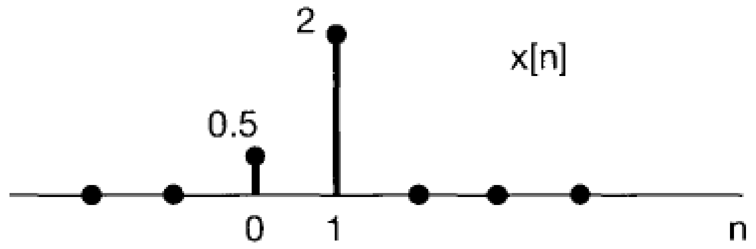
- and then sum all values of g[k]

The sum of all values of $g[k]$ is the output value at the selected time n.

Example 3

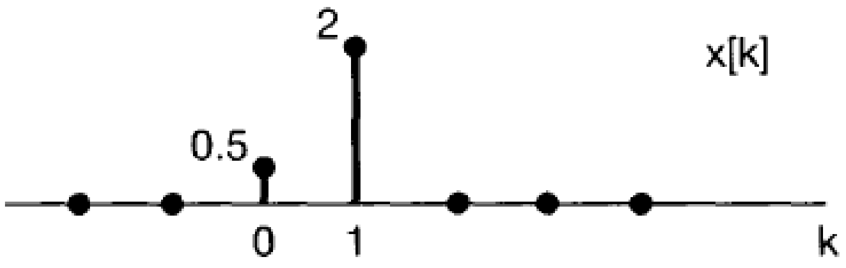
Consider an LTI system with impulse response $h[n]$ and input $x[n]$, as illustrated below

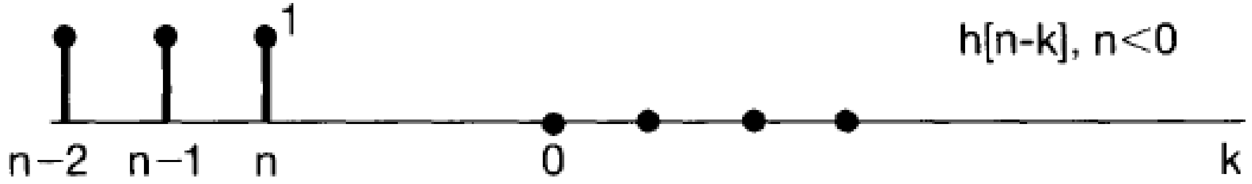




$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k].$$

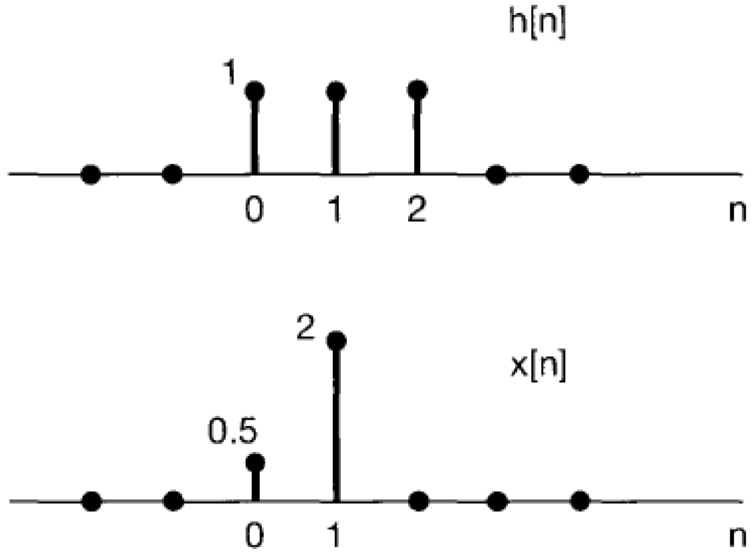
$$y[n] = 0 \text{ for } n < 0$$





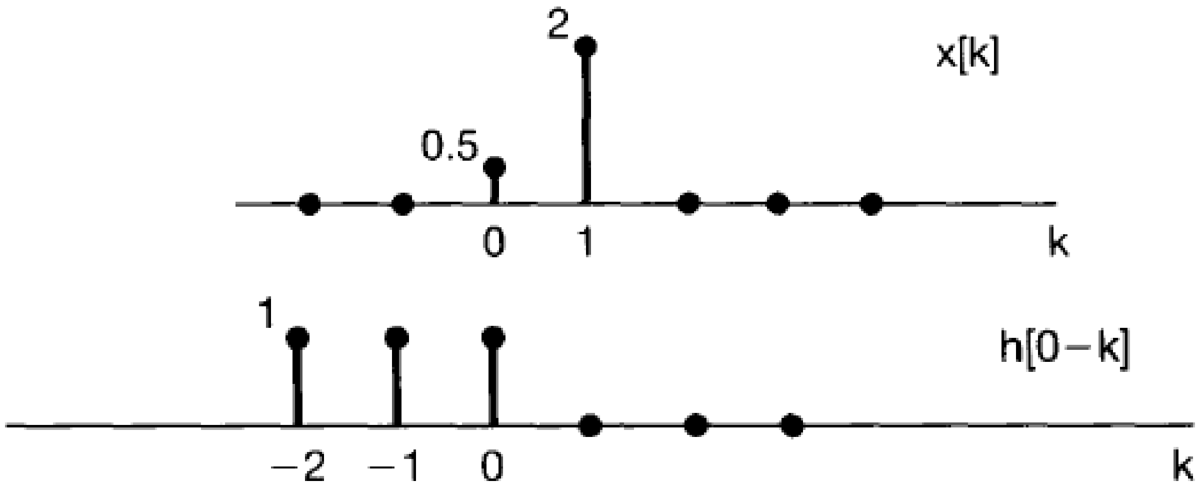
For $n < 0$, we see that $x[k]h[n-k] = 0$ for all k, since the nonzero values of $x[k]$ and $h[n-k]$ do not overlap. Consequently, $y[n] = 0$ for $n < 0$.

Example 3- continued



$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k].$$

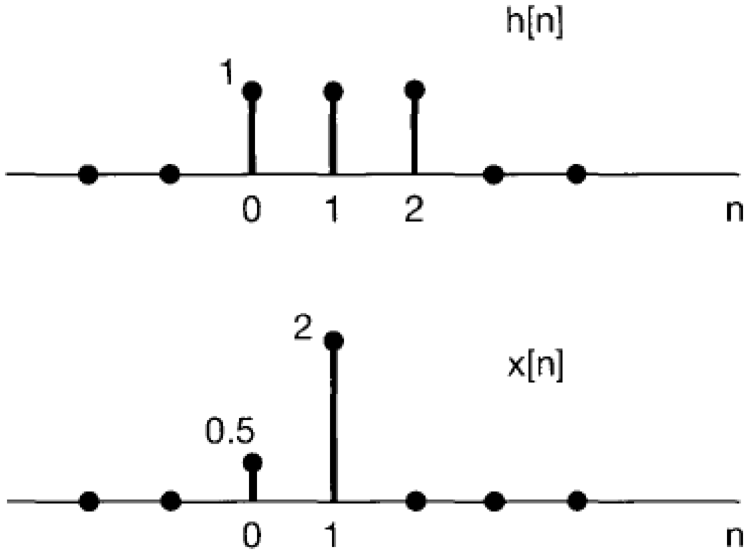
$n = 0$



For $n = 0$, the product of the sequence $x[k]$ with the sequence $h[O - k]$ has only one nonzero sample.

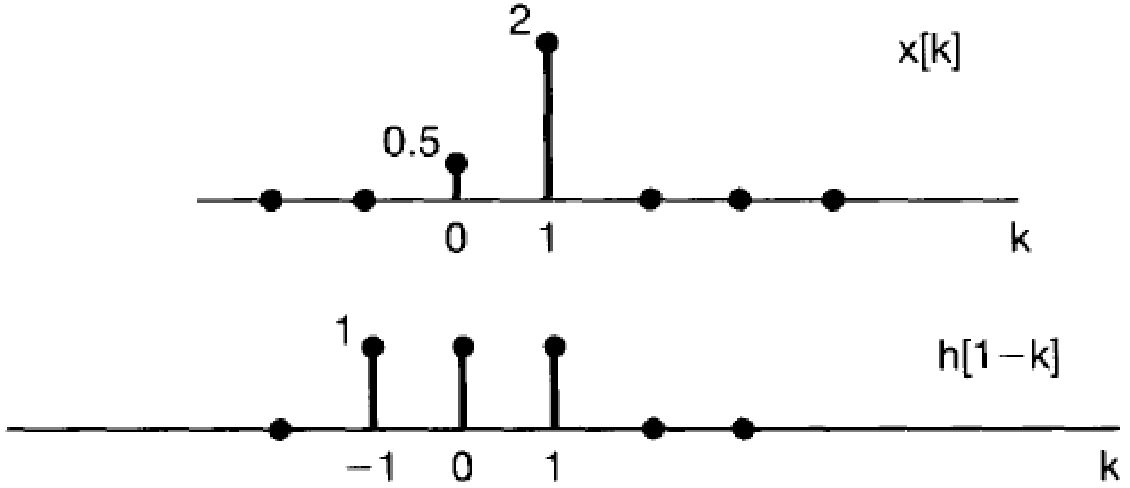
$$y[0] = \sum_{k=-\infty}^{\infty} x[k]h[0-k] = 0.5.$$

Example 3 - continued



$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k].$$

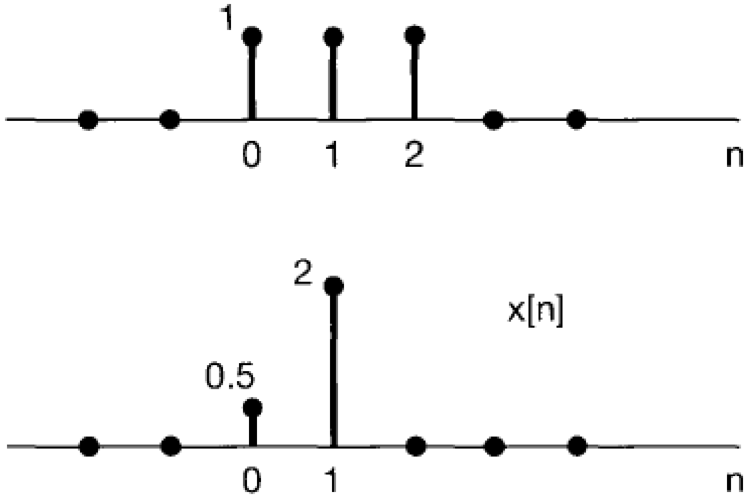
$n = 1$



For $n = 1$, the product of the sequence $x[k]$ with the sequence $h[1 - k]$ has two nonzero samples, which may be summed to obtain

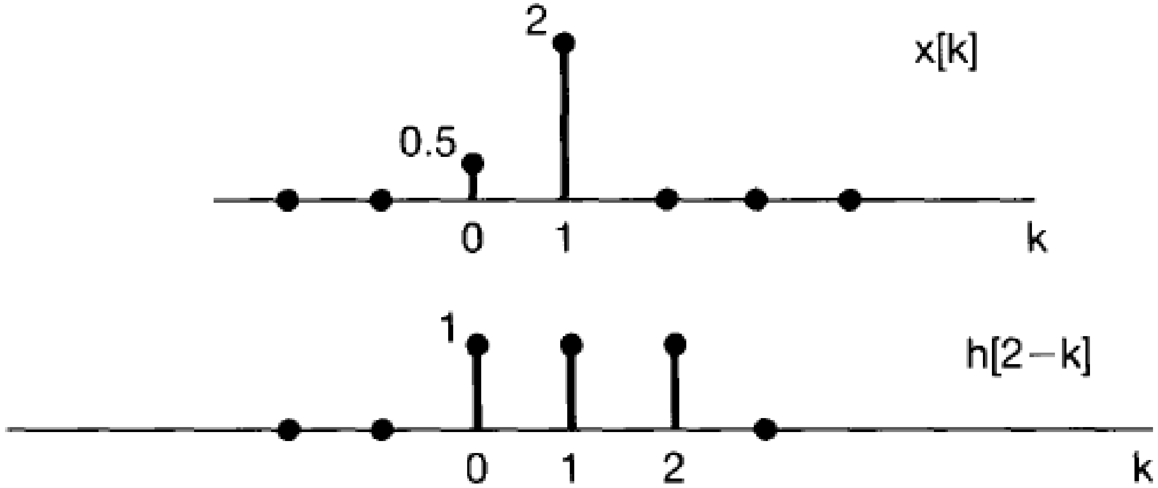
$$y[1] = \sum_{k=-\infty}^{\infty} x[k]h[1-k] = 0.5 + 2.0 = 2.5.$$

Example 3 - continued



$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k].$$

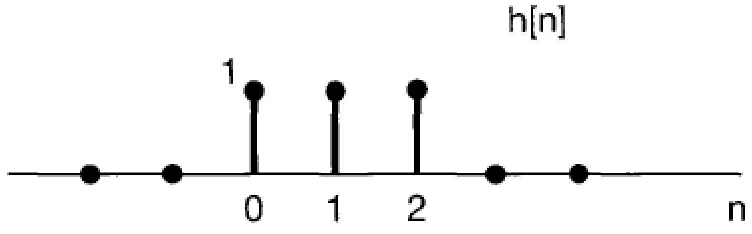
$n = 2$

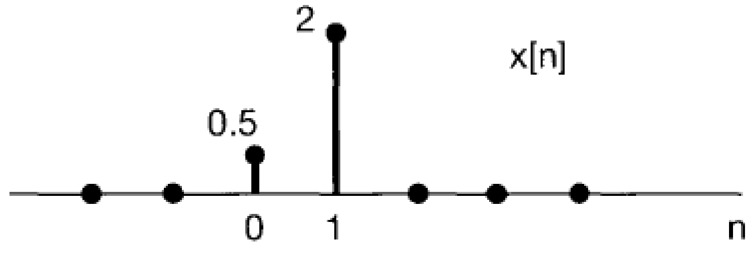


For $n = 2$, the product of the sequence $x[k]$ with the sequence $h[2 - k]$ has two nonzero samples, which may be summed to obtain

$$y[2] = \sum_{k=-\infty}^{\infty} x[k]h[2-k] = 0.5 + 2.0 = 2.5$$

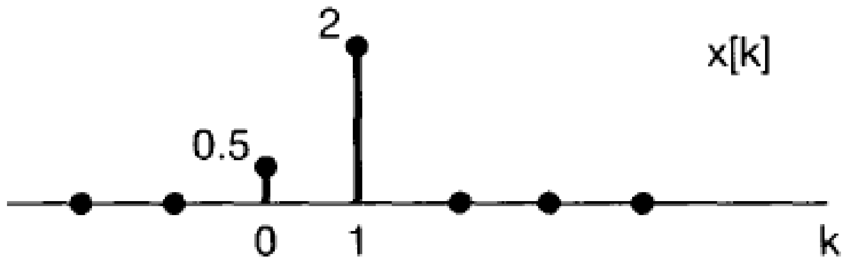
Example 3 - continued

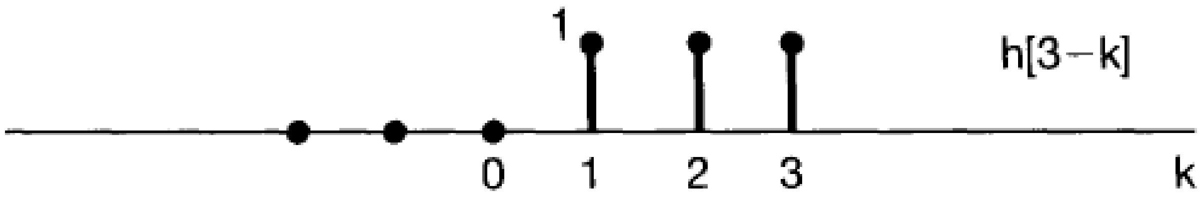




$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k].$$

$n = 3$



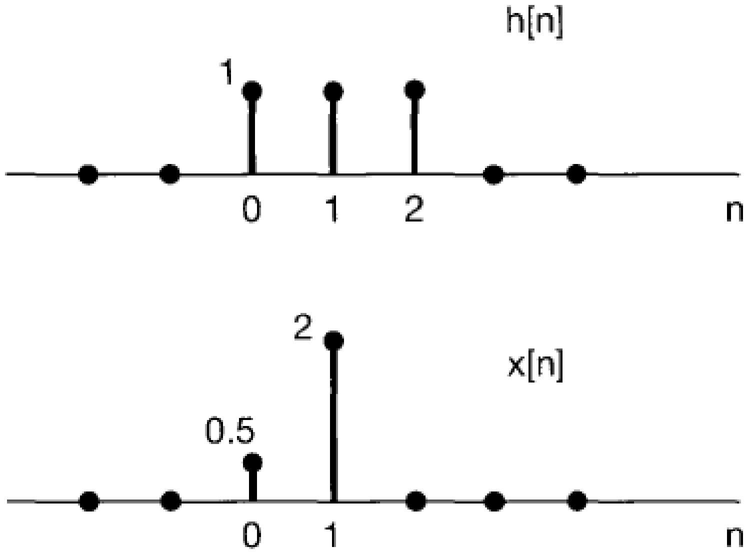


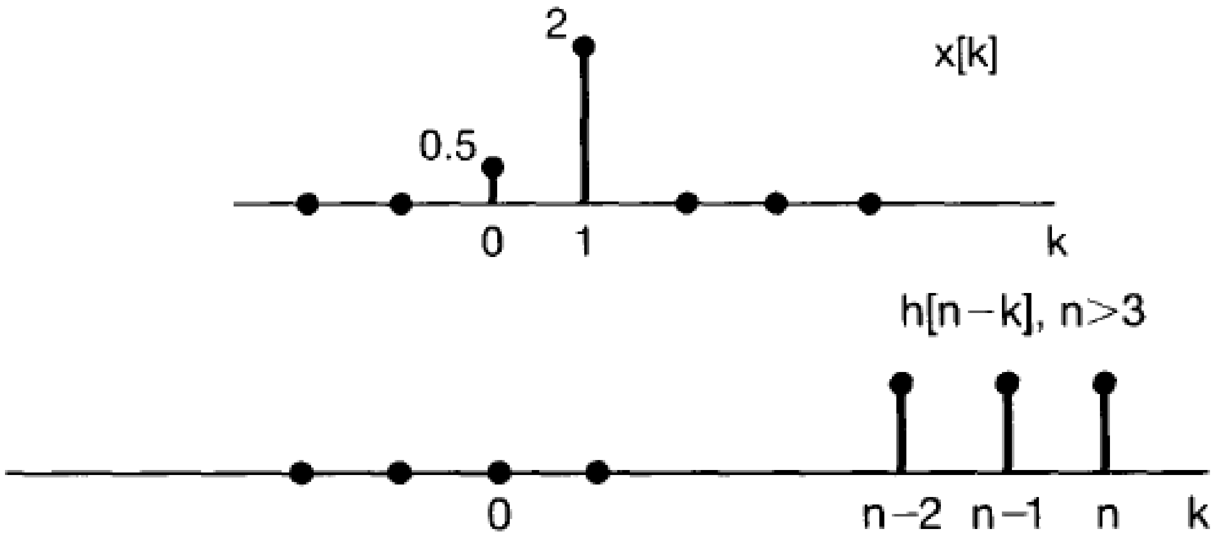
For $n = 3$, the product of the sequence $x[k]$ with the sequence $h[3 - k]$ has one nonzero sample.

$$y[3] = \sum_{k=-\infty}^{\infty} x[k]h[3-k] = 2.0.$$

Example 3 - continued

$$y[n] = 0 \text{ for } \quad n > 3$$

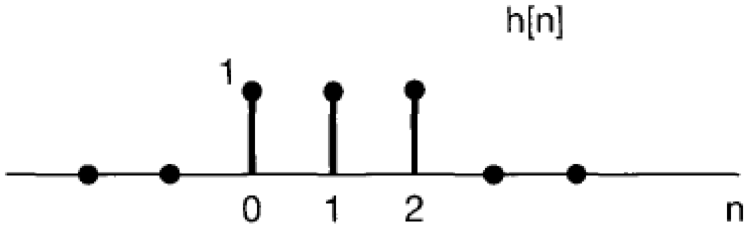


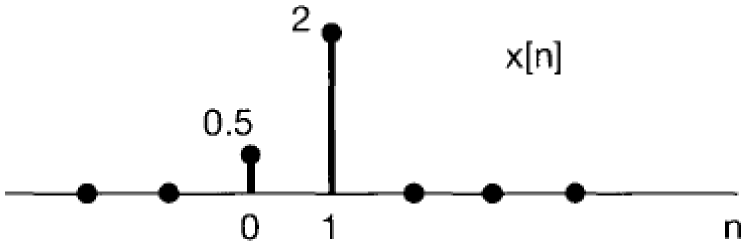


$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k].$$

For $n > 3$, we see that $x[k]h[n-k] = 0$ for all k, since the nonzero values of $x[k]$ and $h[n-k]$ do not overlap. Consequently, $y[n] = 0$ for $n > 3$.

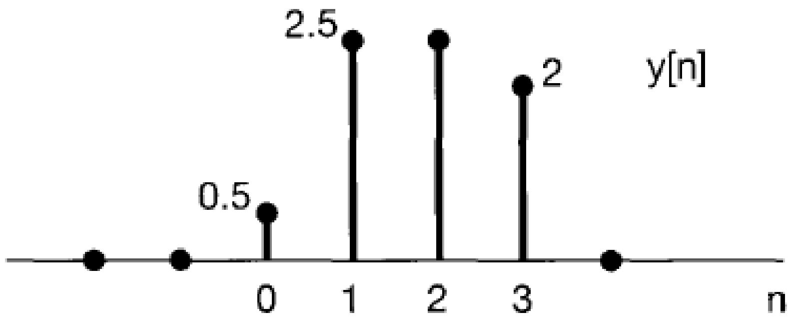
Example 3 - continued





$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k].$$

The resulting output signal $y[n]$ of the system:

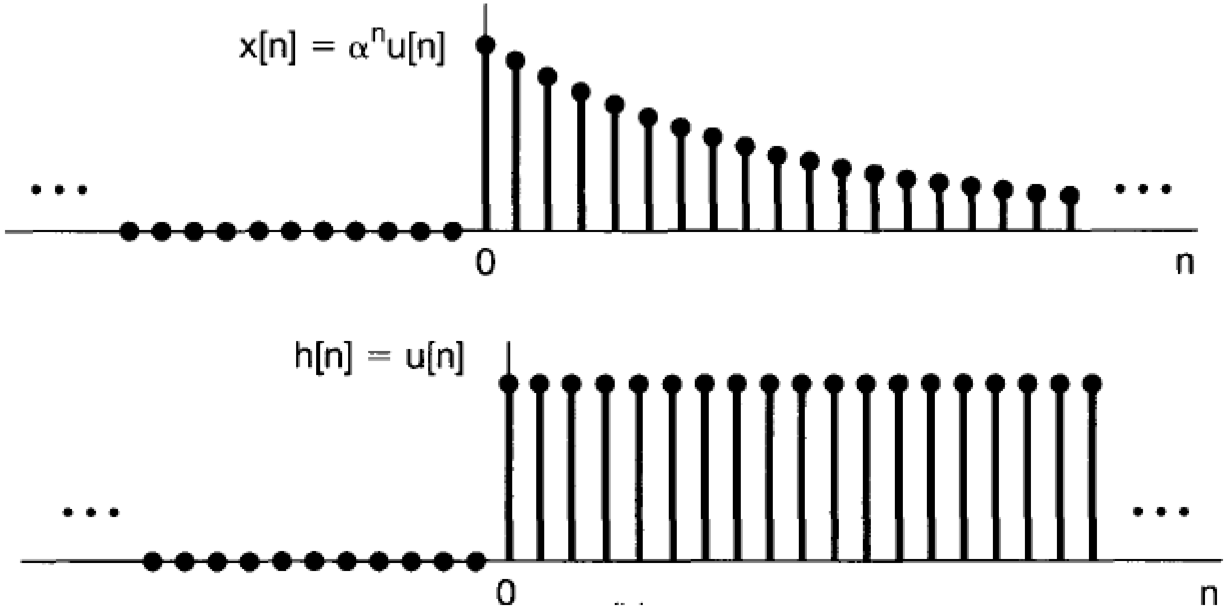


Example 4

Consider an input $x[n]$ and a unit impulse response $h[n]$ given by

$$x[n] = \alpha^n u[n],$$
$$h[n] = u[n],$$

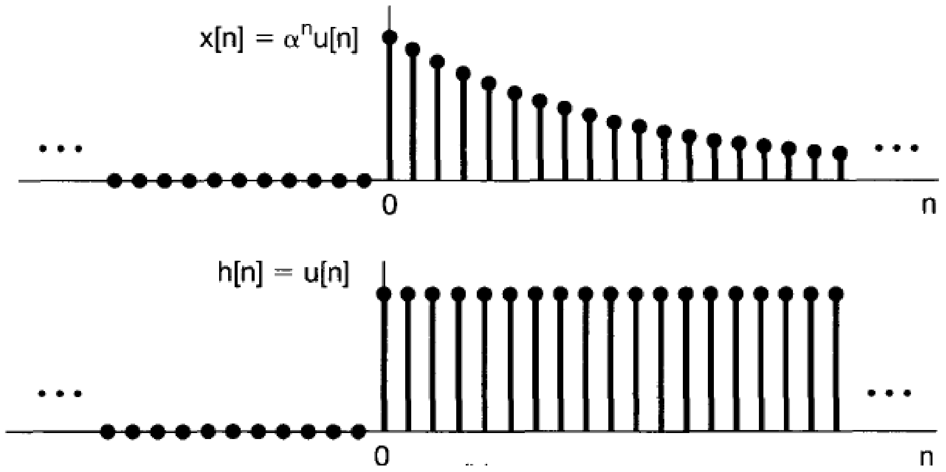
with $0 < a < 1$.

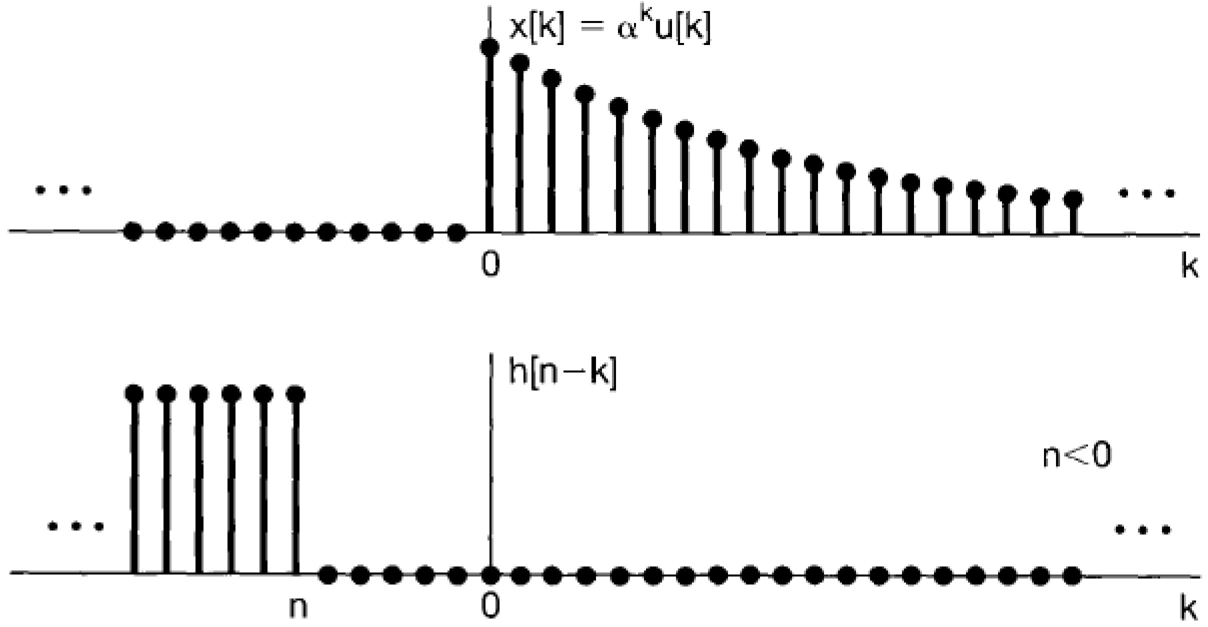


$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k].$$

Example 4 - continued

$y[n] = 0$ for $n < 0$

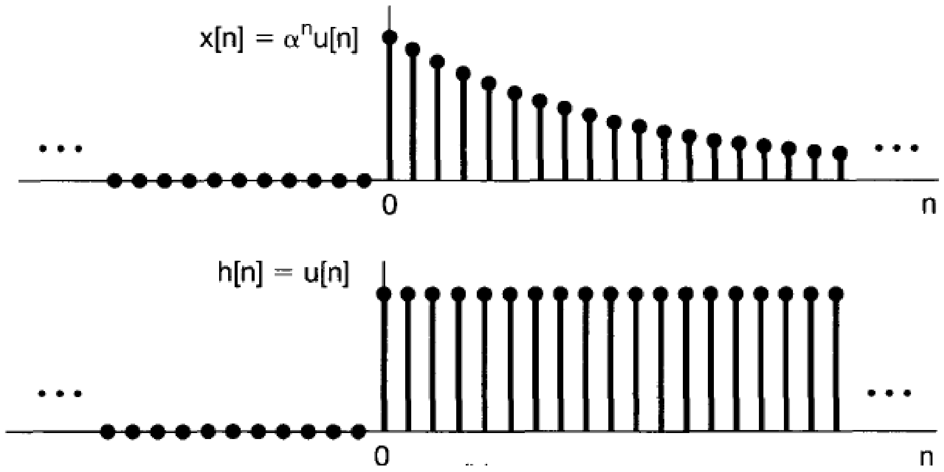




$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

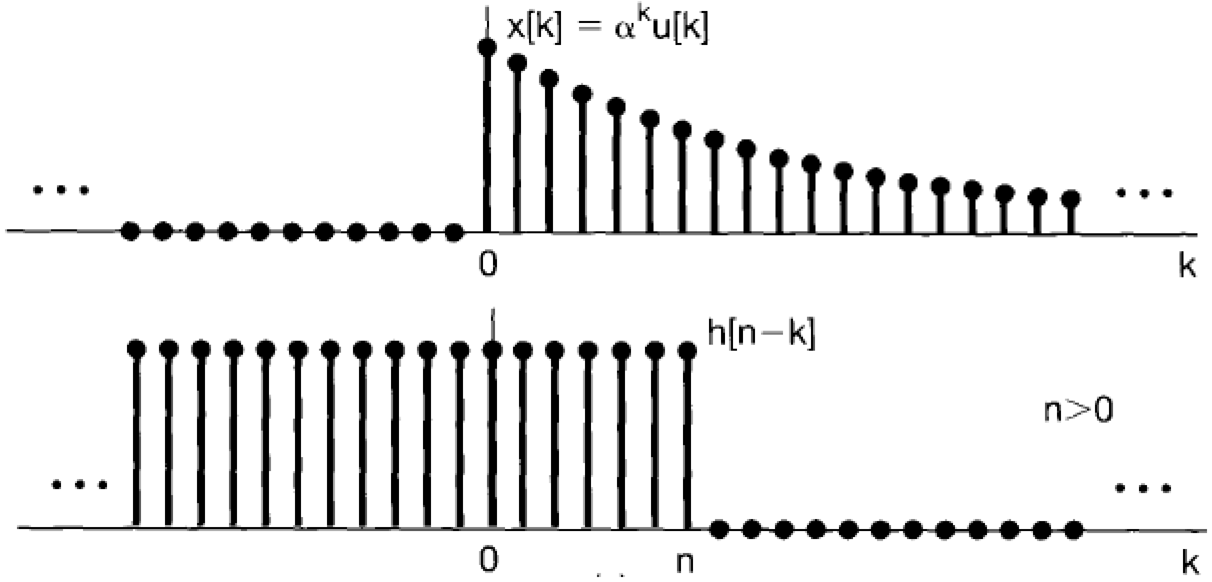
For $n < 0$, there is no overlap between the nonzero points in $x[k]$ and $h[n - k]$.
Thus, for $n < 0$, $x[k]h[n- k] = 0$ for all values of k.
$y[n] = 0$ for $n < 0$

Example 4 - continued



$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

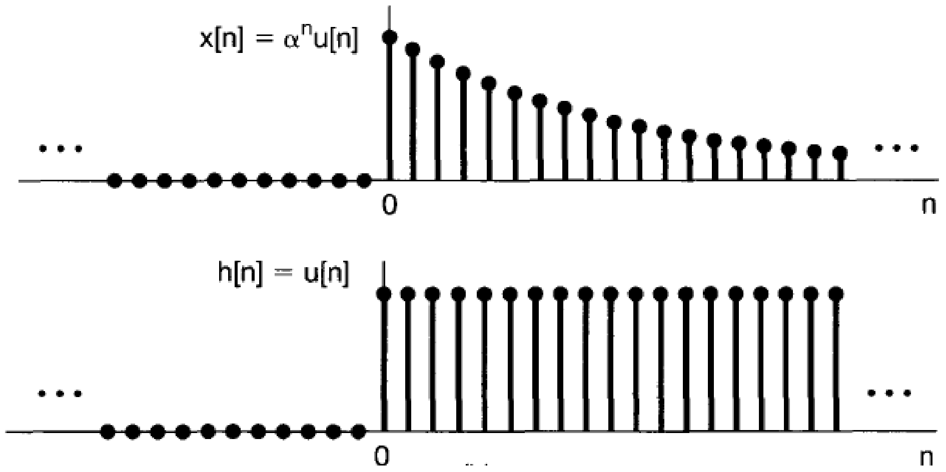
$n \geq 0$



For $n \geq 0$, $x[k]h[n-k] = \begin{cases} \alpha^k, & 0 \leq k \leq n \\ 0, & \text{otherwise} \end{cases}$

$$y[n] = \sum_{k=0}^{n} \alpha^k = \frac{1-\alpha^{n+1}}{1-\alpha} \qquad \text{for } n \geq 0.$$

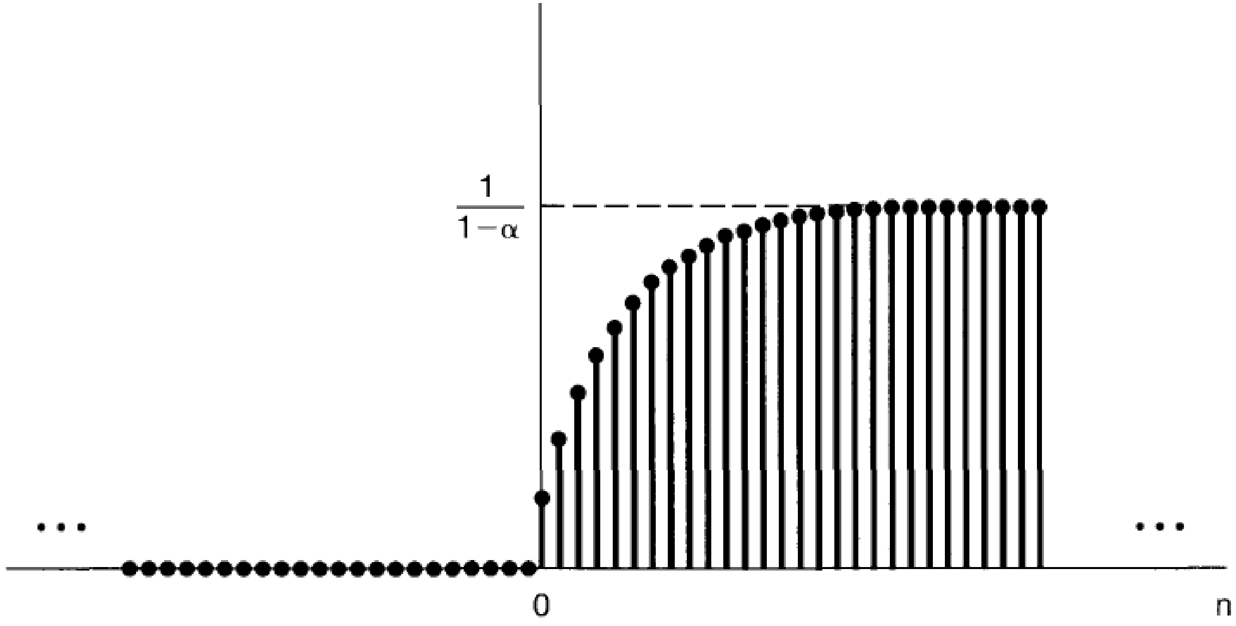
Example 4 - continued



$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k].$$

The resulting output signal $y[n]$ of the system:

$$y[n] = \left(\frac{1-\alpha^{n+1}}{1-\alpha}\right)u[n].$$



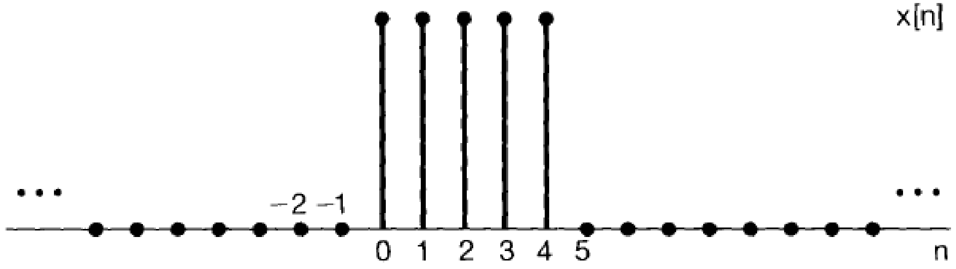
Example 5

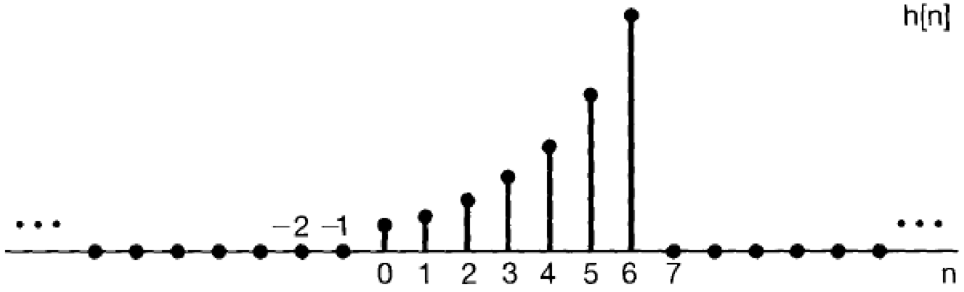
Consider an input $x[n]$ and a unit impulse response $h[n]$ given by

$$x[n] = \begin{cases} 1, & 0 \leq n \leq 4 \\ 0, & \text{otherwise} \end{cases}$$

$$h[n] = \begin{cases} \alpha^n, & 0 \leq n \leq 6 \\ 0, & \text{otherwise} \end{cases}$$

with $a > 1$





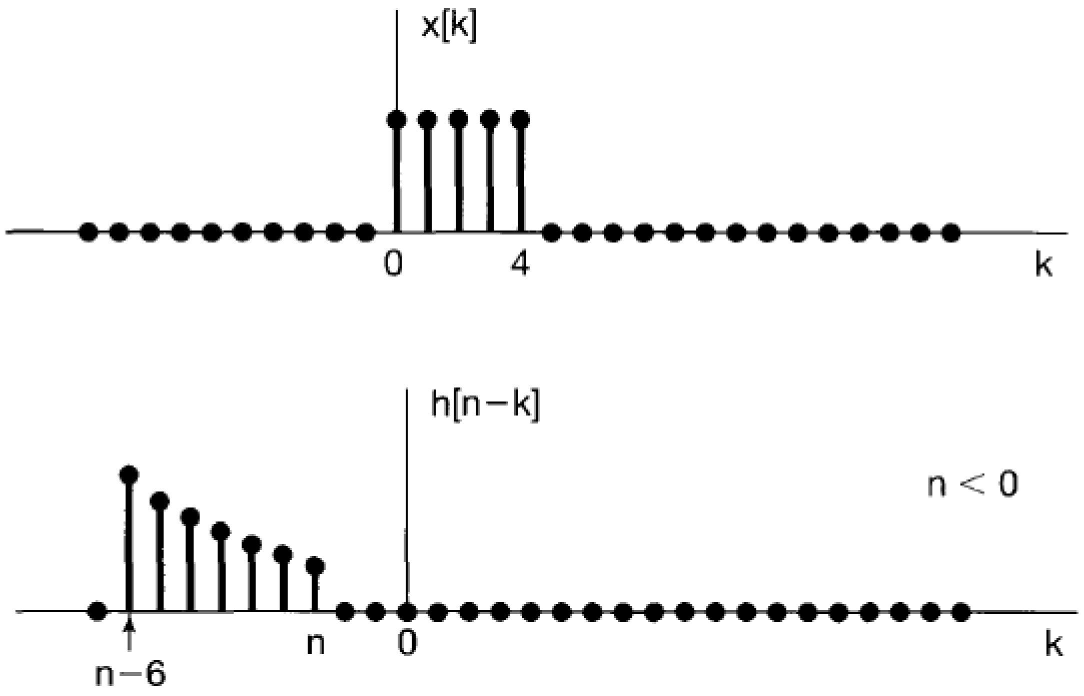
$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k].$$

In order to calculate the convolution of the two signals, it is convenient to consider five separate intervals for n.

Example 5 - continued

Interval 1.

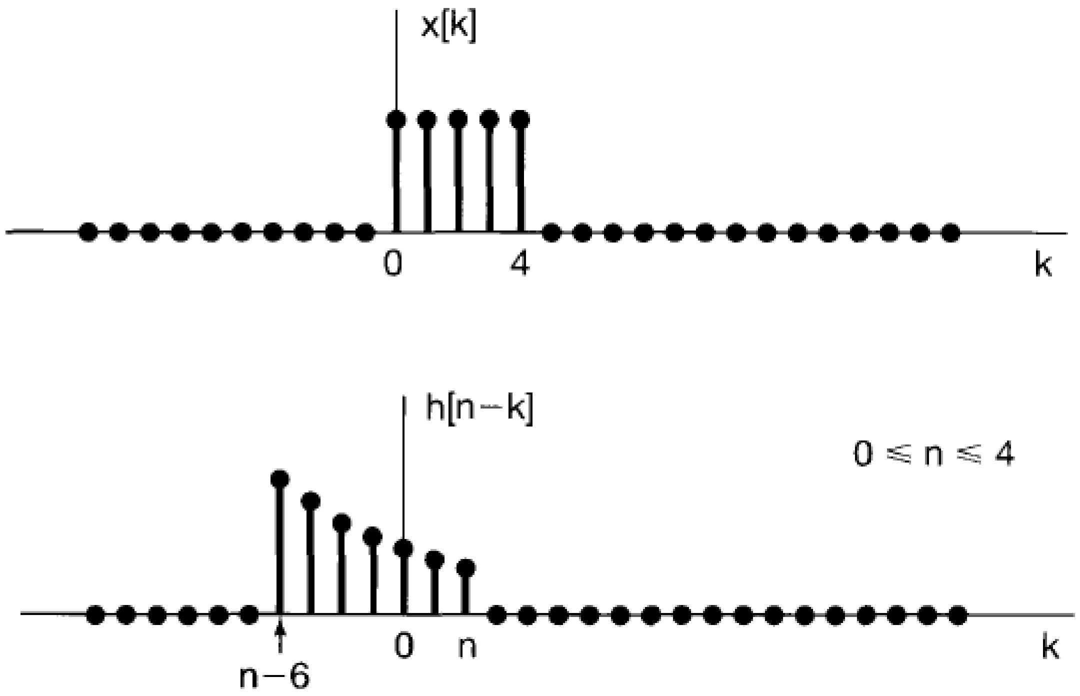
For $n < 0$, there is no overlap between the nonzero portions of $x[k]$ and $h[n - k]$, and consequently, $y[n] = 0$ for $n < 0$.



Example 5 - continued

Interval 2.

For $0 \le n \le 4$,



$$x[k]h[n-k] = \begin{cases} \alpha^{n-k}, & 0 \le k \le n \\ 0, & \text{otherwise} \end{cases}$$

$$y[n] = \sum_{k=0}^{n} \alpha^{n-k}, \qquad \text{for } 0 \le n \le 4$$

Changing the variable of summation from k to $r = n - k$, we obtain

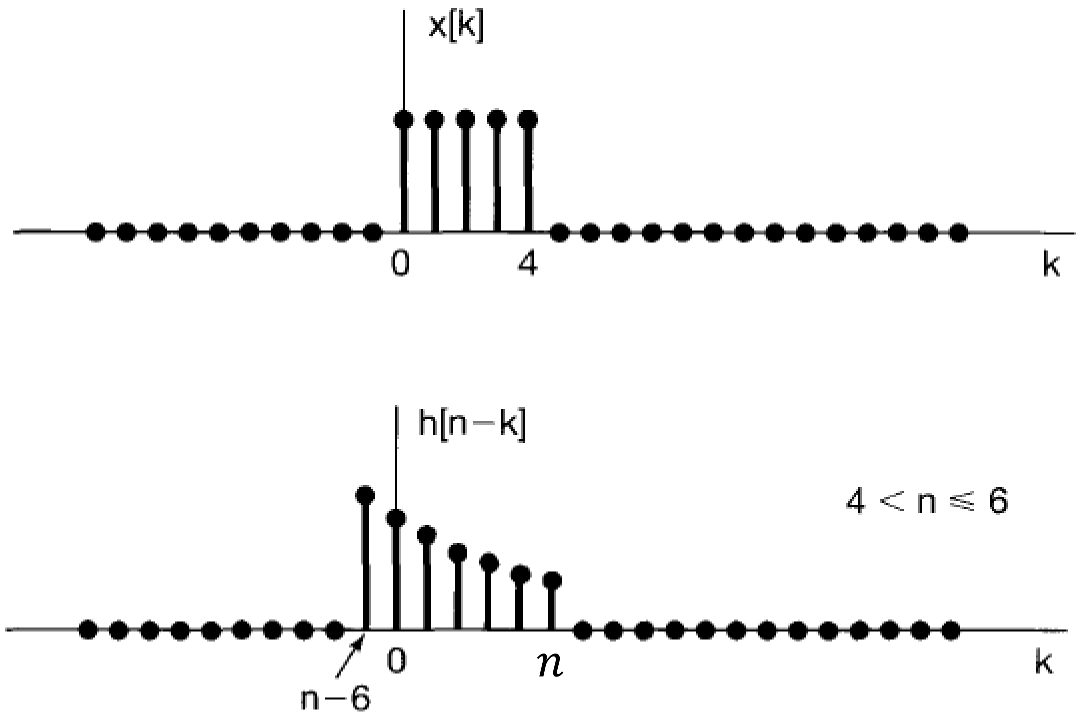
$$y[n] = \sum_{r=0}^{n} \alpha^{r} = \frac{1-\alpha^{n+1}}{1-\alpha}. \qquad \text{for } 0 \le n \le 4$$

Example 5 - continued

Interval 3.

For $4 < n \leq 6$,

$$x[k]h[n-k] = \begin{cases} \alpha^{n-k}, & 0 \leq k \leq 4 \\ 0, & \text{otherwise} \end{cases}$$



$$y[n] = \sum_{k=0}^{4} \alpha^{n-k} \quad \text{for } 4 < n \leq 6$$

$$y[n] = \alpha^n \sum_{k=0}^{4} (\alpha^{-1})^k = \alpha^n \frac{1-(\alpha^{-1})^5}{1-\alpha^{-1}}$$

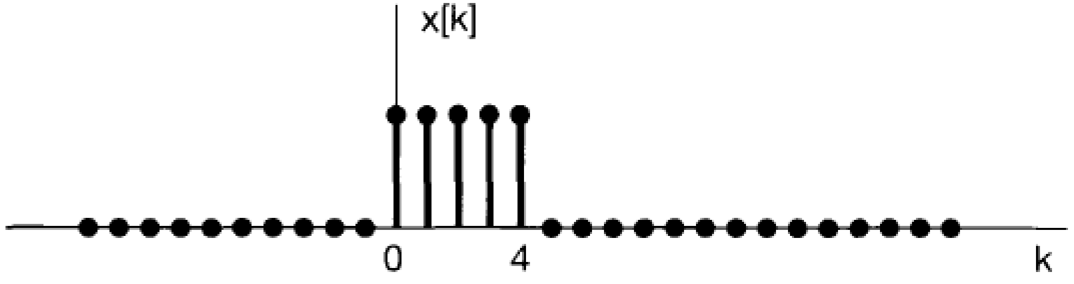
$$y[n] = \frac{\alpha^{n-4} - \alpha^{n+1}}{1-\alpha}. \quad \text{for } 4 < n \leq 6$$

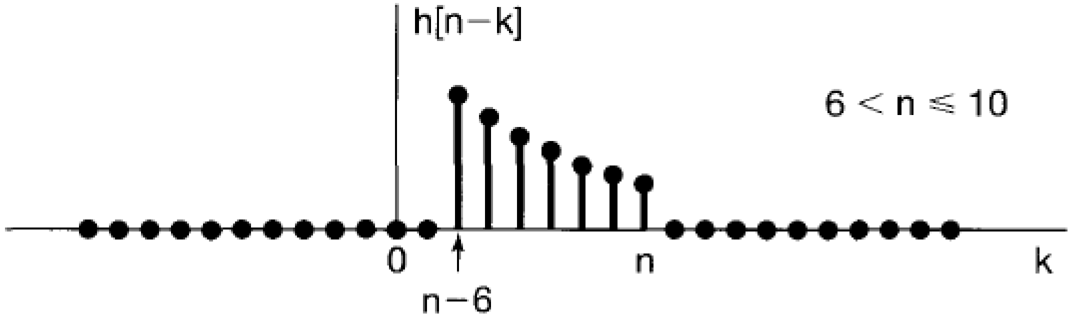
Example 5 - continued

Interval 4.

For $6 < n \le 10$,

$$x[k]h[n-k] = \begin{cases} \alpha^{n-k}, & (n-6) \le k \le 4 \\ 0, & \text{otherwise} \end{cases}$$





$$y[n] = \sum_{k=n-6}^{4} \alpha^{n-k}, \qquad \text{for } 6 < n \le 10$$

Letting $r = k - n + 6$, we obtain

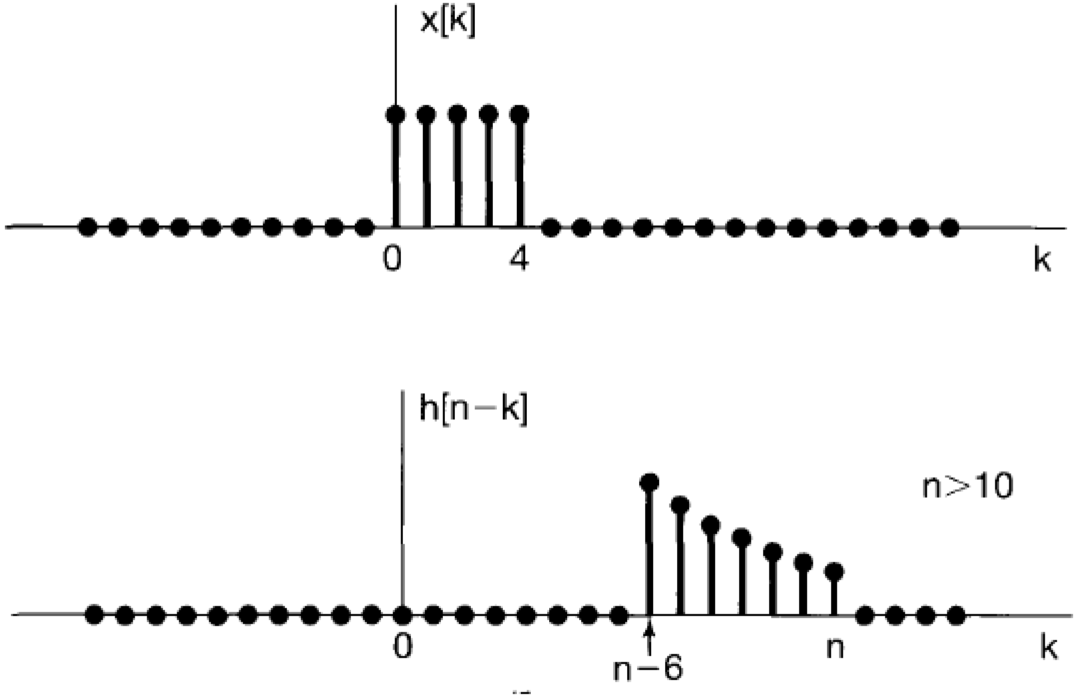
$$y[n] = \sum_{r=0}^{10-n} \alpha^{6-r} = \alpha^{6} \sum_{r=0}^{10-n} (\alpha^{-1})^{r} = \alpha^{6} \frac{1 - \alpha^{n-11}}{1 - \alpha^{-1}}$$

$$y[n] = \frac{\alpha^{n-4} - \alpha^{7}}{1 - \alpha} \qquad \text{for } 6 < n \le 10$$

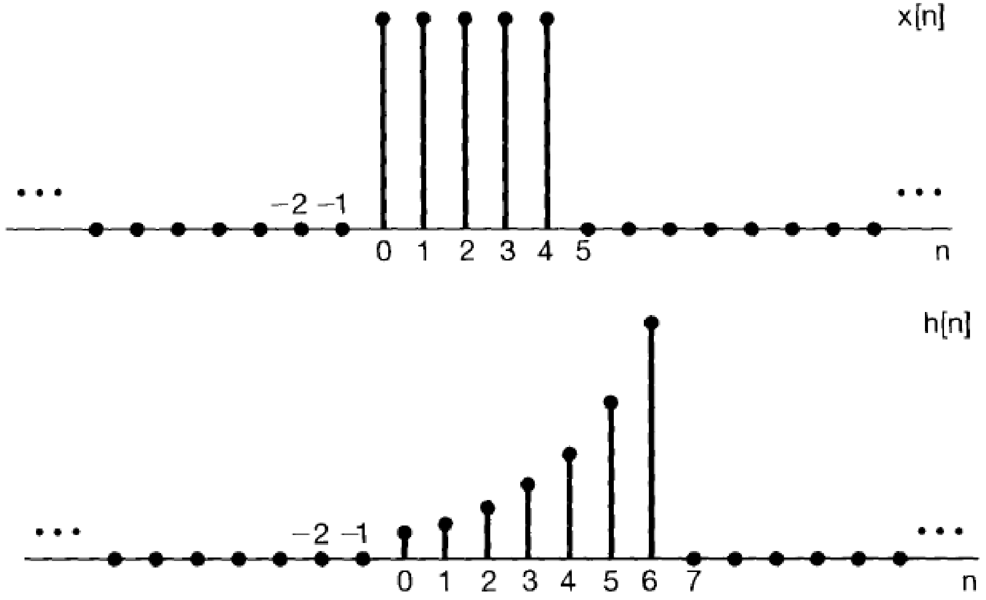
Example 5 - continued

Interval 5.

For $n > 10$, there is no overlap between the nonzero portions of $x[k]$ and $h[n - k]$, and consequently, $y[n] = 0$ for $n > 10$.



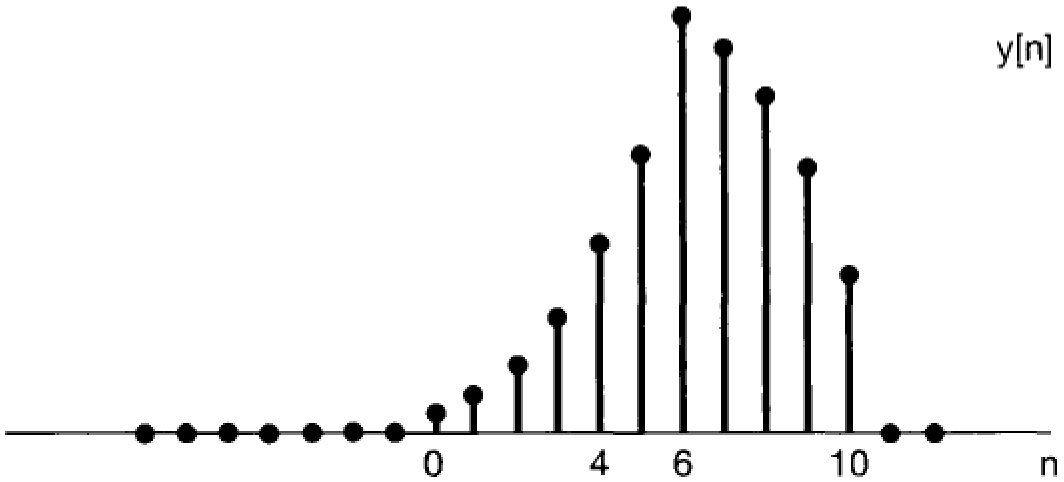
Example 5 - continued



$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k].$$

The resulting output signal $y[n]$ of the system:

$$y[n] = \begin{cases} 0, & n < 0 \\ \dfrac{1-\alpha^{n+1}}{1-\alpha}, & 0 \le n \le 4 \\ \dfrac{\alpha^{n-4}-\alpha^{n+1}}{1-\alpha}, & 4 < n \le 6 \\ \dfrac{\alpha^{n-4}-\alpha^{7}}{1-\alpha}, & 6 < n \le 10 \\ 0, & 10 < n \end{cases},$$



CONTINUOUS-TIME LTI SYSTEMS:

THE CONVOLUTION INTEGRAL

The Representation of Continuous-Time Signals in Terms of Impulses

A continuous-time signal can be viewed as a linear combination of continuous impulses:

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau$$

The Continuous-Time Unit Impulse Response and the Convolution Integral Representation of LTI Systems

$$\delta(t) \longrightarrow h(t)$$

$$\delta(t-\tau) \longrightarrow h_\tau(t)$$

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau \longrightarrow y(t) = \int_{-\infty}^{+\infty} x(\tau)h_\tau(t)d\tau.$$

Let $h_\tau(t)$ denote the response at time t to a unit impulse $\delta(t-\tau)$ located at time τ.

Then, for a linear system the output y(t) to an input signal x(t) can be obtained using the following integral:

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h_\tau(t)d\tau.$$

The Continuous-Time Unit Impulse Response

$$\delta(t) \longrightarrow h(t)$$

$$\delta(t-\tau) \longrightarrow h_\tau(t)$$

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau \longrightarrow y(t) = \int_{-\infty}^{+\infty} x(\tau)h_\tau(t)d\tau.$$

If, in addition to being linear, the system is also time invariant, then $h_\tau(t) = h(t-\tau)$ i.e., the response of an LTI system to the unit impulse $\delta(t-\tau)$, which is shifted by τ seconds from the origin, is a similarly shifted version of the response to the unit impulse function $\delta(t)$.

$h(t)$ is the response of the system to the input $\delta(t)$
An LTI system is completely determined by its impulse response.

The Convolution Integral

Since, by time-invariance, $h_\tau(t) = h(t-\tau)$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h_\tau(t)d\tau.$$

becomes

$$\boxed{y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau.}$$

This equation is referred to as the *convolution integral* or the *superposition integral*

The convolution of two signals $x(t)$ and $h(t)$ will be represented symbolically as

$$y(t) = x(t) * h(t).$$

Convolution Integral

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau.$$

To evaluate the convolution integral

for a fixed t

- draw the signals $x(\tau)$ and $h(t-\tau)$ as functions of τ
- multiply them to form the signal $g(\tau)$

$$g(\tau) = x(\tau)h(t-\tau)$$

- integrate $g(\tau)$ from $\tau = -\infty$ to $\tau = \infty$

The result of the integration is the output value at the selected time t.

Example 1

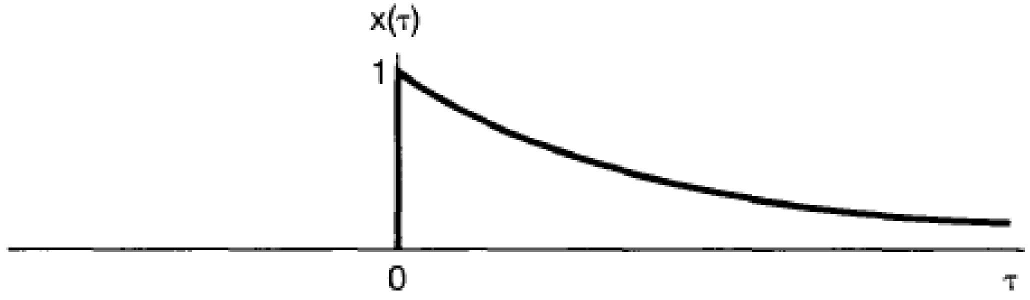
Let $x(t)$ be the input to an LTI system with unit impulse response $h(t)$, where

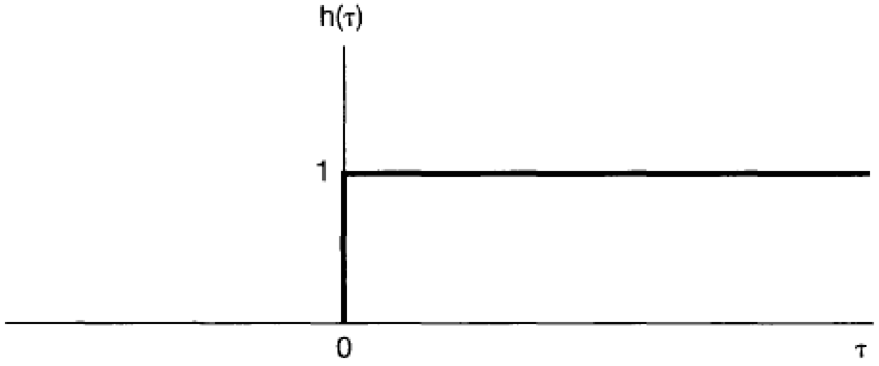
$$x(t) = e^{-at}u(t), \quad a > 0$$

$$h(t) = u(t)$$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau.$$

To find $y(t)$, first sketch $x(\tau)$ and $h(t-\tau)$ as functions of τ

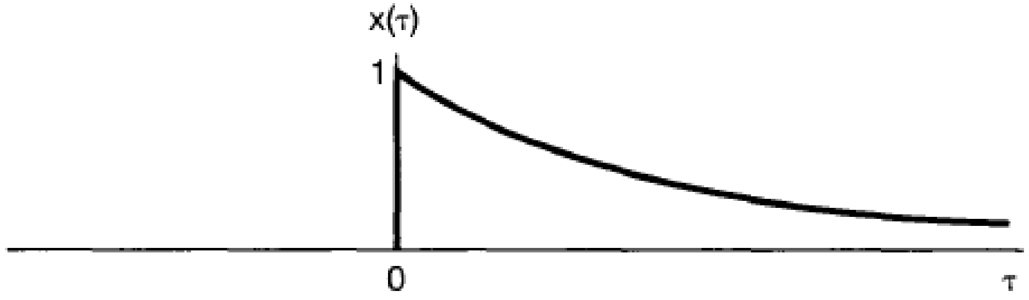


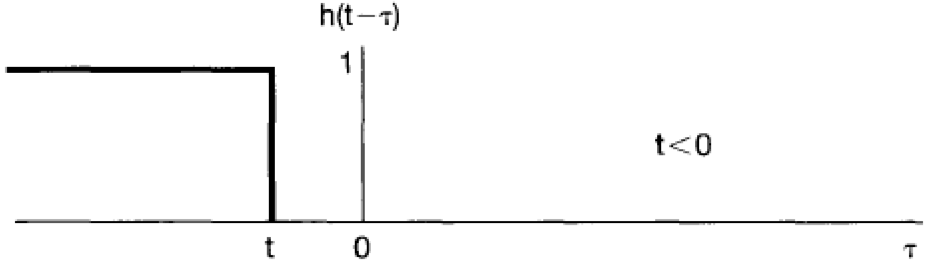


Example 1 - continued

For a fixed t, draw the signal $h(t-\tau)$ from $h(\tau)$ by a reflection about the origin and a shift to the right by t if $t > 0$ or a shift to the left by $|t|$ for $t < 0$.

For $t < 0$





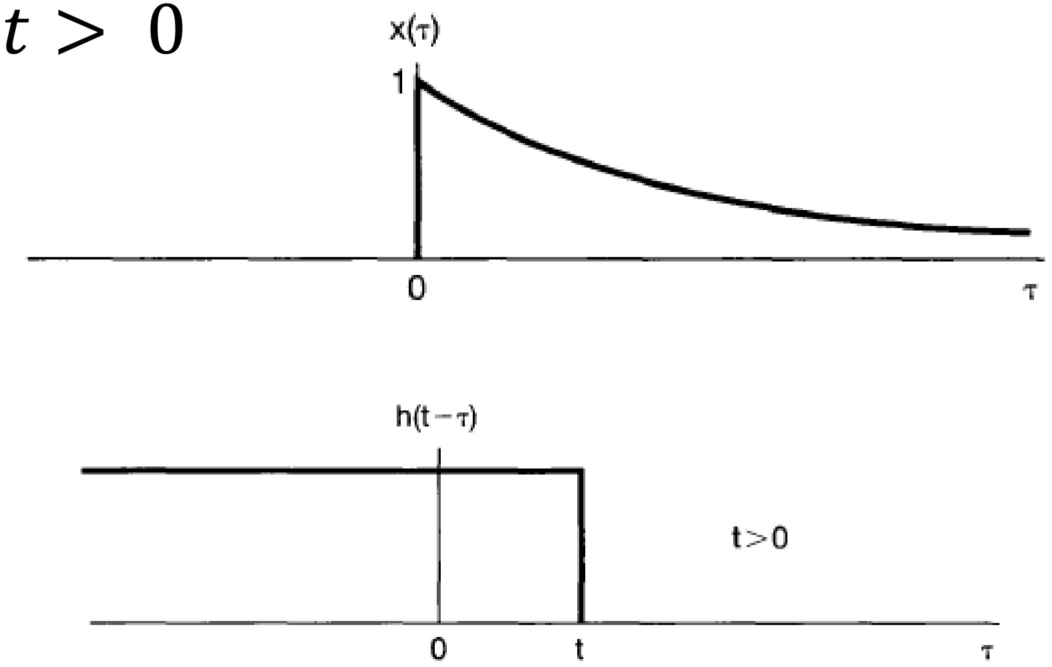
For $t < 0$, the product of $x(\tau)$ and $h(t-\tau)$ is zero, and consequently, $y(t)$ is zero.

$y(t) = 0 \quad$ for $t < 0$

Example 1 - continued

For a fixed t, draw the signal $h(t-\tau)$ from $h(\tau)$ by a reflection about the origin and a shift to the right by t if $t > 0$ or a shift to the left by $|t|$ for $t < 0$.

For $t > 0$



For $t > 0$, the product of $x(\tau)$ and $h(t-\tau)$ is

$$x(\tau)h(t-\tau) = \begin{cases} e^{-a\tau}, & 0 < \tau < t \\ 0, & \text{otherwise} \end{cases}$$

$$y(t) = \int_0^t e^{-a\tau} d\tau = -\frac{1}{a} e^{-a\tau}\Big|_0^t$$

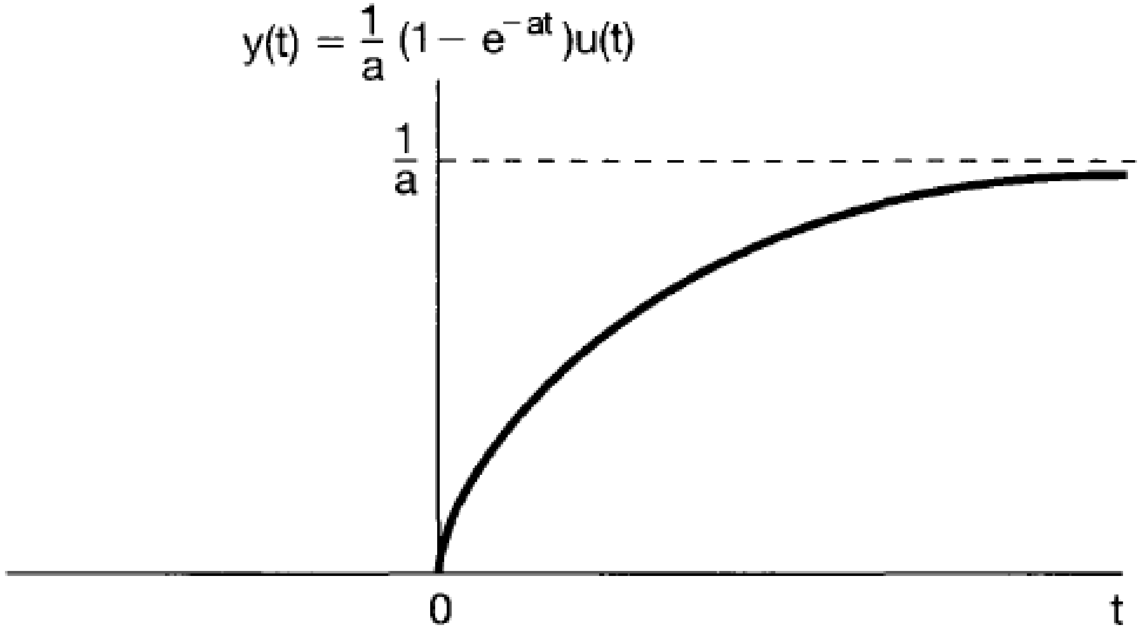
$$= \frac{1}{a}(1 - e^{-at})$$

for $t > 0$

Example 1 - continued

The output of the system for all t:

$$y(t) = \frac{1}{a}(1 - e^{-at})u(t),$$



Example 2

Let $x(t)$ be the input to an LTI system with unit impulse response $h(t)$, where

$$x(t) = \begin{cases} 1, & 0 < t < T \\ 0, & \text{otherwise} \end{cases},$$

$$h(t) = \begin{cases} t, & 0 < t < 2T \\ 0, & \text{otherwise} \end{cases}$$

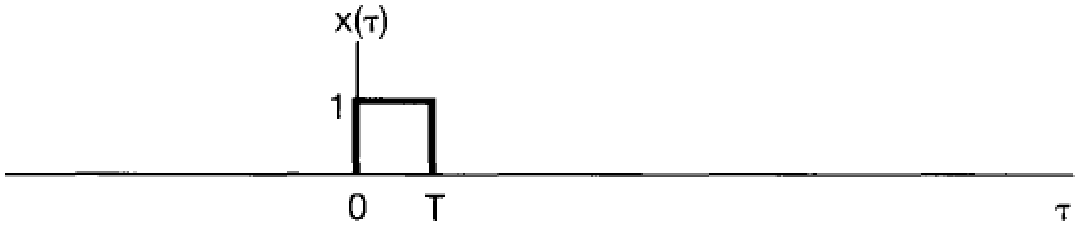
It is convenient to consider the evaluation of $y(t)$ in separate intervals.

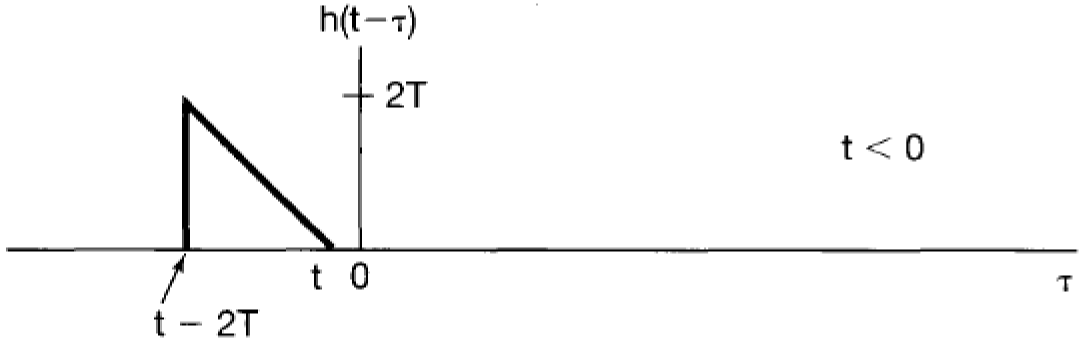
We will consider five intervals in this example:

Interval 1: $t < 0$
Interval 2: $0 < t < T$
Interval 3: $T < t < 2T$
Interval 4: $2T < t < 3T$
Interval 5: $3T < t$

Example 2 - continued

Interval 1: $t < 0$



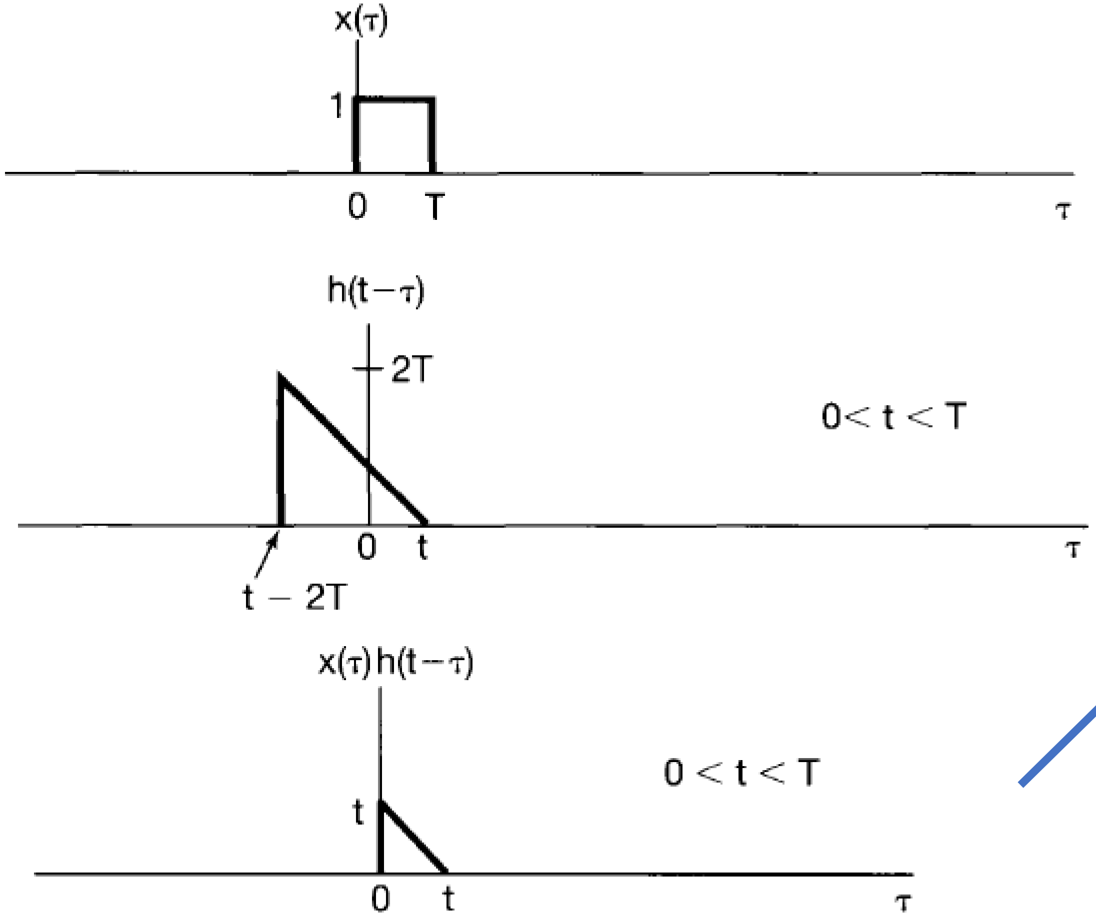


For $t < 0$, the product of $x(\tau)$ and $h(t-\tau)$ is zero, and consequently, $y(t)$ is zero.

$$y(t) = 0 \quad \text{for } t < 0$$

Example 2 - continued

Interval 2: $0 < t < T$



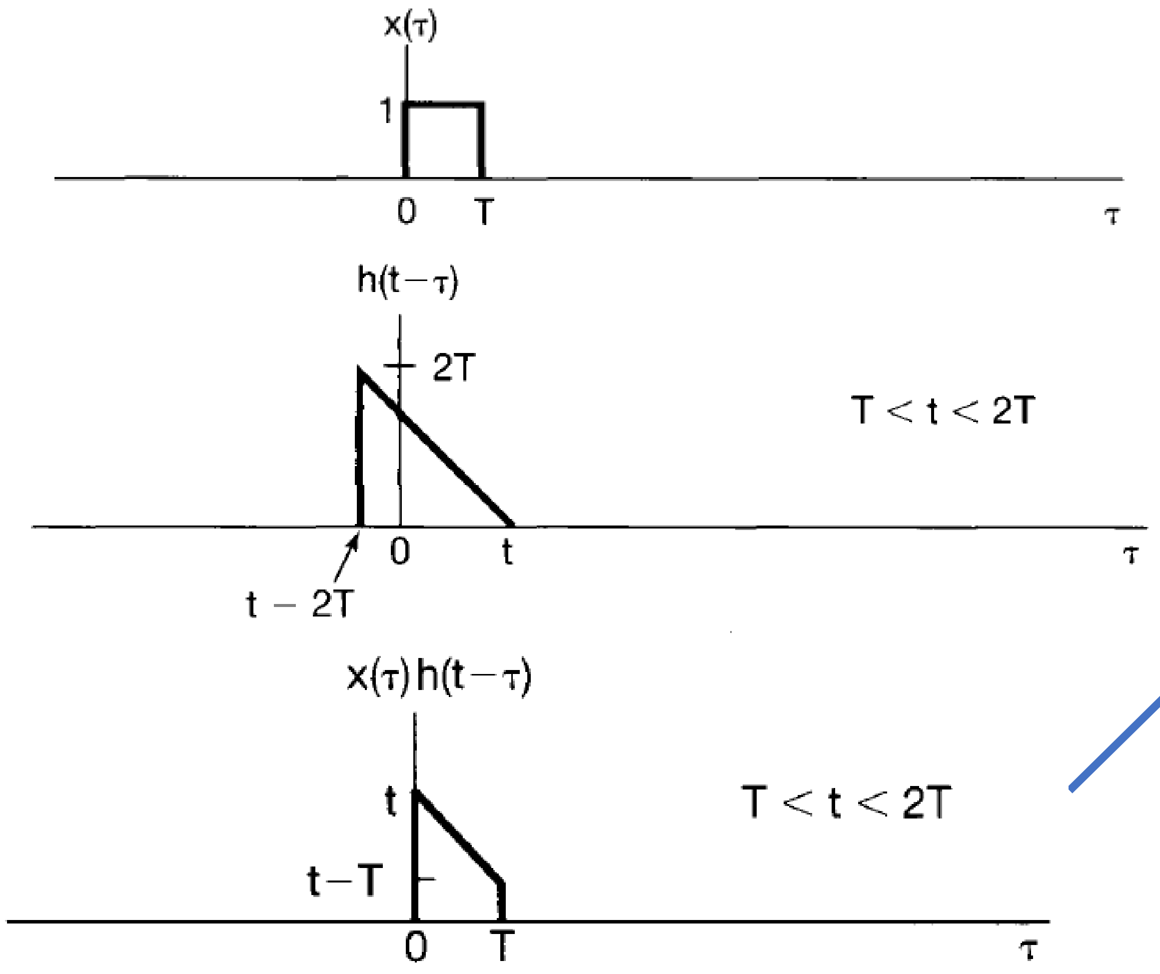
Integrate $x(\tau)h(t-\tau)$ from $\tau = -\infty$ to $\tau = \infty$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau.$$

$$y(t) = \frac{1}{2}t^2 \quad \text{for } 0 < t < T$$

Example 2 - continued

Interval 3: $T < t < 2T$



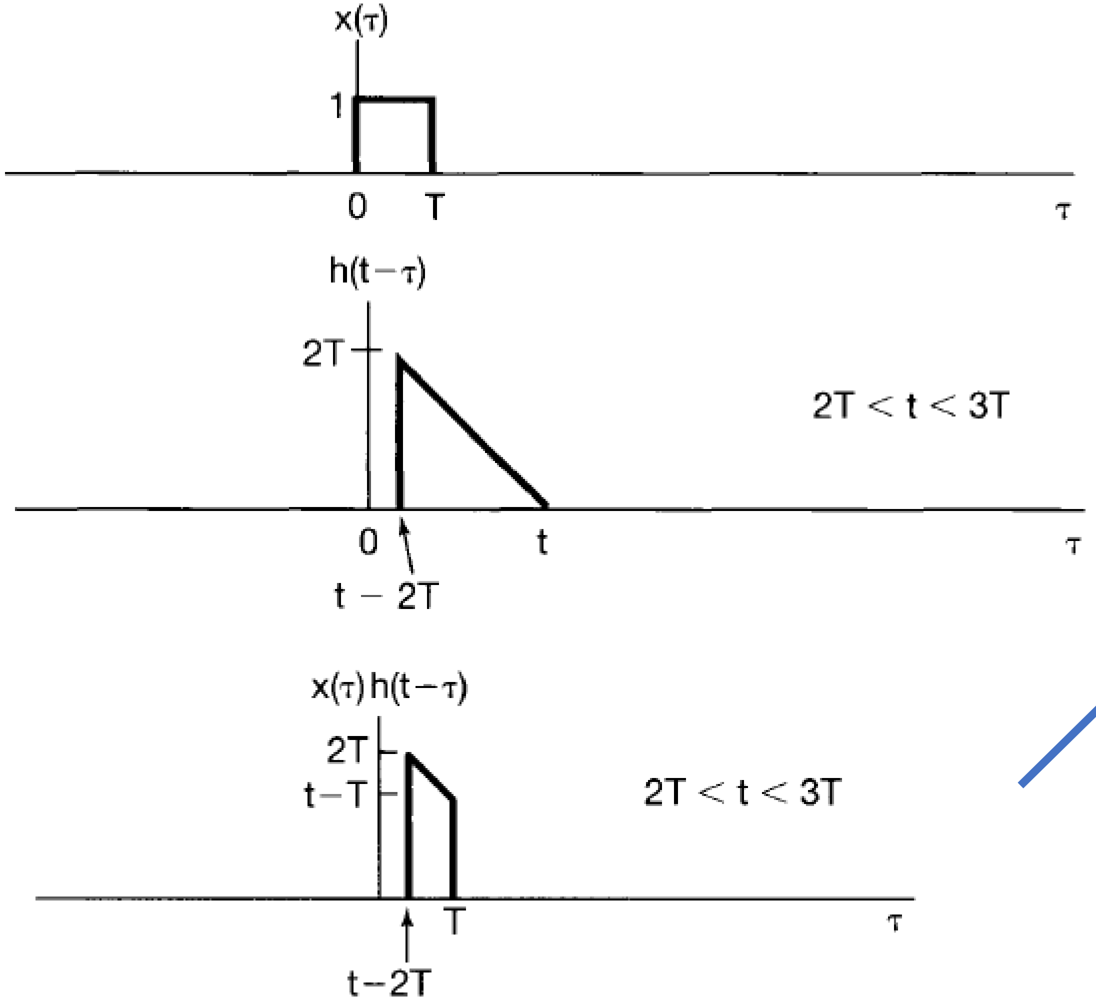
Integrate $x(\tau)h(t-\tau)$ from $\tau = -\infty$ to $\tau = \infty$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau.$$

$$y(t) = Tt - \frac{1}{2}T^2 \quad \text{for } T < t < 2T$$

Example 2 - continued

Interval 4: $2T < t < 3T$



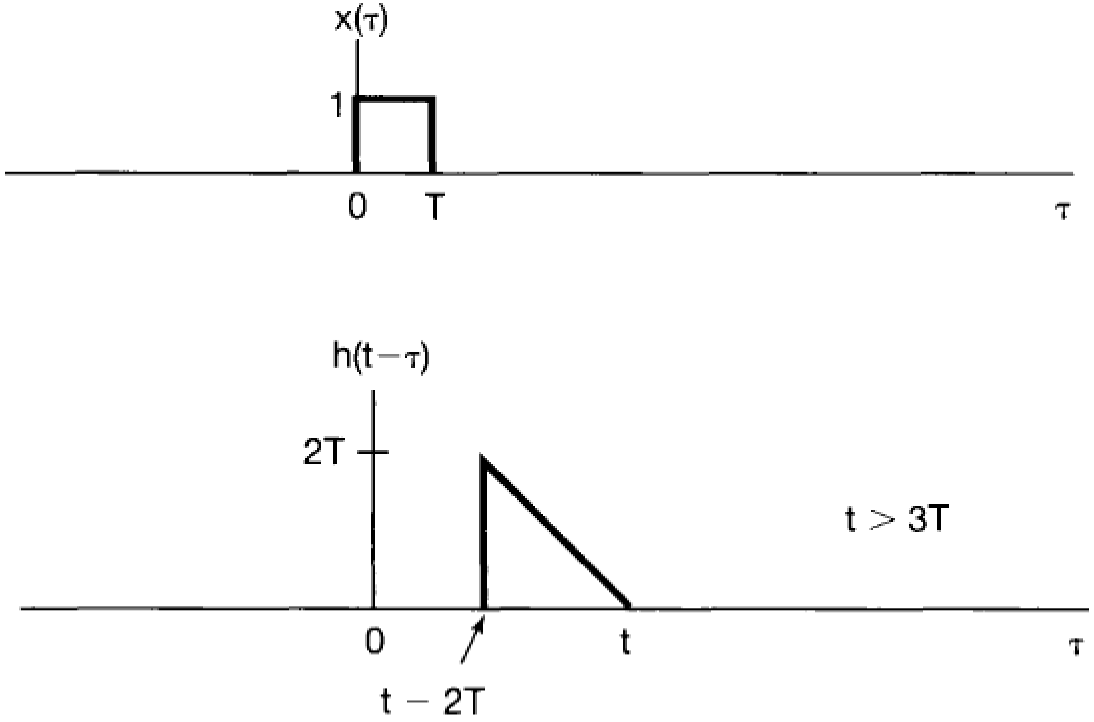
Integrate $x(\tau)h(t-\tau)$ from $\tau = -\infty$ to $\tau = \infty$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau.$$

$$y(t) = -\frac{1}{2}t^2 + Tt + \frac{3}{2}T^2 \quad \text{for } 2T < t < 3T$$

Example 2 - continued

Interval 5: $t > 3T$



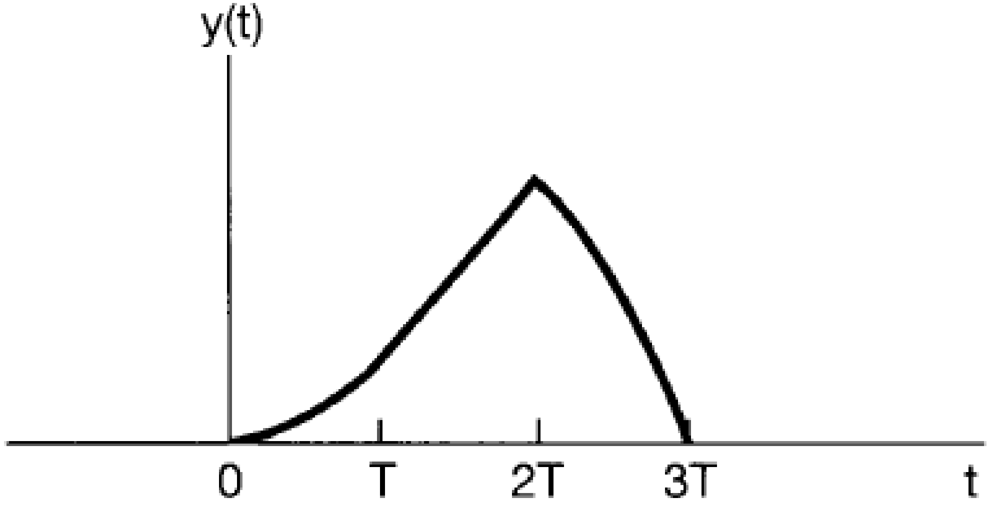
For $t > 3T$, the product of $x(\tau)$ and $h(t-\tau)$ is zero, and consequently, $y(t)$ is zero.

$$y(t) = 0 \quad \text{for } t > 3T$$

Example 2 - continued

The output of the system:

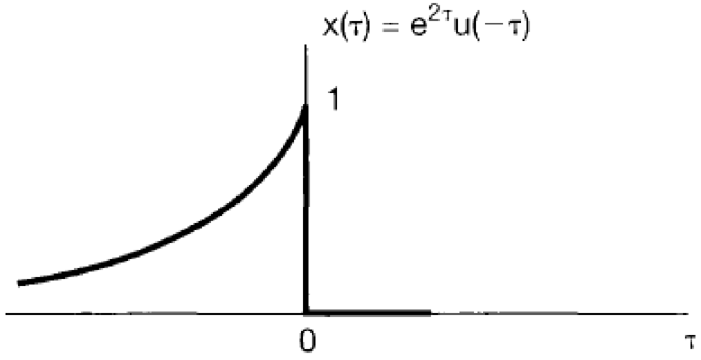
$$y(t) = \begin{cases} 0, & t < 0 \\ \frac{1}{2}t^2, & 0 < t < T \\ Tt - \frac{1}{2}T^2, & T < t < 2T \\ -\frac{1}{2}t^2 + Tt + \frac{3}{2}T^2, & 2T < t < 3T \\ 0, & 3T < t \end{cases},$$

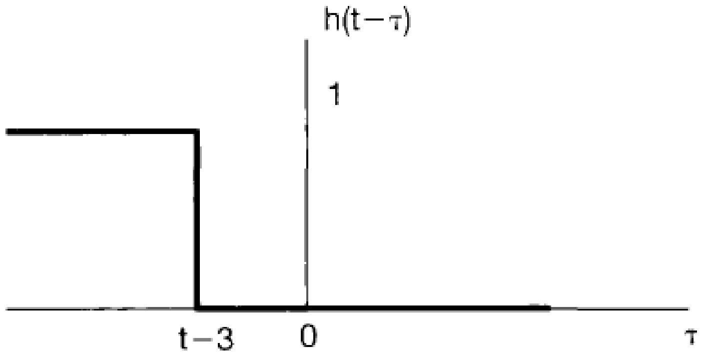


Example 3

Let $y(t)$ denote the convolution of the following two signals:

$$x(t) = e^{2t}u(-t),$$
$$h(t) = u(t-3).$$





When $t-3 \leq 0$, the product of $x(\tau)$ and $h(t-\tau)$ is nonzero for $-\infty < \tau < t-3$, and the convolution integral becomes

$$y(t) = \int_{-\infty}^{t-3} e^{2\tau} d\tau = \frac{1}{2} e^{2(t-3)}.$$

When $t-3 \geq 0$, the product of $x(\tau)$ and $h(t-\tau)$ is is nonzero for $-\infty < \tau < 0$, and the convolution integral becomes

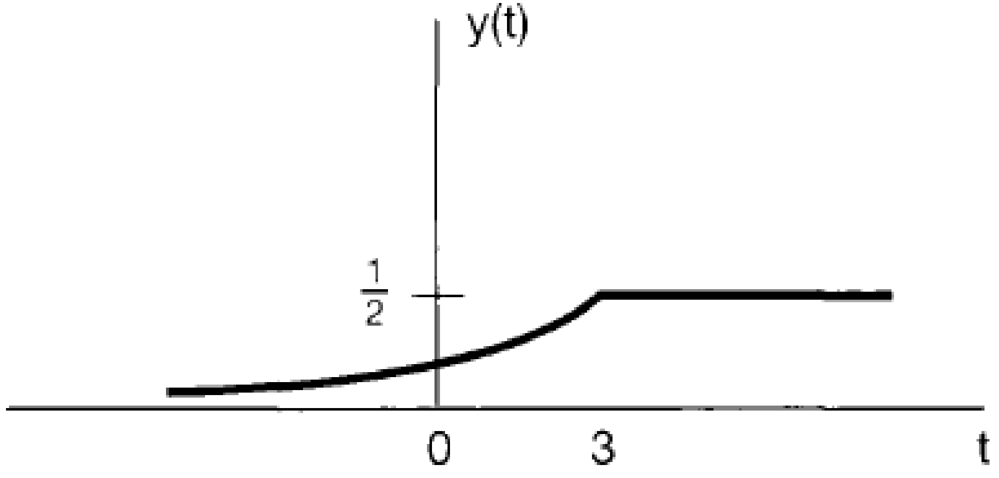
$$y(t) = \int_{-\infty}^{0} e^{2\tau} d\tau = \frac{1}{2}.$$

Example 3 - continued

Let $y(t)$ denote the convolution of the following two signals:

$$x(t) = e^{2t}u(-t),$$
$$h(t) = u(t-3).$$

The resulting signal $y(t)$:



Properties of LTI systems and convolution

Convolution

Continuous-time and discrete-time LTI systems can be represented in terms of their unit impulse responses.

In discrete time the representation takes the form of the convolution sum, while its continuous-time counterpart is the convolution integral.

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = x[n] * h[n]$$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$

The Commutative Property

Convolution in both continuous and discrete time is a commutative operation.

Discrete time:

$$x[n] * h[n] = h[n] * x[n] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k],$$

$$x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = \sum_{r=-\infty}^{+\infty} x[n-r]h[r] = h[n] * x[n]$$

Continuous time:

$$x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau.$$

The Distributive Property

Convolution in both continuous and discrete time has the distributive property.
Convolution distributes over addition

Discrete time:

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$$

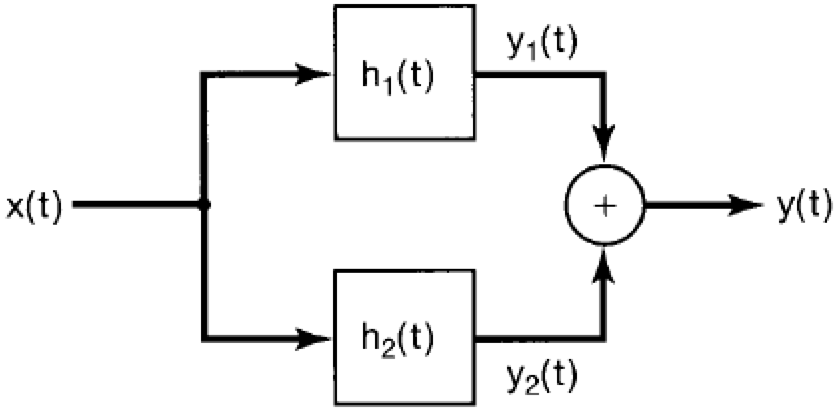
Continuous time:

$$x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t).$$

The Distributive Property

The distributive property has a useful interpretation in terms of system interconnections.

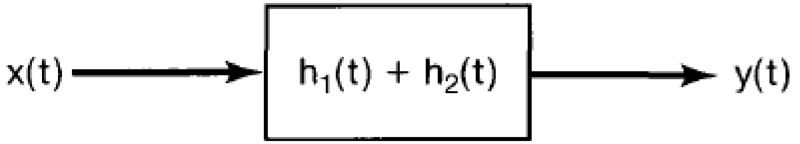
Consider two continuous-time LTI systems in parallel with impulse responses $h_1(t)$ and $h_2(t)$.



$$y_1(t) = x(t) * h_1(t)$$

$$y_2(t) = x(t) * h_2(t)$$

$$y(t) = x(t) * h_1(t) + x(t) * h_2(t)$$



$$\begin{aligned} y(t) &= x(t) * [h_1(t) + h_2(t)] \\ &= x(t) * h_1(t) + x(t) * h_2(t). \end{aligned}$$

Due to the distributive property

The two systems are identical.

Both in continuous time and discrete time, a parallel combination of LTI systems can be replaced by a single LTI system whose unit impulse response is the sum of the individual unit impulse responses in the parallel combination.

The Associative Property

Convolution in both continuous and discrete time is associative.

Discrete time:

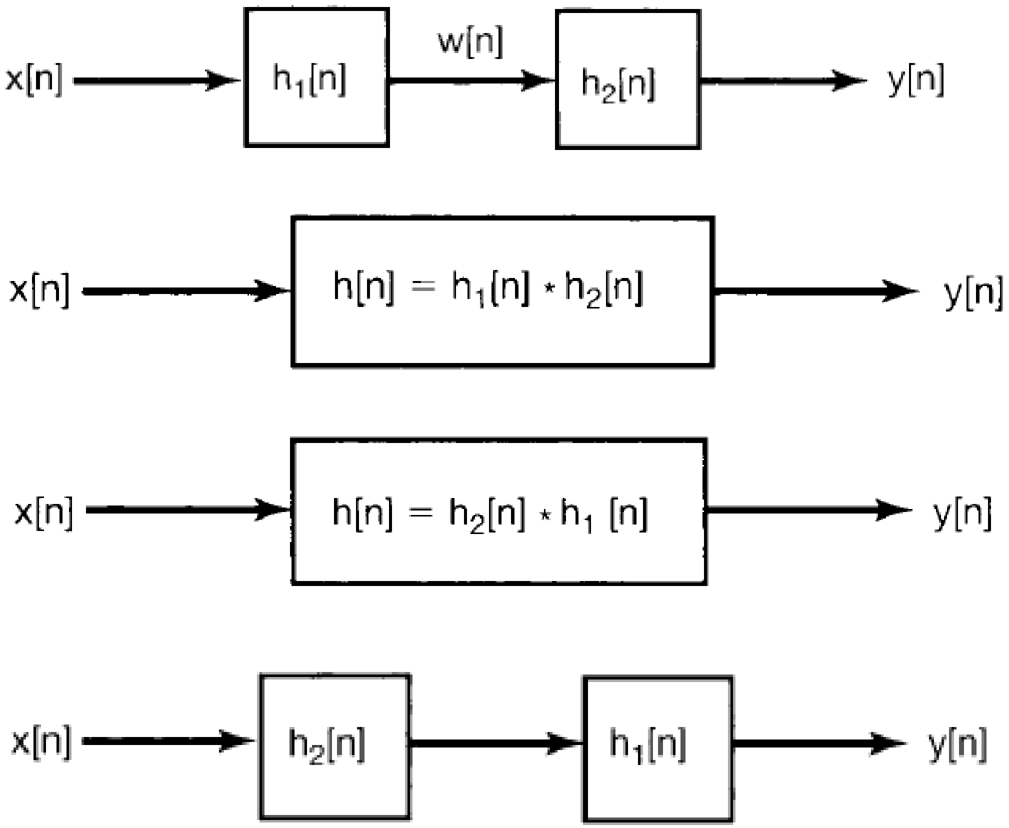
$$x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n]$$

Continuous time:

$$x(t) * [h_1(t) * h_2(t)] = [x(t) * h_1(t)] * h_2(t)$$

The Associative Property

The impulse response of the cascade of two LTI systems is the convolution of their individual impulse responses.



$$\begin{aligned} y[n] &= w[n] * h_2[n] \\ &= (x[n] * h_1[n]) * h_2[n]. \end{aligned}$$

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= x[n] * (h_1[n] * h_2[n]) \end{aligned}$$

The unit impulse response of a cascade of two LTI systems does not depend on the order in which they are cascaded.

In fact, this holds for an arbitrary number of LTI systems in cascade: The order in which they are cascaded does not matter as far as the overall system impulse response is concerned.

The same conclusions hold in continuous time as well.

All four systems are equivalent.