Signals and Systems

Lecture 5

FOURIER SERIES REPRESENTATION OF CONTINUOUS-TIME PERIODIC SIGNALS

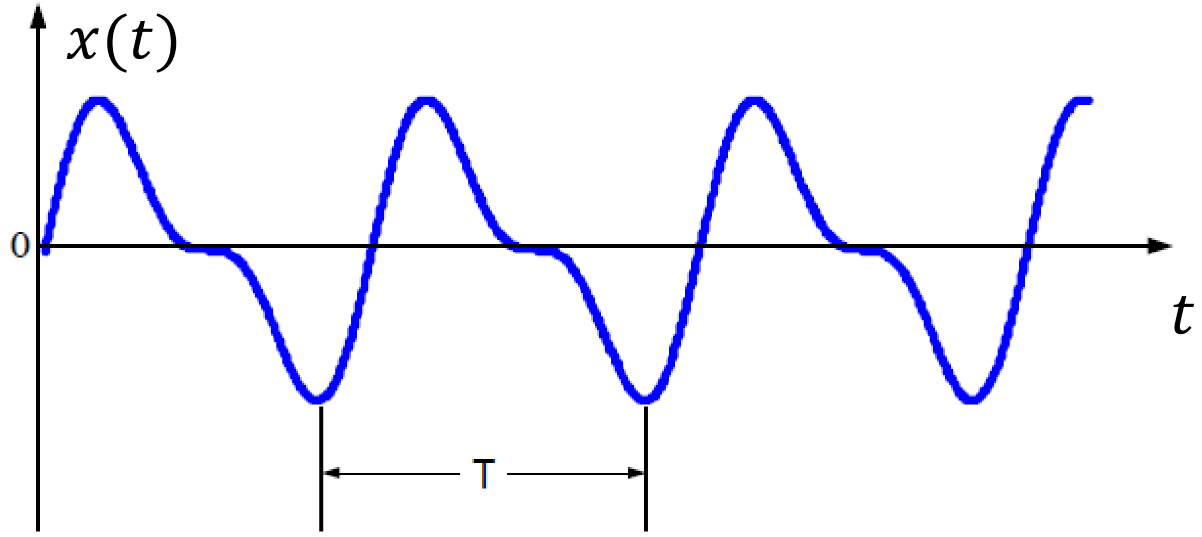
Continuous-time periodic signals

Remember periodic signals. A signal is periodic if, for some positive value of T,

$$x(t) = x(t+T) \qquad \text{for all } t$$

The fundamental period of $x(t)$ is the minimum positive, nonzero value of T for which the above equations is satisfied.
The value $\omega_0 = 2\pi/T$ is referred to as the fundamental frequency.

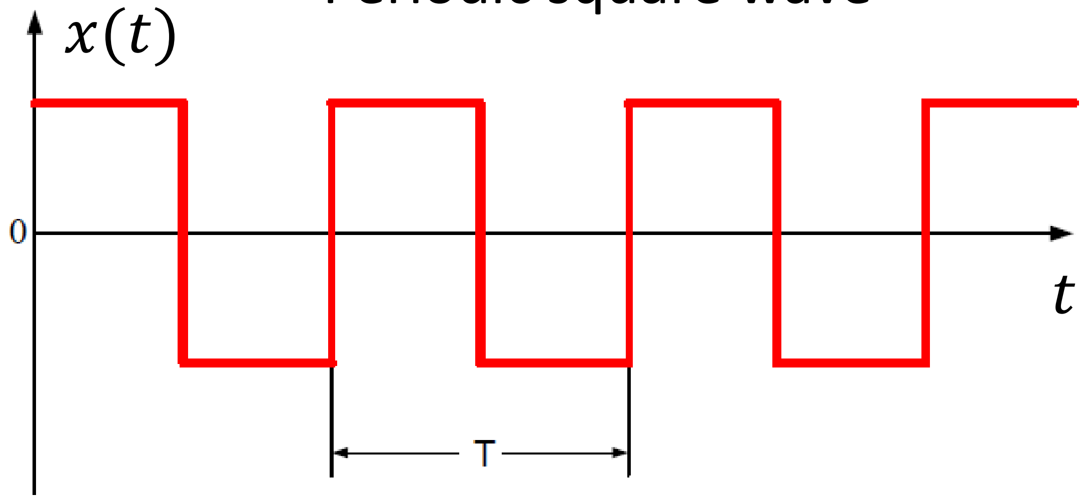
Example



Continuous-time periodic signals

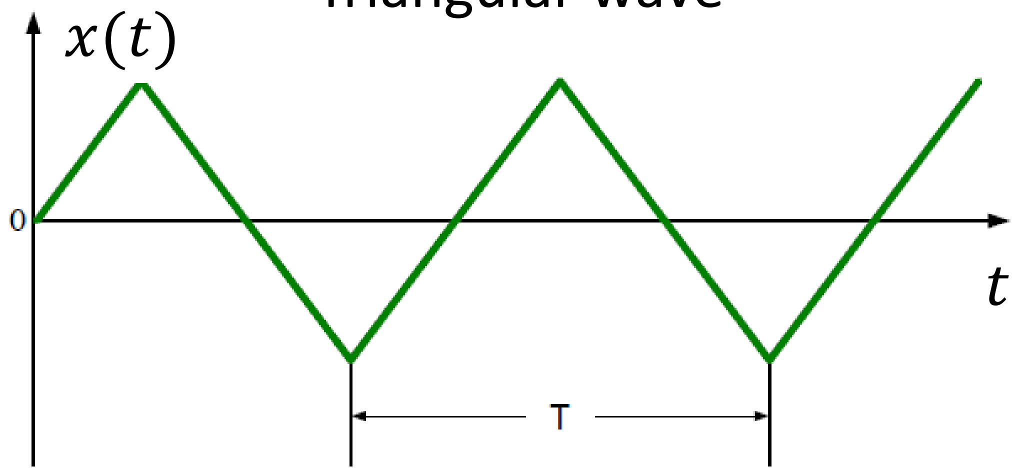
Example

Periodic square wave



Example

Triangular wave



Harmonically Related Complex Exponentials

Remember the two basic periodic signals:

The sinusoidal signal: $x(t) = \cos \omega_0 t$

The periodic complex exponential: $x(t) = e^{j\omega_0 t}$

Both of these signals are periodic with fundamental frequency ω_0 and fundamental period $T = 2\pi/\omega_0$.
Associated with the periodic complex exponential is the set of *harmonically related* complex exponentials:

$$\phi_k(t) = e^{jk\omega_0 t} = e^{jk(2\pi/T)t}, \quad k = 0, \pm 1, \pm 2, \ldots.$$

Each of these signals has a fundamental frequency that is a multiple of ω_0, and therefore, each is periodic with period T.

Linear Combinations of Harmonically Related Complex Exponentials

The set of *harmonically related* complex exponentials:

$$\phi_k(t) = e^{jk\omega_0 t} = e^{jk(2\pi/T)t}, \quad k = 0, \pm 1, \pm 2, \ldots .$$

Each of these signals has a fundamental frequency that is a multiple of ω_0 , and therefore, each is periodic with period T.
A linear combination of harmonically related complex exponentials is also periodic with T.

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$$

Linear Combinations of Harmonically Related Complex Exponentials

A linear combination of harmonically related complex exponentials is also periodic with T.

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$$

- The term for $k = 0$ is a constant signal.
- The terms for $k = +1$ and $k = -1$ both have fundamental frequency equal to ω_0 and are collectively referred to as the *fundamental components* or the *first harmonic components.*
- The two terms for $k = +2$ and $k = -2$ are periodic with half the period (or, equivalently, twice the frequency) of the fundamental components and are referred to as the *second harmonic components.*
- More generally, the components for $k = +N$ and $k = -N$ are referred to as the *Nth* harmonic components.

Linear Combinations of Harmonically Related Complex Exponentials

Example

Consider a periodic signal $x(t)$, with period $T = 1$ and with fundamental frequency $\omega_0 = 2\pi$, that is expressed in the form

$$x(t) = \sum_{k=-3}^{+3} a_k e^{jk2\pi t}$$

$$a_0 = 1,$$

$$a_1 = a_{-1} = \frac{1}{4},$$

$$a_2 = a_{-2} = \frac{1}{2},$$

$$a_3 = a_{-3} = \frac{1}{3}.$$

$$x(t) = 1 + \frac{1}{4}(e^{j2\pi t} + e^{-j2\pi t}) + \frac{1}{2}(e^{j4\pi t} + e^{-j4\pi t}) + \frac{1}{3}(e^{j6\pi t} + e^{-j6\pi t}).$$

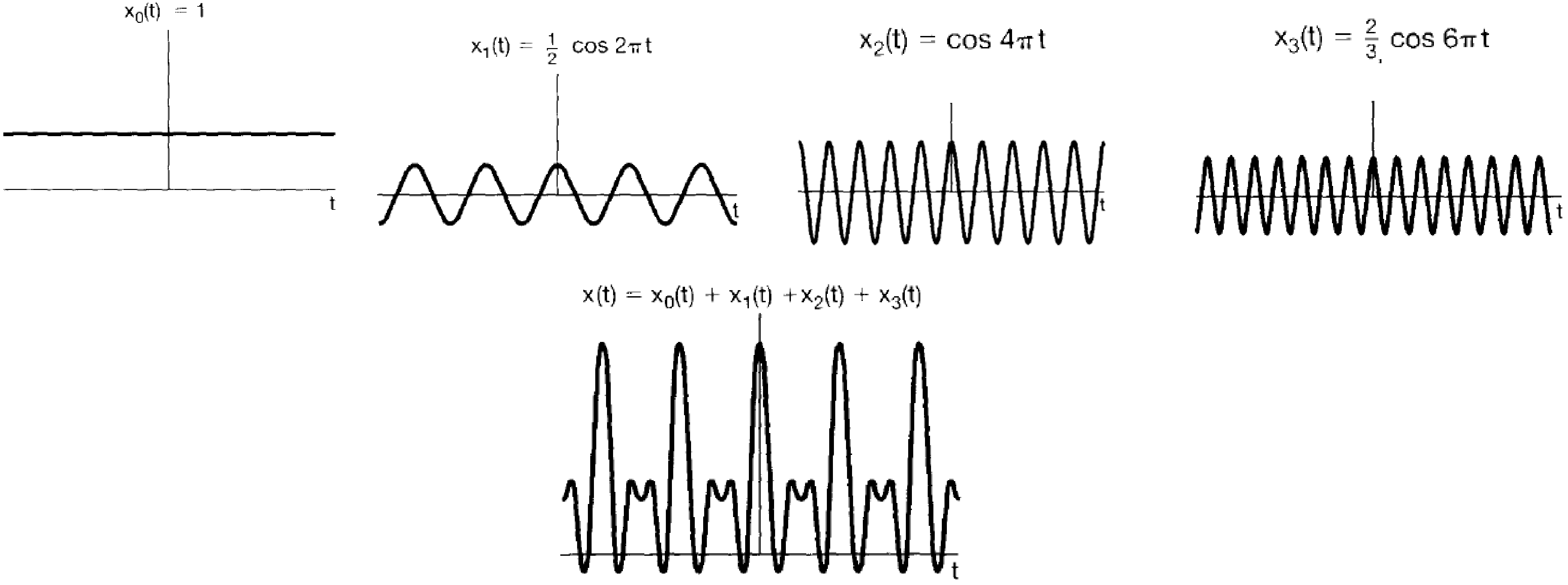
Using Euler's relation:

$$x(t) = 1 + \frac{1}{2}\cos 2\pi t + \cos 4\pi t + \frac{2}{3}\cos 6\pi t.$$

Linear Combinations of Harmonically Related Complex Exponentials

Example - continued

$$x(t) = \sum_{k=-3}^{+3} a_k e^{jk2\pi t} \qquad x(t) = 1 + \frac{1}{2}\cos 2\pi t + \cos 4\pi t + \frac{2}{3}\cos 6\pi t.$$



Fourier Series representation of periodic continuous-time signals

We can represent a continuous-time periodic signal $x(t)$ with period T as:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$$

This representation is called the **Fourier Series representation** of the signal.

The a_k are the **Fourier Series coefficients.**

Determination of the Fourier Series Representation of a Continuous-time Periodic Signal

Fourier Series representation of the periodic signal $x(t)$ with period T:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$$

The *synthesis* equation

How do we determine the Fourier Series coefficients a_k ?

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}\,dt.$$

The *analysis* equation

The set of coefficients $\{a_k\}$ are called the *Fourier series coefficients* or the *spectral coefficients* of $x(t)$.

These complex coefficients measure the portion of the signal *x(t)* that is at each harmonic of the fundamental component.

Fourier Series Coefficients

The set of coefficients $\{a_k\}$ are called the *Fourier series coefficients* or the *spectral coefficients* of $x(t)$.

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}\,dt.$$

The coefficient a_0 is the dc or constant component of $x(t)$ and is calculated as:

$$a_0 = \frac{1}{T}\int_T x(t)\,dt,$$

which is simply the average value of $x(t)$ over one period.

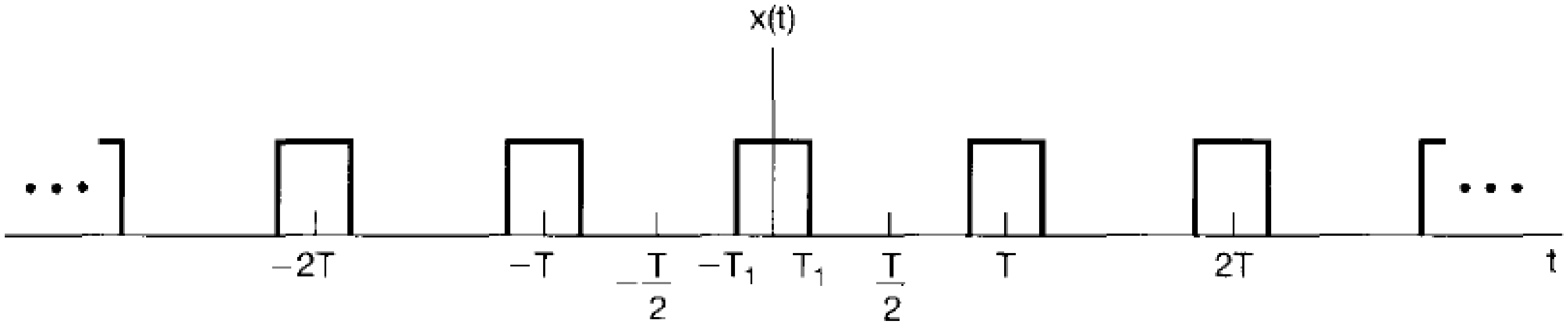
Fourier Series Representation

Example 1:

Determine the Fourier Series coefficients of the periodic square wave defined over one period as:

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$

This signal is periodic with fundamental period *T* and fundamental frequency $\omega_0 = 2\pi/T$

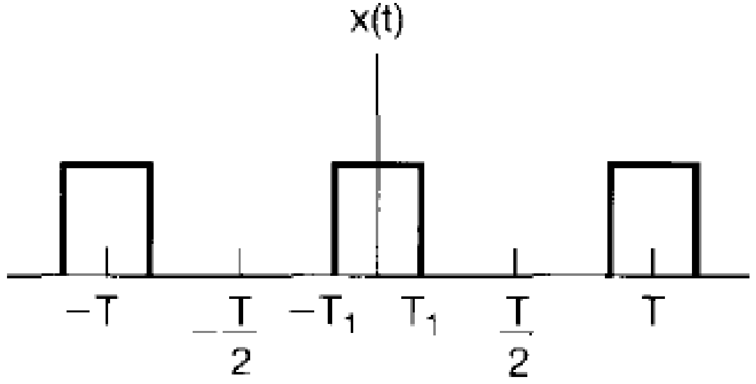


Fourier Series Representation

Example 1 – continued :

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}\,dt.$$

Because of the symmetry of $x(t)$ about $t = 0$, it is convenient to choose $-\frac{T}{2} < t < \frac{T}{2}$ as the interval over which the integration is performed, although any interval of length *T* is equally valid and thus will lead to the same result.



First calculate a_0:

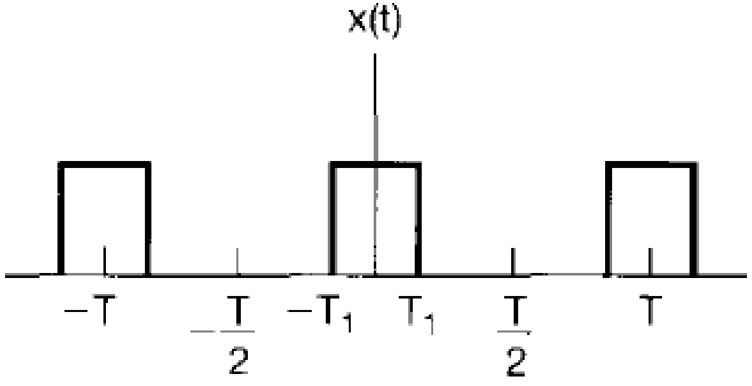
$$a_0 = \frac{1}{T}\int_{-T_1}^{T_1} dt = \frac{2T_1}{T}.$$

For $\mathrm{k} \neq 0$, we obtain

$$a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}\,dt = -\frac{1}{jk\omega_0 T}e^{-jk\omega_0 t}\Big|_{-T_1}^{T_1}$$

Fourier Series Representation

Example 1 – continued :



For $k \neq 0$, we obtain

$$a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}\,dt = -\frac{1}{jk\omega_0 T}e^{-jk\omega_0 t}\Big|_{-T_1}^{T_1}$$

$$a_k = \frac{2}{k\omega_0 T}\left[\frac{e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{2j}\right]$$

$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} = \frac{\sin(k\omega_0 T_1)}{k\pi}, \quad k \neq 0$$

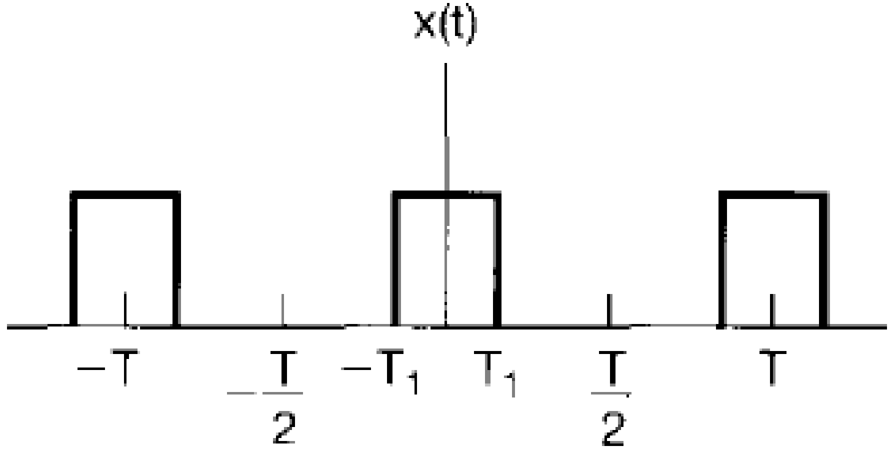
$$\omega_0 T = 2\pi$$

Fourier Series Representation

Example 1 – continued :

$$a_0 = \frac{1}{T}\int_{-T_1}^{T_1} dt = \frac{2T_1}{T}.$$

$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} = \frac{\sin(k\omega_0 T_1)}{k\pi}, \quad k \neq 0$$



When $T = 4T_1$:

$$a_0 = \frac{1}{2}$$

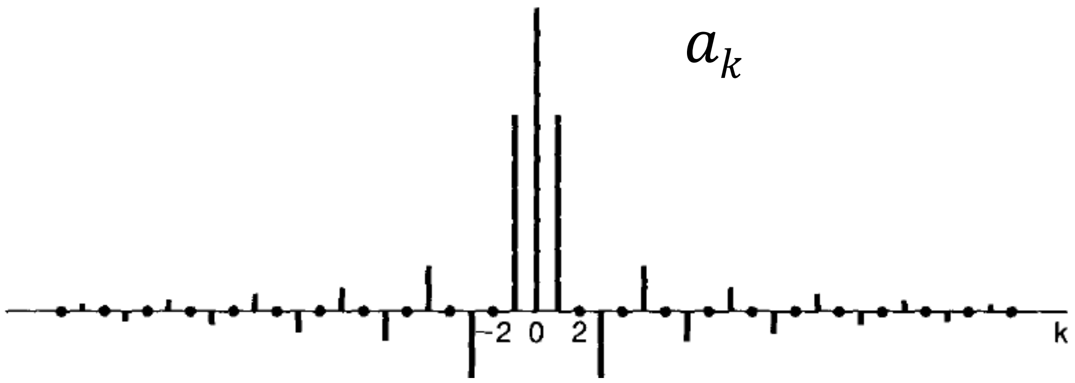
$$a_k = \frac{\sin(\pi k/2)}{k\pi}, \quad k \neq 0$$

$$a_1 = a_{-1} = \frac{1}{\pi},$$

$$a_3 = a_{-3} = -\frac{1}{3\pi},$$

$$a_5 = a_{-5} = \frac{1}{5\pi},$$

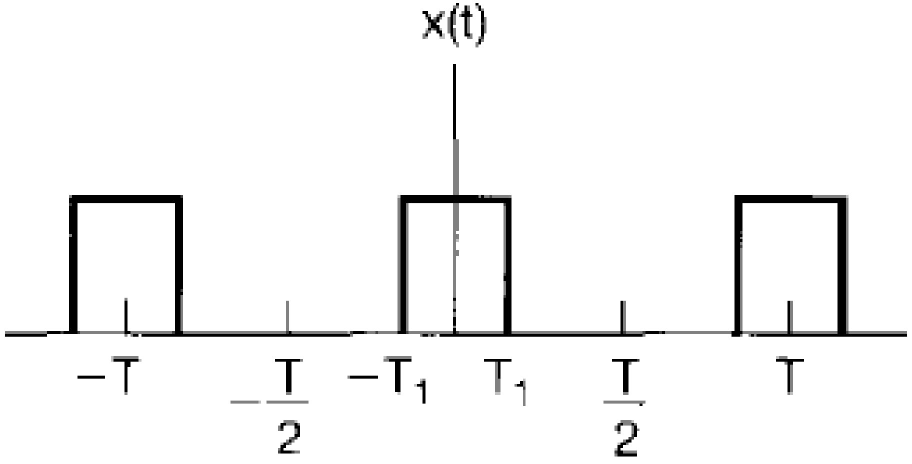
$$\vdots$$



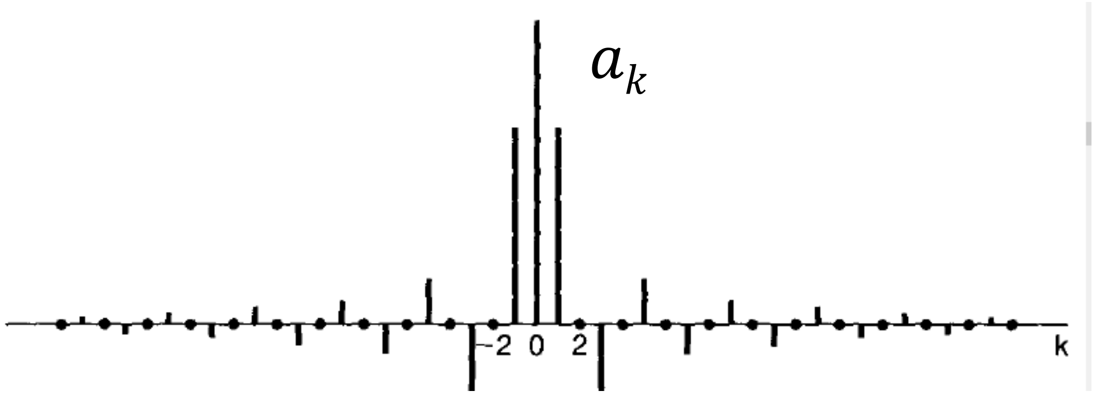
a_k are 0 when k is even.

Fourier Series Representation

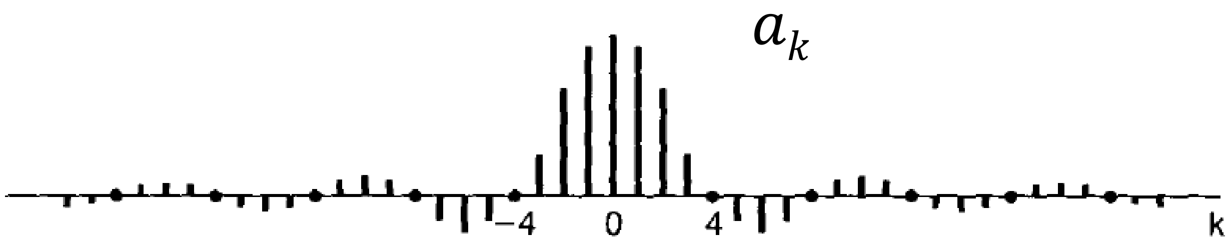
Example 1 – continued :



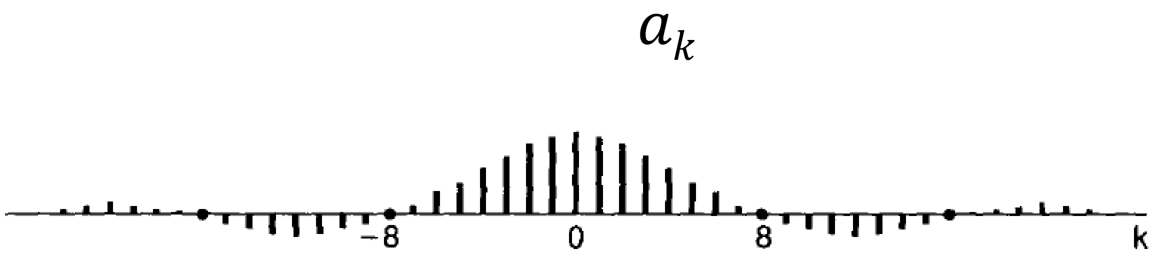
When $T = 4T_1$:



When $T = 8T_1$:



When $T = 16T_1$:



Fourier Series Representation

Given a periodic signal $x(t)$ its Fourier series representation:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

let us examine the problem of **approximating** a given periodic signal $x(t)$ by a linear combination of **a finite number** of harmonically related complex exponentials-that is, by a finite series of the form:

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$

For a large class of periodic functions satisfying the Dirichlet conditions, for any other value of t, say, $t = t_1$, we are guaranteed that

$$\lim_{N \to \infty} x_N(t_1) = x(t_1)$$

Fourier Series Representation

Given a periodic signal $x(t)$ its Fourier series representation:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

let us examine the problem of **approximating** a given periodic signal $x(t)$ by a linear combination of **a finite number** of harmonically related complex exponentials-that is, by a finite series of the form:

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$

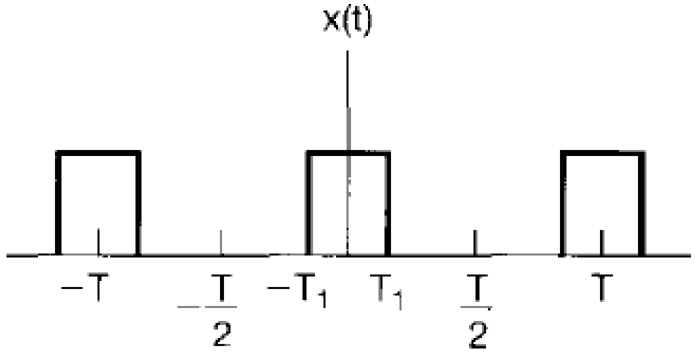
For a large class of periodic functions satisfying the Dirichlet conditions, for any other value of t, say, $t = t_1$, we are guaranteed that

$$\lim_{N \to \infty} x_N(t_1) = x(t_1)$$

Fourier Series Representation

Example 1

Consider the signal $x(t)$

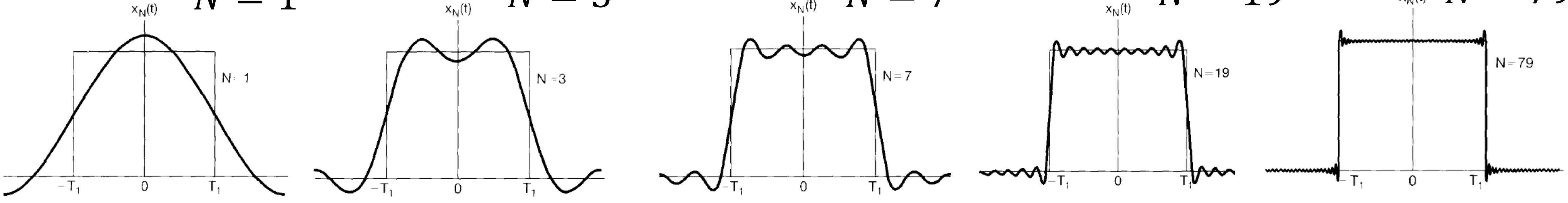


and its approximation:

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$

$x_N(t)$

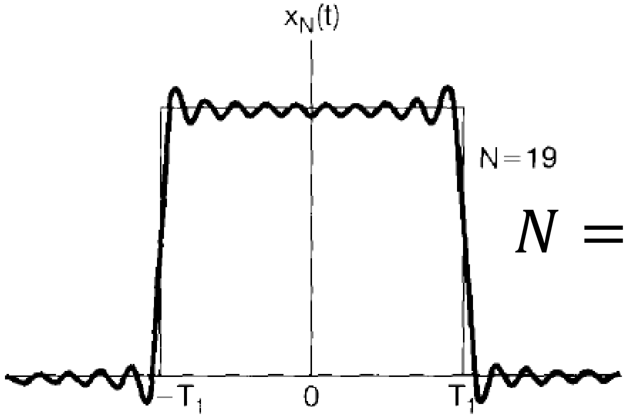
$N = 1$ $N = 3$ $N = 7$ $N = 19$ $N = 79$



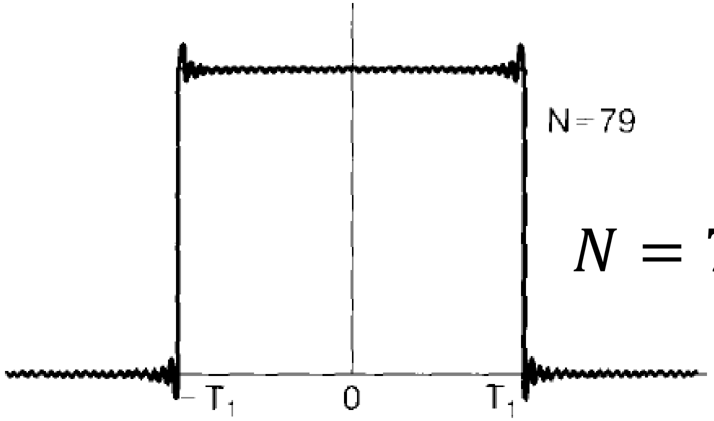
Gibbs phenomenon

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$



$N = 19$



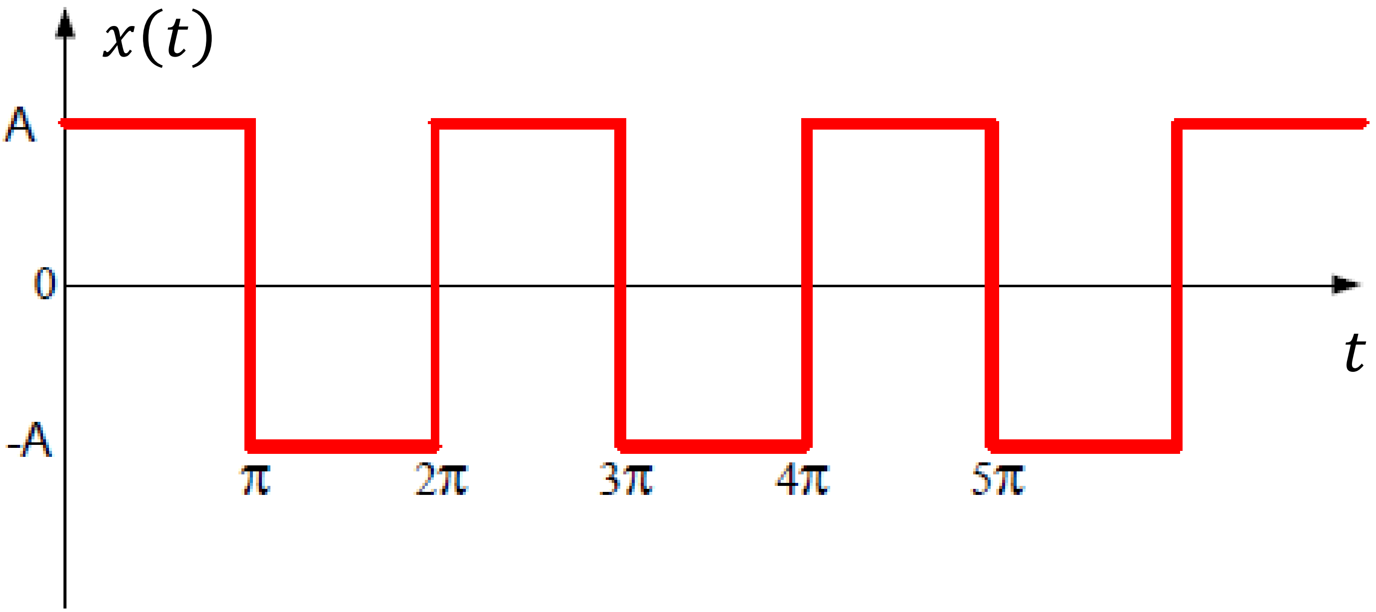
$N = 79$

At the points of discontinuity, *N* must be large in order to reduce the error.
As *N* increases, the ripples in the partial sums become compressed toward the discontinuity.
But for *any* finite value of *N,* the peak amplitude of the ripples remains constant.
This behavior is called the *Gibbs phenomenon.*
The truncated Fourier series approximation $x_N(t)$ of a discontinuous signal $x(t)$ will in general exhibit high-frequency ripples and overshoot $x(t)$ near the discontinuities.

In the limit, as *N* goes to infinity, the energy in the approximation error vanishes and the Fourier series representation of a discontinuous signal such as the square wave converges.

Example 2

Find the Fourier series coefficients of the following periodic signal $x(t)$.

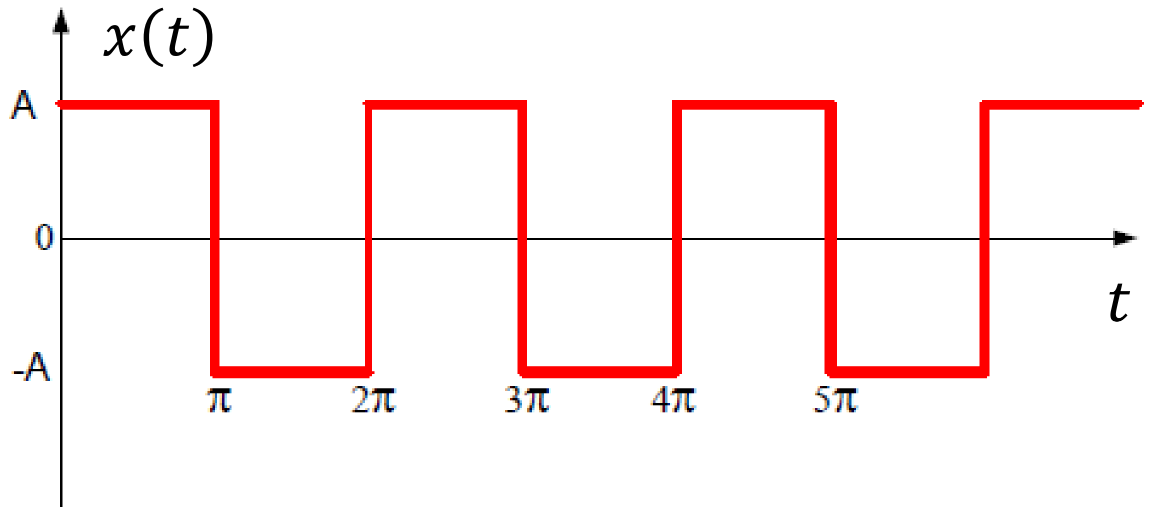


$x(t)$ is periodic with $T = 2\pi$. The fundamental frequency is $\omega_0 = \frac{2\pi}{T} = 1$.

$$a_0 = \frac{1}{T}\int_T x(t)\,dt, \qquad a_0 = \frac{1}{2\pi}\int_0^{2\pi} x(t)dt = \frac{1}{2\pi}\int_0^{\pi} Adt - \frac{1}{2\pi}\int_{\pi}^{2\pi} Adt = 0$$

Example 2 - continued

$\omega_0 = 1 \qquad T = 2\pi$



$a_0 = 0$

For $k \neq 0$, we obtain

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt.$$

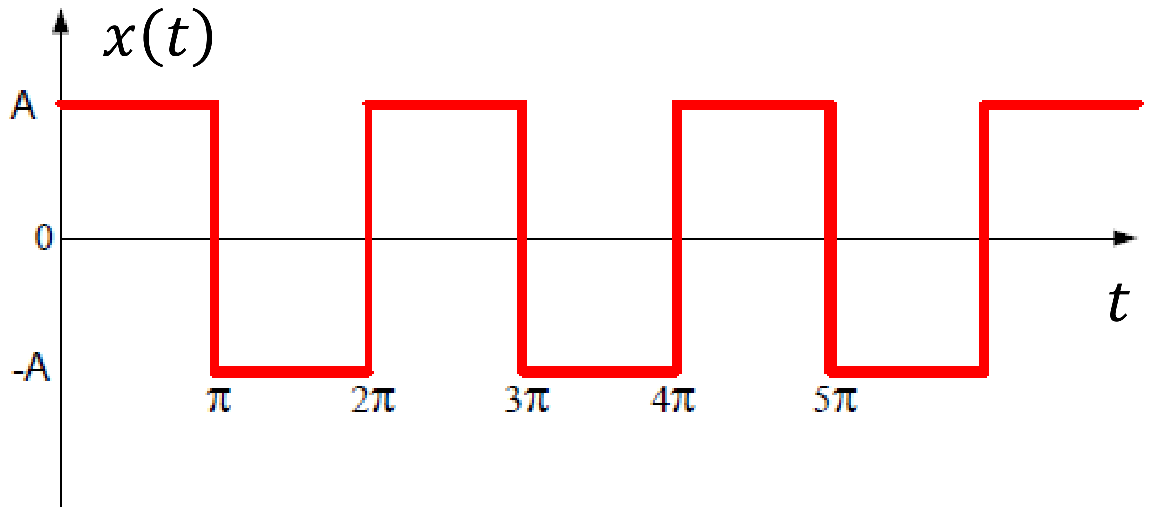
$$a_k = \frac{1}{2\pi}\int_0^{2\pi} x(t)e^{-jkt}dt = \frac{1}{2\pi}\int_0^{\pi} Ae^{-jkt}dt - \frac{1}{2\pi}\int_{\pi}^{2\pi} Ae^{-jkt}dt$$

$$a_k = \frac{1}{2\pi}\frac{A}{(-jk)}e^{-jkt}\Big|_0^{\pi} - \frac{1}{2\pi}\frac{A}{(-jk)}e^{-jkt}\Big|_{\pi}^{2\pi} = \frac{-A}{2\pi jk}\left(e^{-jk\pi} - 1 - 1 + e^{-jk\pi}\right)$$

$$a_k = \frac{A(1 - e^{-jk\pi})}{\pi jk}$$

Example 2 - continued

$$\omega_0 = 1 \qquad T = 2\pi$$



$$a_k = \frac{A(1 - e^{-jk\pi})}{\pi jk} \qquad \text{for k} \neq 0$$

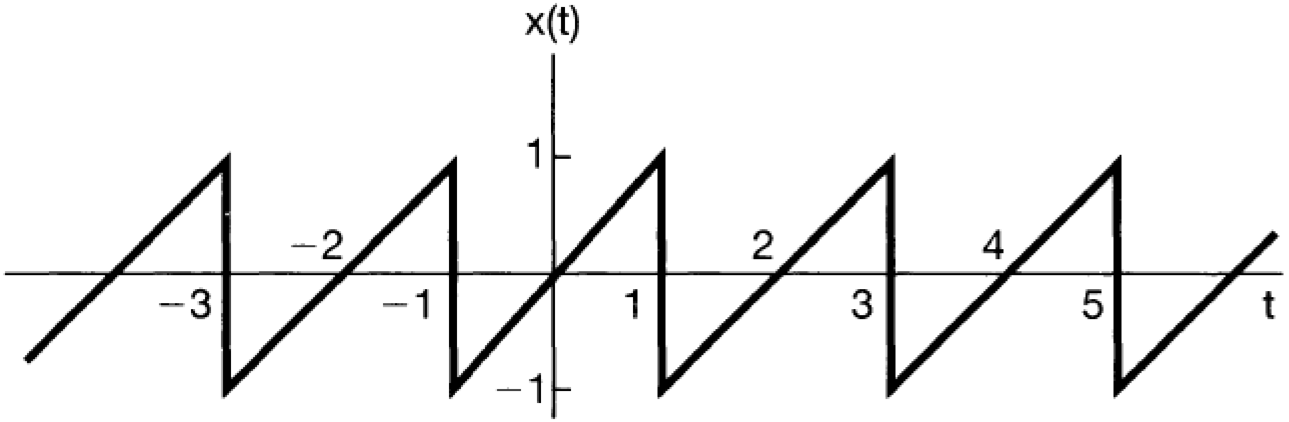
For k odd: $$a_k = -j\frac{2A}{\pi k}$$

For k even: $$a_k = 0$$

$$a_0 = 0$$

Example 3

Find the Fourier series coefficients of the following periodic signal $x(t)$.



$x(t)$ is periodic with $T = 2$. The fundamental frequency is $\omega_0 = \frac{2\pi}{2} = \pi$.

$$a_0 = \frac{1}{T}\int_T x(t)\,dt,$$

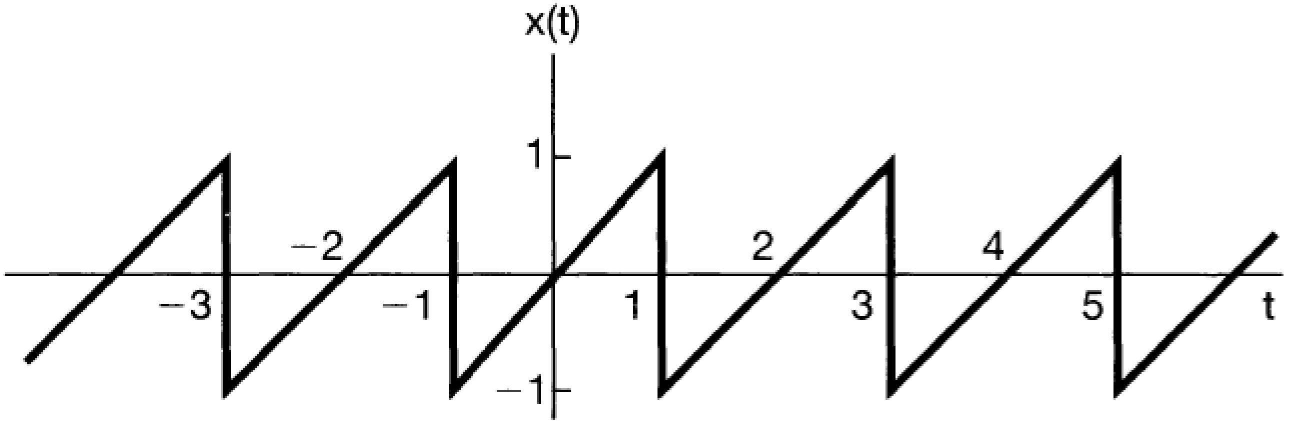
$x(t) = t$ for $-1 < t < 1$

$$a_0 = \frac{1}{2}\int_{-1}^{1} x(t)dt = \frac{1}{2}\int_{-1}^{1} tdt = \frac{1}{4}t^2\Big|_{-1}^{1} = 0$$

Example 3 - continued

$T = 2 \qquad \omega_0 = \pi$

$a_0 = 0$



For $\mathrm{k} \neq 0$, we obtain

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt.$$

$$a_k = \frac{1}{2}\int_{-1}^{1} x(t)e^{-jk\pi t}dt = \frac{1}{2}\int_{-1}^{1} te^{-jk\pi t}dt$$

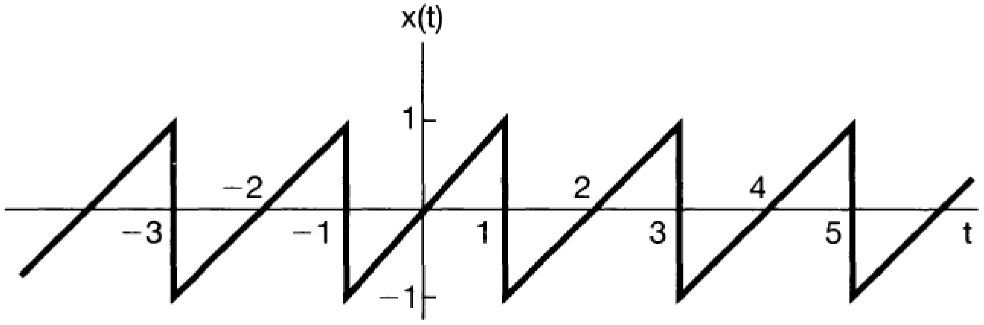
Integration by parts

$$a_k = \frac{1}{2}\frac{1}{(-j\pi k)}te^{-jk\pi t}\Big|_{-1}^{1} - \frac{1}{2}\frac{1}{(-j\pi k)}\int_{-1}^{1} e^{-jk\pi t}dt$$

$$a_k = \frac{1}{2}\frac{1}{(-j\pi k)}te^{-jk\pi t}\Big|_{-1}^{1} - \frac{1}{2}\frac{1}{(-j\pi k)^2}e^{-jk\pi t}\Big|_{-1}^{1}$$

Example 3 - continued

$T = 2 \qquad \omega_0 = \pi$



For $k \neq 0$, we obtain

$$a_k = \frac{1}{2}\frac{1}{(-j\pi k)} t e^{-jk\pi t}\Big|_{-1}^{1} - \frac{1}{2}\frac{1}{(-j\pi k)^2} e^{-jk\pi t}\Big|_{-1}^{1}$$

$$a_k = \frac{1}{2}\frac{1}{(-j\pi k)}\left(e^{-jk\pi} + e^{jk\pi}\right) - \frac{1}{2}\frac{1}{(-j\pi k)^2}\left(e^{-jk\pi} - e^{jk\pi}\right)$$

$$a_k = \frac{j}{\pi k}\cos(\pi k) - \frac{j}{(\pi k)^2}\sin(\pi k)$$

$\sin(\pi k) = 0$ for all k

$$\boxed{a_k = \frac{j}{\pi k}\cos(\pi k) = \frac{j}{\pi k}(-1)^k \quad \text{for } k \neq 0 \qquad a_0 = 0}$$

Example 4

Find the Fourier series coefficients of the following periodic signal $x(t)$.

$$x(t) = \sin \omega_0 t$$

We can determine the Fourier series coefficients for this signal by applying the analysis equation:

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt$$

For this simple case, however, it is easier to expand the sinusoidal signal as a linear combination of complex exponentials and identify the Fourier series coefficients by inspection. Specifically, we can express

$$\sin \omega_0 t = \frac{1}{2j}e^{j\omega_0 t} - \frac{1}{2j}e^{-j\omega_0 t}$$

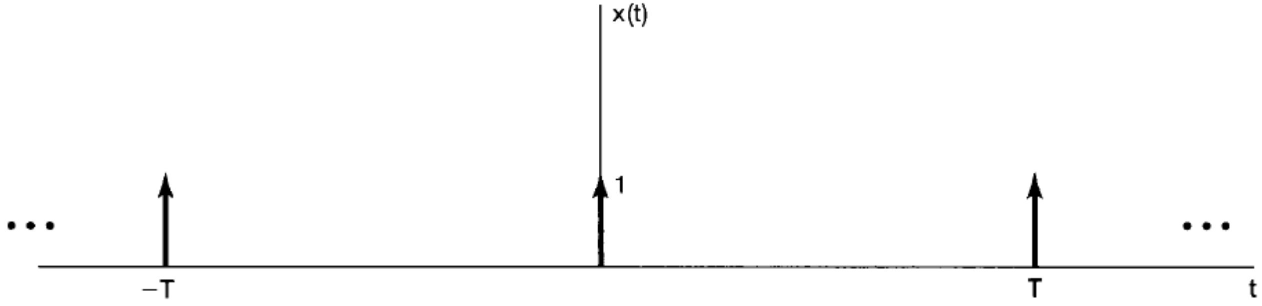
$$\sin \omega_0 t = \frac{1}{2j}e^{j\omega_0 t} - \frac{1}{2j}e^{-j\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \Rightarrow$$

$$\boxed{\begin{aligned} a_1 &= \frac{1}{2j}, & a_{-1} &= -\frac{1}{2j} \\ a_k &= 0, & k &\neq +1 \text{ or } -1 \end{aligned}}$$

Example 5

Find the Fourier series coefficients of the periodic train of impulses (impulse train), which is expressed as

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$



$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt.$$

Select the interval of integration to be $-T/2 < t < T/2$. Within this interval, $x(t)$ is the same as $\delta(t)$.

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)e^{-jk2\pi t/T}\,dt = \frac{1}{T}$$

All the Fourier series coefficients of the impulse train are identical.

Trigonometric Fourier Series

- Trigonometric Fourier Series is common to represent **real-valued** continuous-time periodic signals.
- Given a real-valued signal $x(t)$ with period T and $\omega_0 = 2\pi/T$, we can represent the signal as:

$$x(t) = \frac{A_0}{2} + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t) + \sum_{k=1}^{\infty} B_k \sin(k\omega_0 t)$$

Fourier Series coefficients of the trignonometric form are calculated using:

$$A_k = \frac{2}{T}\int_T x(t)cos(k\omega_0 t)dt \qquad \text{for } k = 0,1,2,\ldots$$

$$B_k = \frac{2}{T}\int_T x(t)sin(k\omega_0 t)dt \qquad \text{for } k = 1,2,\ldots$$

Fourier Series – Relation between exponential form and trigonometric form for real valued periodic signals

Trigonometric Form

$$x(t) = \frac{A_0}{2} + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t) + \sum_{k=1}^{\infty} B_k \sin(k\omega_0 t)$$

$$A_k = \frac{2}{T}\int_T x(t)cos(k\omega_0 t)dt \qquad \text{for } k = 0,1,2,\dots$$

$$B_k = \frac{2}{T}\int_T x(t)sin(k\omega_0 t)dt \qquad \text{for } k = 1,2,\dots$$

Exponential Form

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$$

$$A_k = 2\mathcal{Re}(a_k)$$

$$B_k = -2\mathcal{Im}(a_k)$$

where $\mathcal{Re}(a_k)$ and $\mathcal{Im}(a_k)$ are the real and imaginary parts of a_k

Fourier Series – Relation between exponential form and trigonometric form for real valued periodic signals

Trigonometric Form

$$x(t) = \frac{A_0}{2} + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t) + \sum_{k=1}^{\infty} B_k \sin(k\omega_0 t)$$

Exponential Form

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$A_k = 2\mathcal{Re}(a_k)$$

$$B_k = -2\mathcal{Im}(a_k)$$

where $\mathcal{Re}(a_k)$ and $\mathcal{Im}(a_k)$ are the real and imaginary parts of a_k

$$a_0 = \frac{A_0}{2}$$

$$a_k = \frac{A_k}{2} - j\frac{B_k}{2} \qquad \text{for } k = 1,2,\ldots$$

$$a_k = \frac{A_k}{2} + j\frac{B_k}{2} \qquad \text{for } k = -1,-2,\ldots$$

Fourier Series –
when $x(t)$ is a **real-valued even** signal.

Exponential Form

a_k is real and even

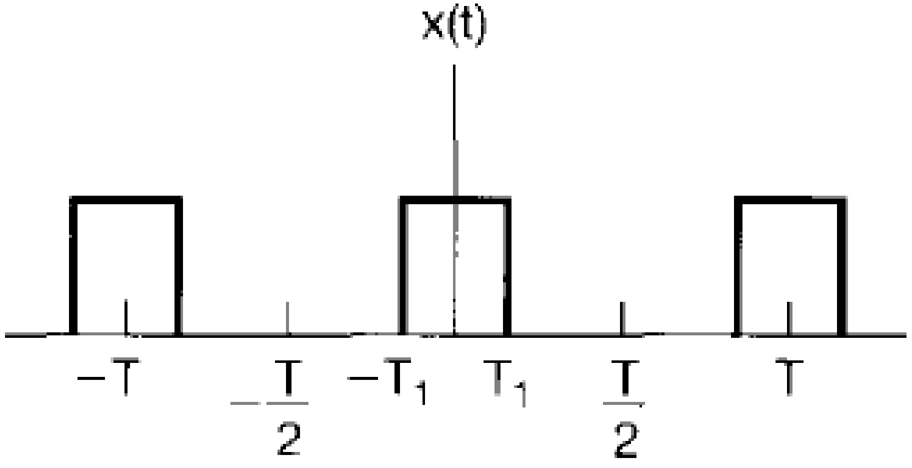
$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

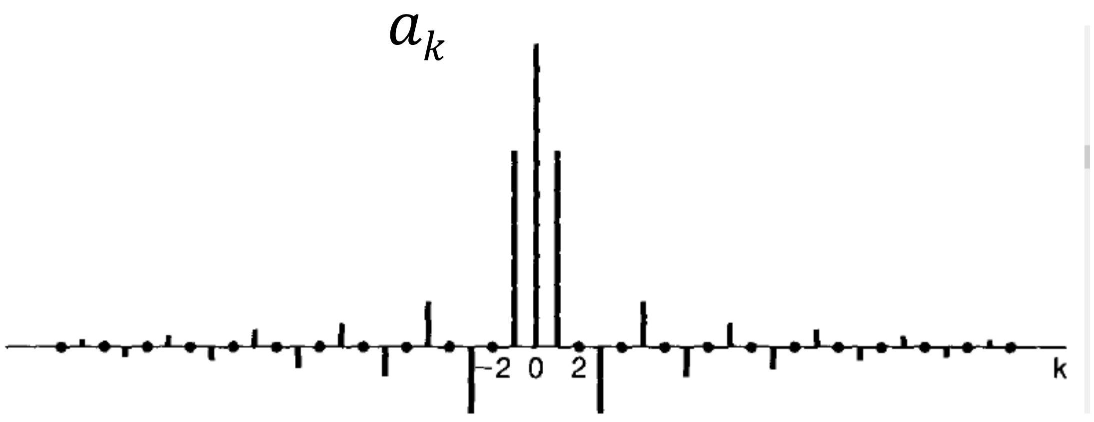
Trigonometric Form

B_k are zero for all k

$$x(t) = \frac{A_0}{2} + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t)$$

E.g.





Fourier Series –
when $x(t)$ is a **real-valued odd** signal.

Exponential Form

a_k is purely imaginary and odd

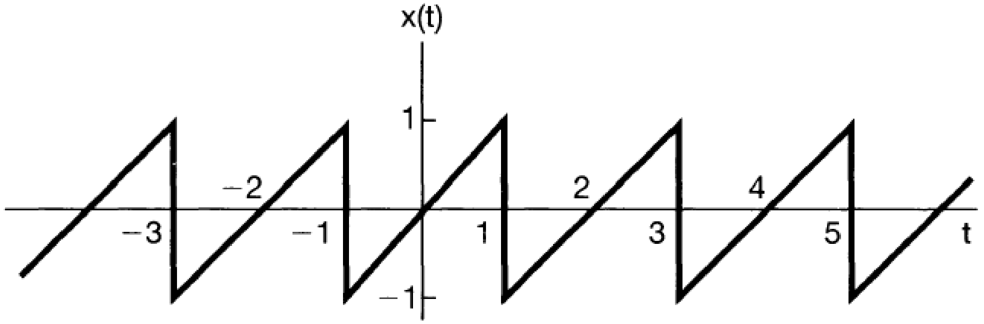
$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Trigonometric Form

A_k are zero for all k

$$x(t) = \sum_{k=1}^{\infty} B_k \sin(k\omega_0 t)$$

E.g.



$$a_0 = 0$$

$$a_k = \frac{j}{\pi k}(-1)^k \qquad \text{for k} \neq 0$$