Signals and Systems

Lecture 6

The Continuous-time Fourier Transform

The Continuous-Time Fourier Transform

- The Continuous-Time Fourier Transform is a representation for both periodic and aperiodic signals.
- Given a continuous-time signal $x(t)$, its Fourier transform $X(j\omega)$ is given by

$$\boxed{X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt.}$$

Analysis Equation

- Given the Fourier transform representation $X(j\omega)$, the signal $x(t)$ in time domain can be recovered by Inverse Fourier transform, given by

$$\boxed{x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega}$$

Synthesis Equation

The Continuous-Time Fourier Transform

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt.$$

Analysis Equation

Fourier Transform (spectrum) of $x(t)$
Frequency-domain representation

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega$$

Synthesis Equation

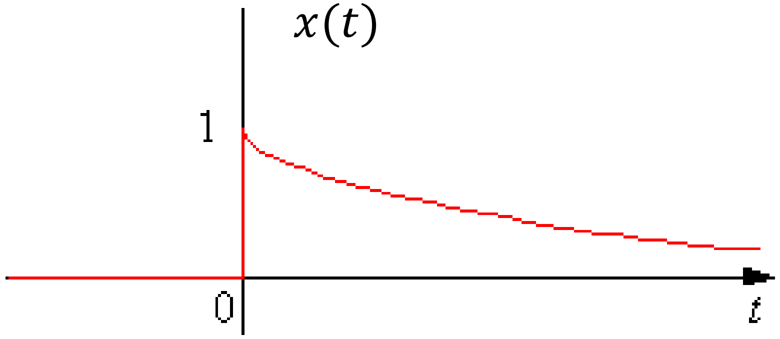
$x(t)$ is represented as a linear combination of complex exponentials.

For aperiodic signals, the complex exponentials occur at a continuum of frequencies ω.

Example 1

Find and sketch the Fourier Transform of the following signal:

$$x(t) = e^{-at}u(t) \quad a > 0$$



$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$$

$$X(j\omega) = \int_0^{\infty} e^{-at}e^{-j\omega t}dt = -\frac{1}{a+j\omega}e^{-(a+j\omega)t}\Big|_0^{\infty}$$

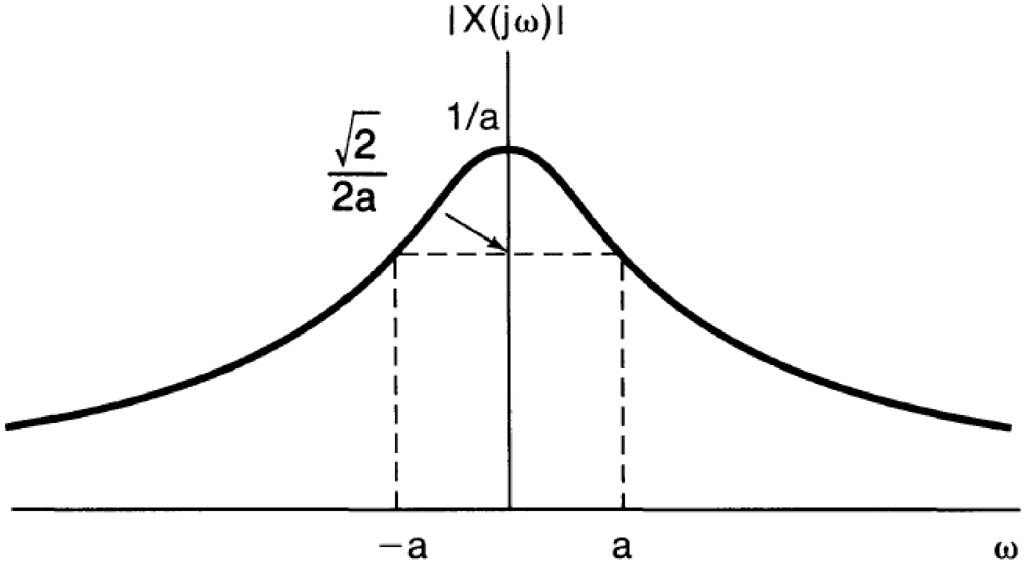
$$X(j\omega) = \frac{1}{a+j\omega}, \quad a > 0$$

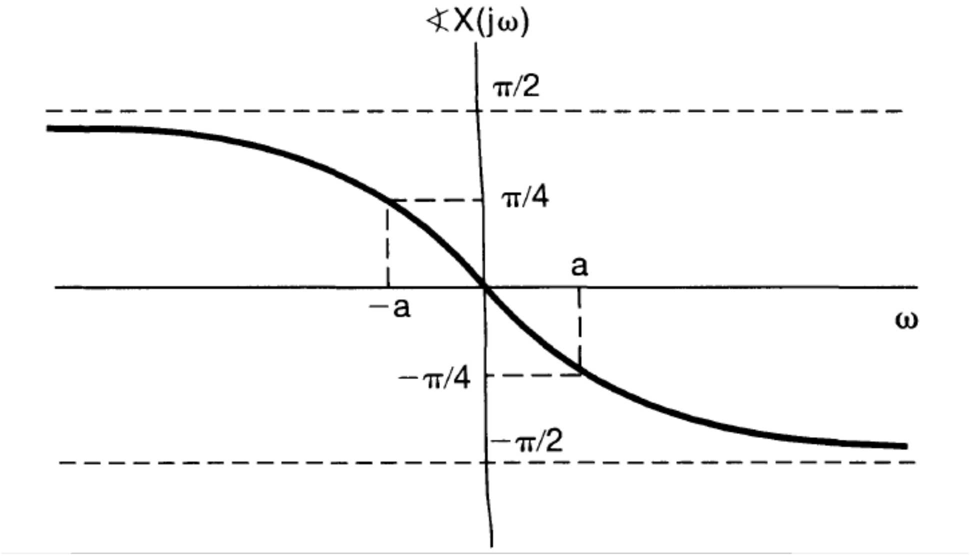
Example 1 - continued

$$X(j\omega) = \frac{1}{a + j\omega}, \quad a > 0.$$

Since this Fourier transform is complex valued, to plot it as a function of ω, we express $X(j\omega)$ in terms of its magnitude and phase.

$$|X(j\omega)| = \frac{1}{\sqrt{a^2 + \omega^2}}, \quad \sphericalangle X(j\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$

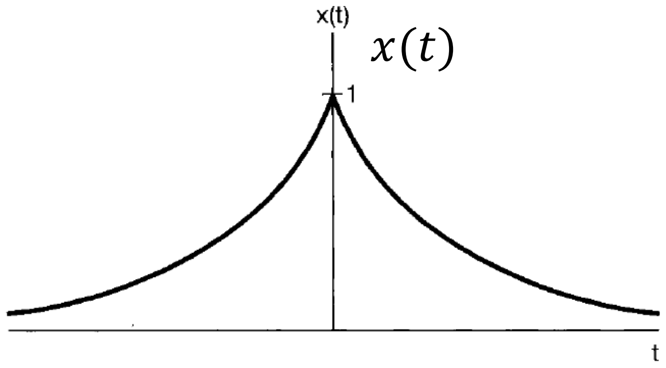




Example 2

Find and sketch the Fourier Transform of the following signal:

$$x(t) = e^{-a|t|}, \quad a > 0$$



$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$$

$$X(j\omega) = \int_{-\infty}^{+\infty} e^{-a|t|}e^{-j\omega t}\,dt = \int_{-\infty}^{0} e^{at}e^{-j\omega t}\,dt + \int_{0}^{\infty} e^{-at}e^{-j\omega t}\,dt$$

$$= \frac{1}{a - j\omega} + \frac{1}{a + j\omega}$$

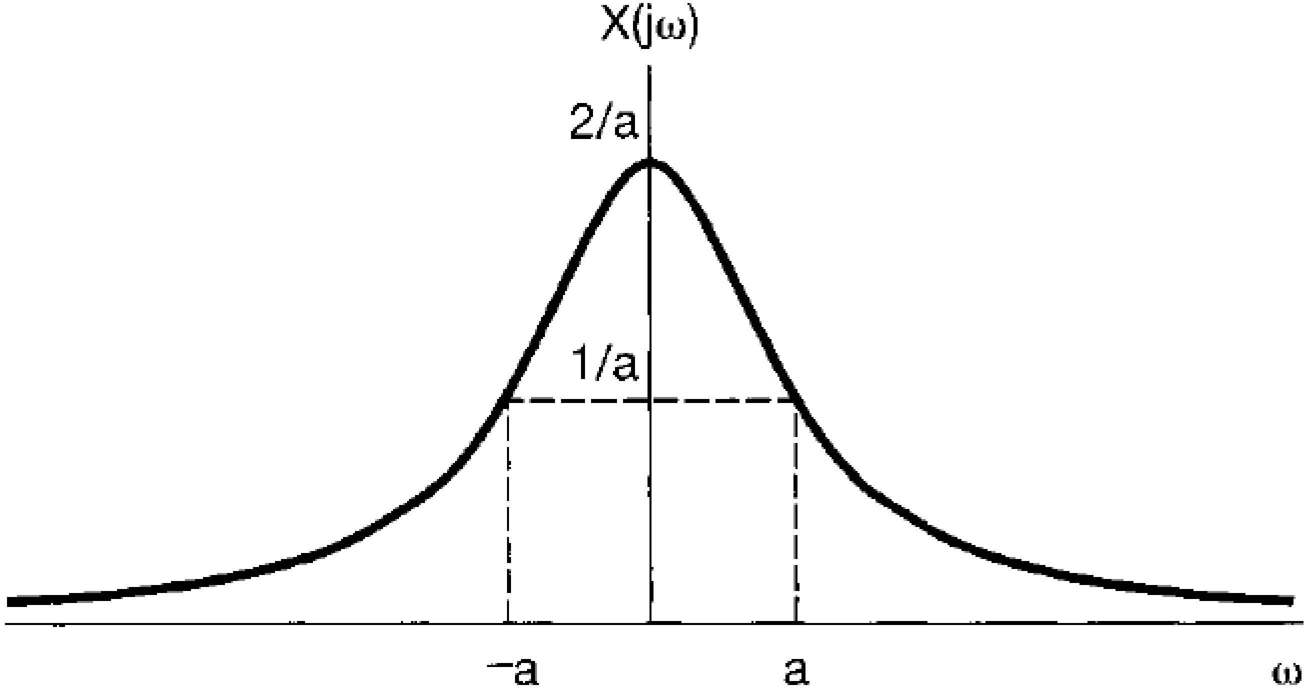
$$= \frac{2a}{a^2 + \omega^2}.$$

In this case $X(j\omega)$ is real

Example 2 - Continued

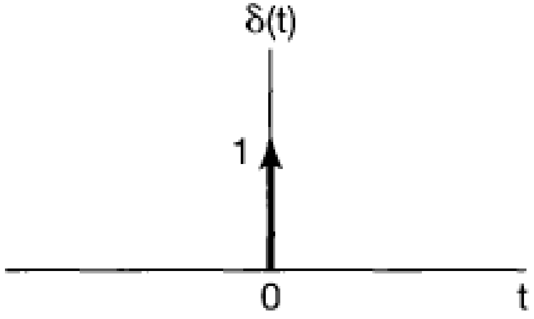
$$X(j\omega) = \frac{2a}{a^2 + \omega^2}$$

In this case $X(j\omega)$ is real.



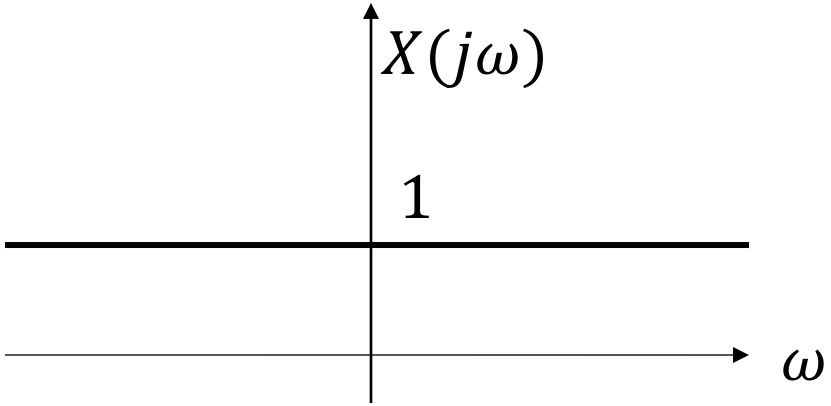
Example 3

Find and sketch the Fourier Transform of the unit impulse function $\delta(t)$.



$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$$

$$X(j\omega) = \int_{-\infty}^{+\infty} \delta(t)e^{-j\omega t}\,dt = 1.$$

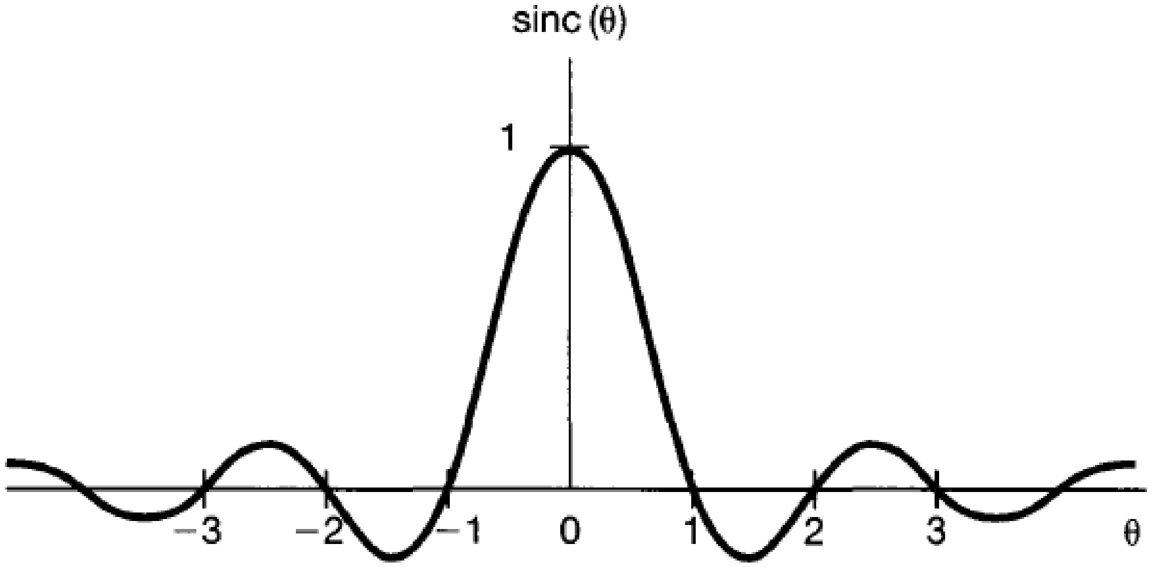


The unit impulse has a Fourier transform consisting of equal contributions at all frequencies.

The sinc function

Sinc functions arise frequently in Fourier analysis and in the study of LTI systems. A commonly used precise form for the sinc function is

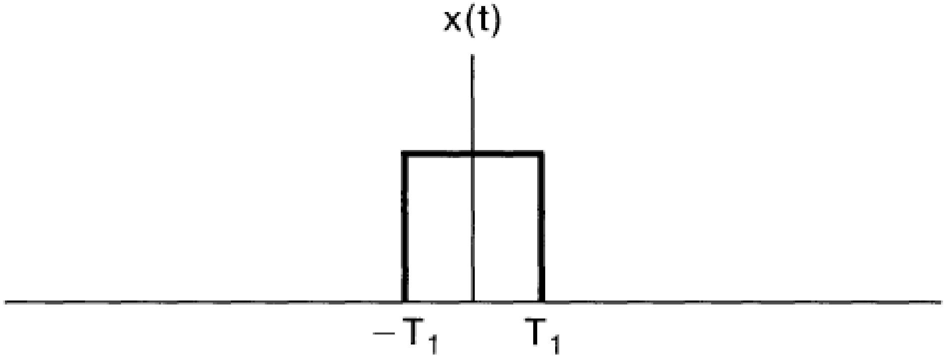
$$\text{sinc}(\theta) = \frac{\sin \pi\theta}{\pi\theta}$$



Example 4

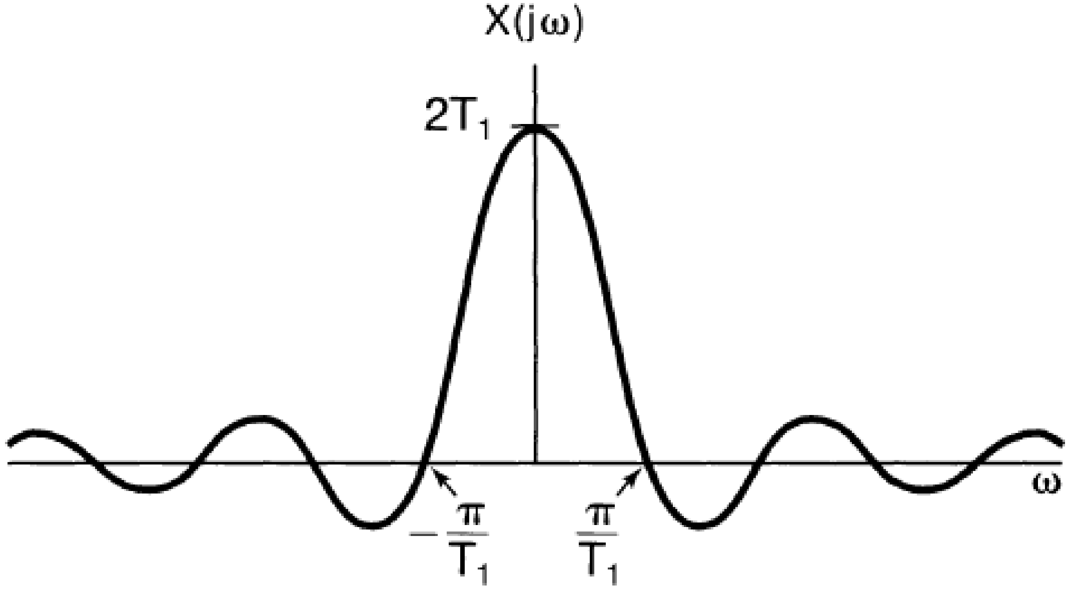
Find and sketch the Fourier Transform of the rectangular pulse signal:

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases}$$



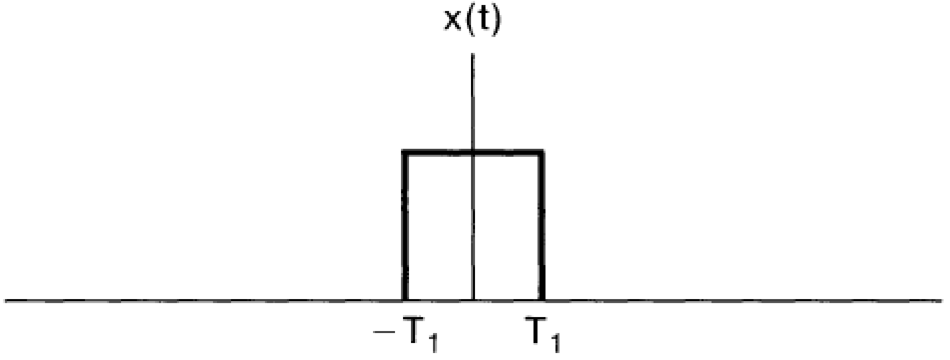
$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$$

$$X(j\omega) = \int_{-T_1}^{T_1} e^{-j\omega t}\,dt = 2\frac{\sin \omega T_1}{\omega}$$



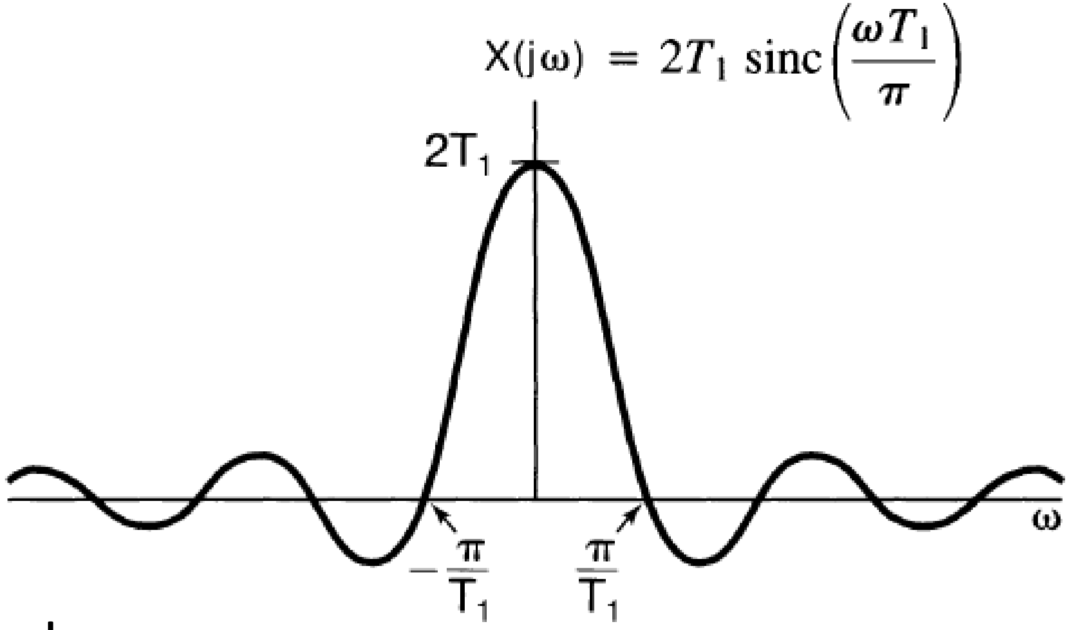
In this case $X(j\omega)$ is real

Example 4 - Continued



$$X(j\omega) = \int_{-T_1}^{T_1} e^{-j\omega t}\,dt = 2\frac{\sin \omega T_1}{\omega}$$

$$\frac{2 \sin \omega T_1}{\omega} = 2T_1 \operatorname{sinc}\left(\frac{\omega T_1}{\pi}\right)$$

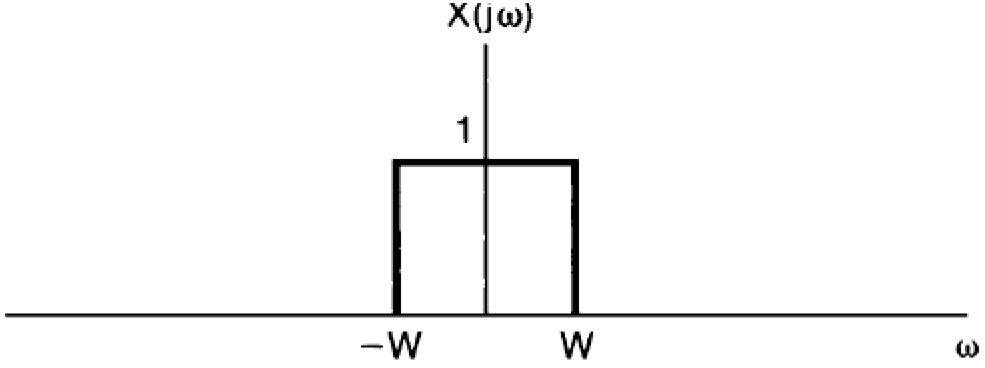


As T_1 increases x(t) becomes broader and $X(j\omega)$ becomes narrower.

Example 5

Find and sketch the inverse Fourier Transform of

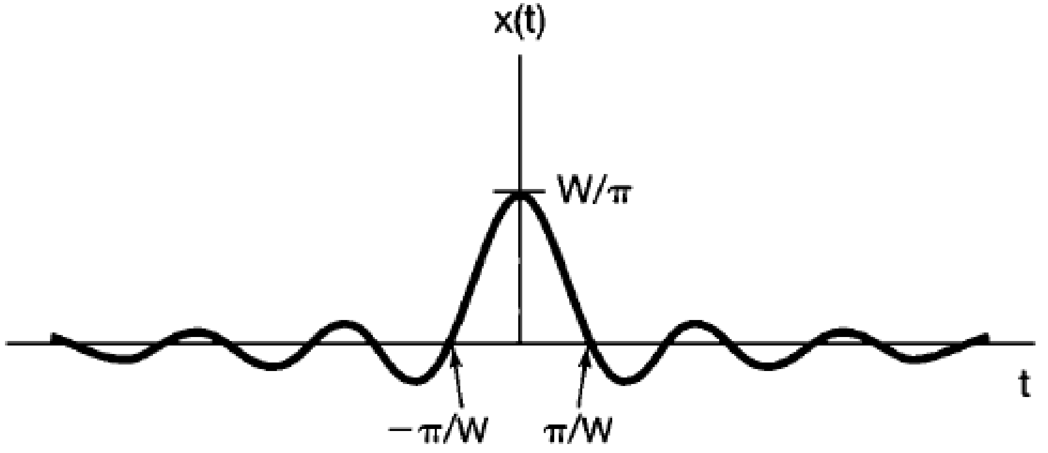
$$X(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}$$



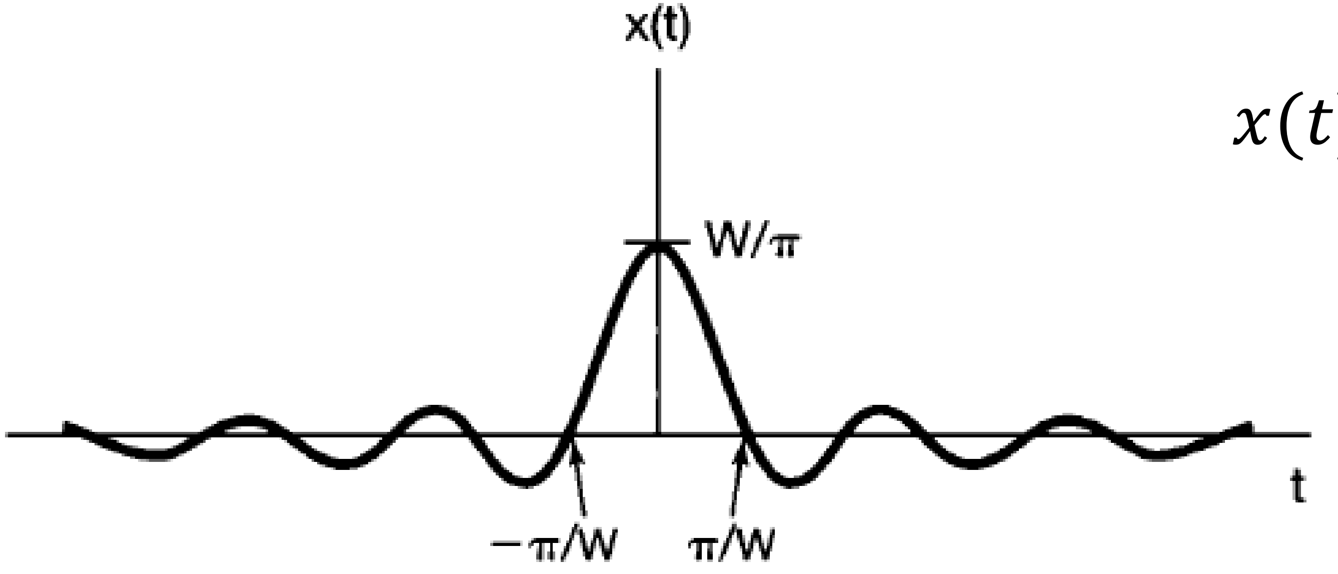
We use the synthesis equation to find $x(t)$:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega$$

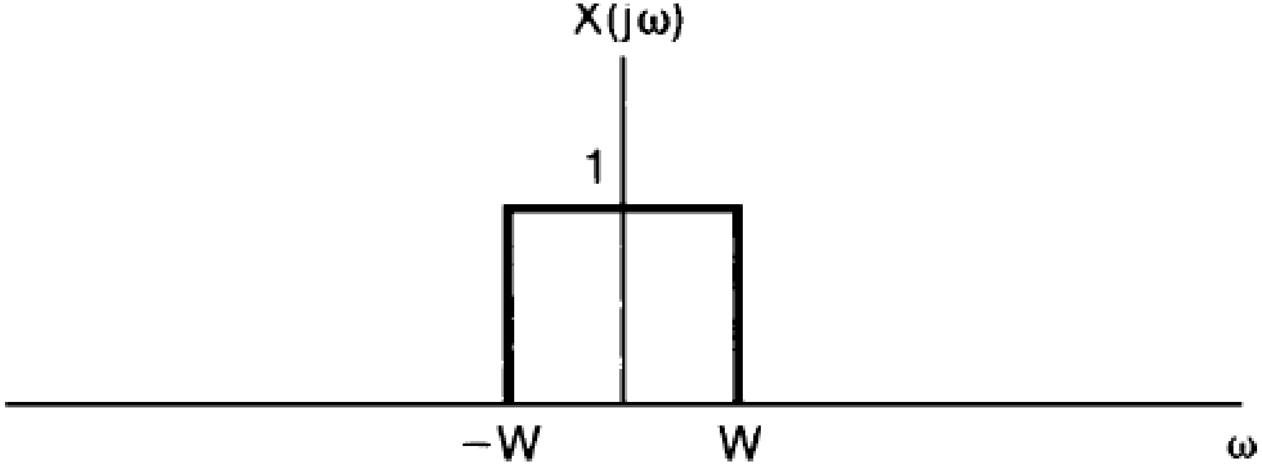
$$x(t) = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t}d\omega = \frac{\sin Wt}{\pi t} = \frac{W}{\pi}\,\mathrm{sinc}\left(\frac{Wt}{\pi}\right)$$



Example 5 - continued

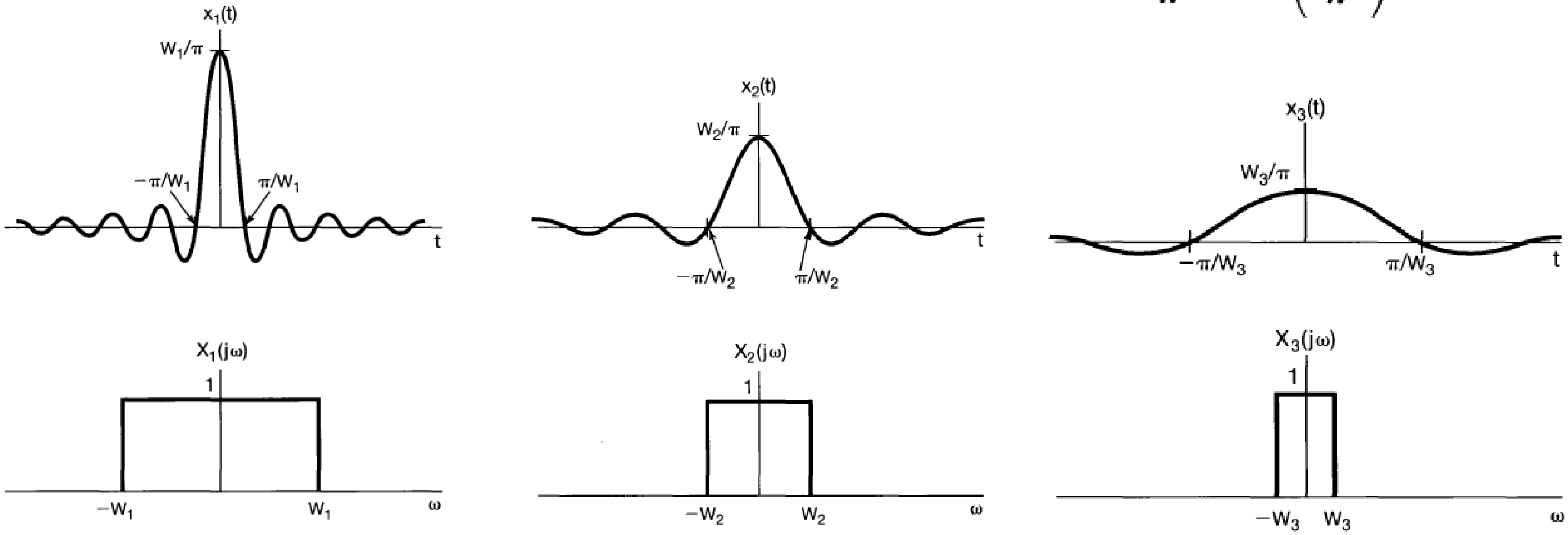


$$x(t) = \frac{W}{\pi}\,\text{sinc}\left(\frac{Wt}{\pi}\right)$$



Example 5 - continued

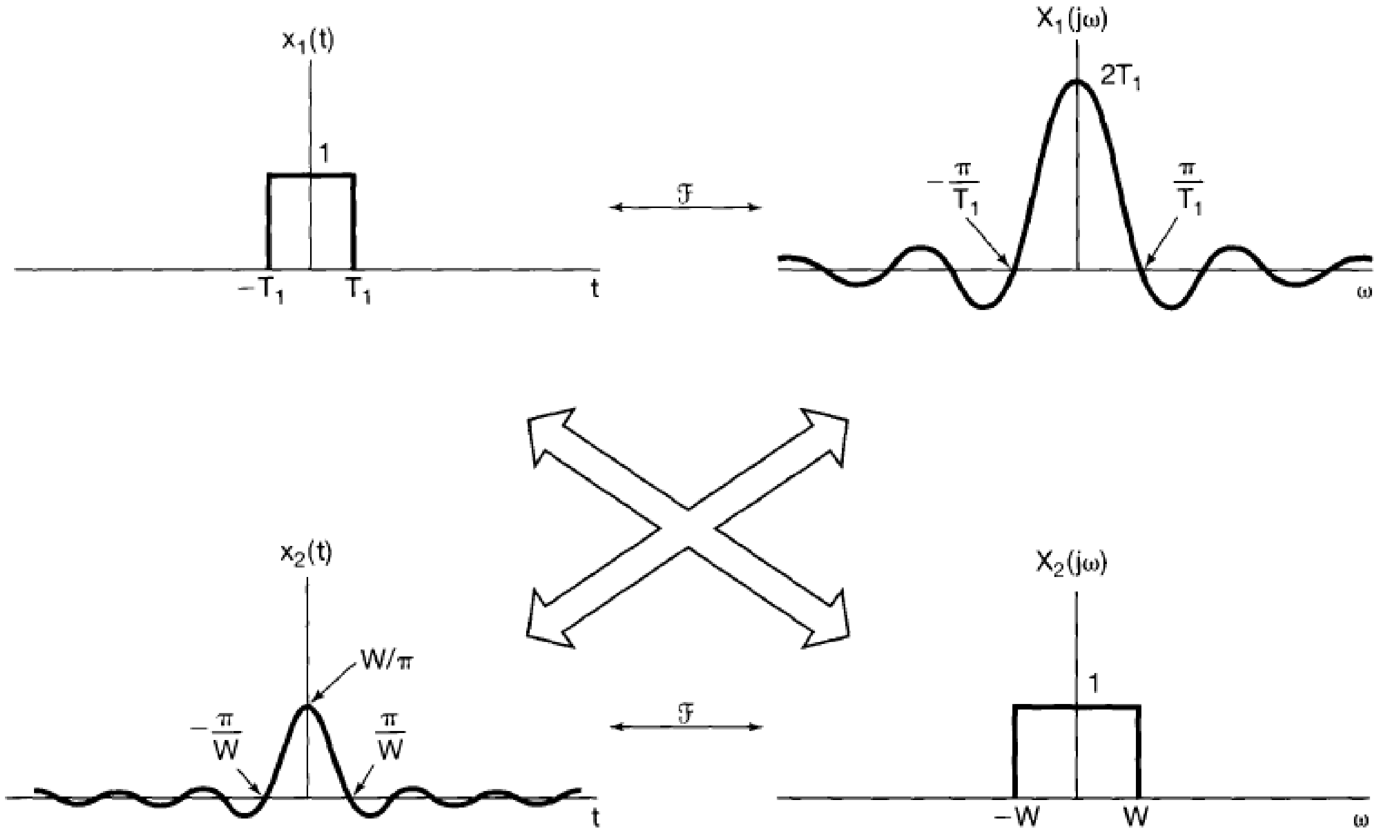
$$x(t) = \frac{W}{\pi}\,\mathrm{sinc}\left(\frac{Wt}{\pi}\right)$$



We see that as W increases, $X(j\omega)$ becomes broader, while the main peak of $x(t)$ at $t = 0$ becomes higher and the width of the first lobe of this signal (i.e., the part of the signal for $|t| < \pi/W$) becomes narrower.

In fact, in the limit as $W \to \infty$, $X(j\omega) = 1$ for all ω, and consequently, we see that $x(t)$ converges to an impulse as $W \to \infty$.

Duality of Continuous-time Fourier Transform



Each of the Fourier transform pair above consists of a function of the form sinc() and a rectangular pulse.
In the first pair the *signal* $x(t)$ that is a pulse and its Fourier transform has the sinc() form.
In the second pair the signal $x(t)$ is in the sinc() form and its Fourier transform is a square function.
The special relationship that is apparent here is a direct consequence of the *duality property* for Fourier transform.

The Fourier Transform of periodic signals

The continuous-time Fourier Transform allows us to consider both periodic and aperiodic signals within a unified context.
We can construct the Fourier transform of a periodic signal directly from its Fourier series representation. The resulting transform consists of a train of impulses in the frequency domain, with the areas of the impulses proportional to the Fourier series coefficients.

$$X(j\omega) = 2\pi\delta(\omega - \omega_0) \quad \Longleftrightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} 2\pi\delta(\omega - \omega_0)e^{j\omega t}\,d\omega = e^{j\omega_0 t}.$$

$$X(j\omega) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(\omega - k\omega_0), \quad \Longleftrightarrow \quad x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}.$$

The Fourier Transform of periodic signals

The Fourier transform of a periodic signal $x(t)$ with Fourier series coefficients $\{a_k\}$ is a train of impulses occurring at the harmonically related frequencies $k\omega_0$.

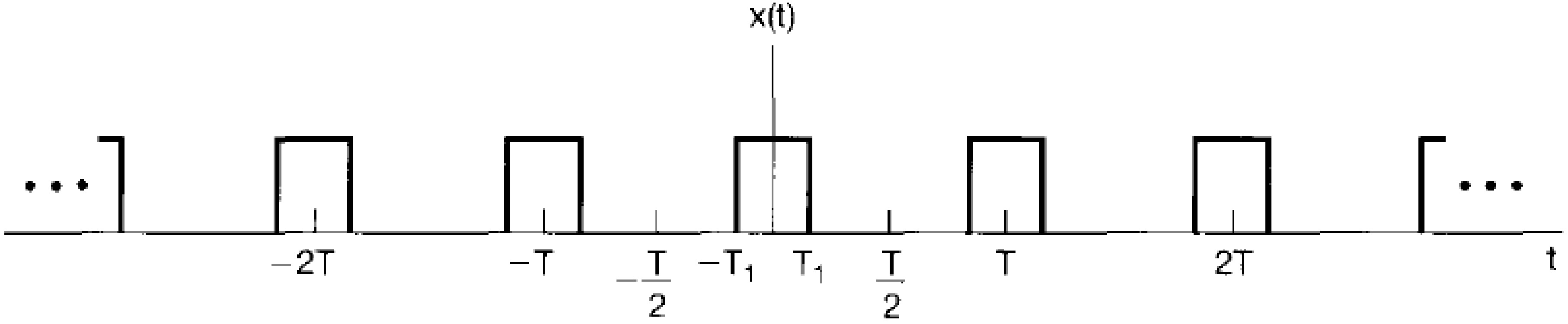
The area of the impulse at the kth harmonic frequency $k\omega_0$ is $2\pi a_k$.

$$X(j\omega) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(\omega - k\omega_0), \quad \Longleftrightarrow \quad x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}.$$

Example 6

Find and sketch the Fourier Transform of the periodic square wave:

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$



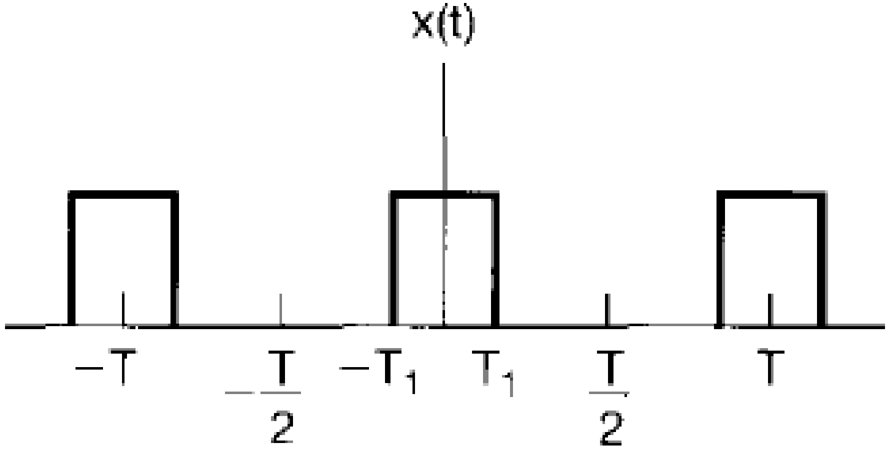
The Fourier series coefficients for this signal are

$$a_k = \frac{\sin k\omega_0 T_1}{\pi k}$$

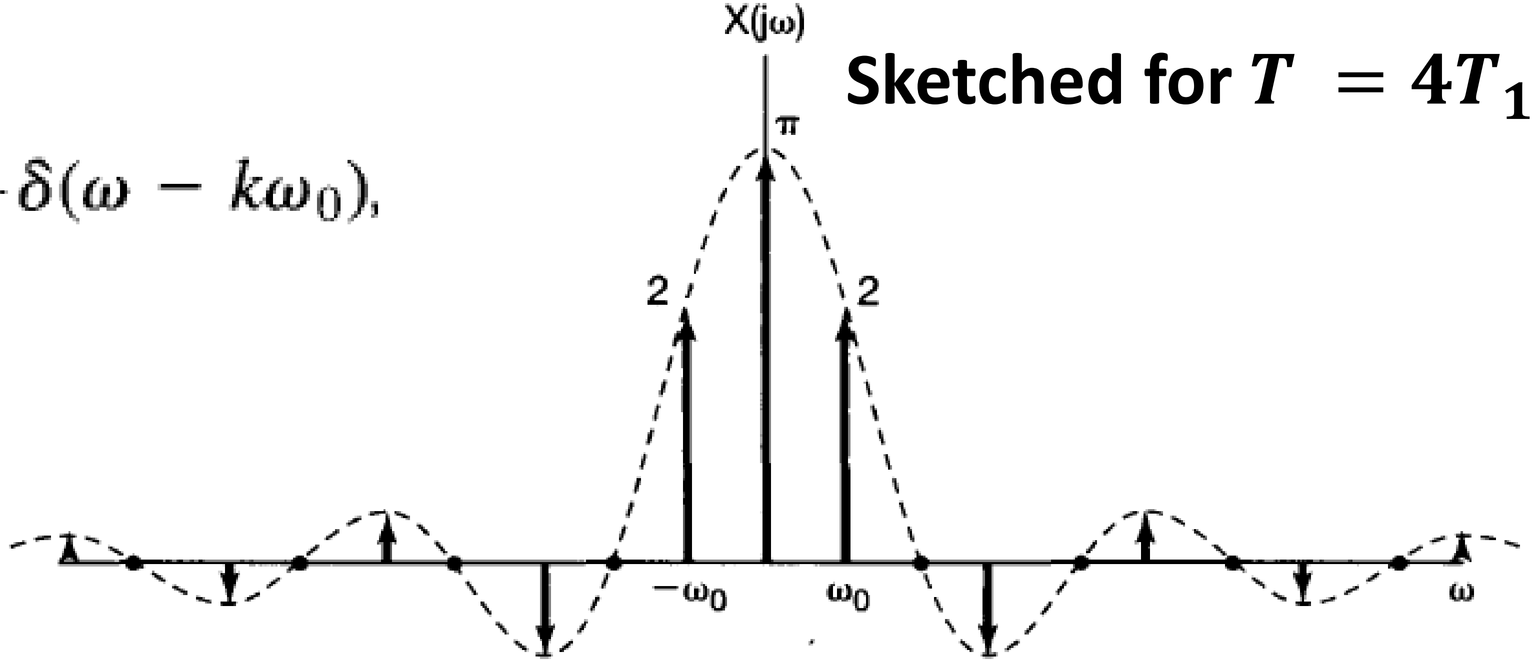
and the Fourier transform of the signal is

$$X(j\omega) = \sum_{k=-\infty}^{+\infty} \frac{2 \sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0),$$

Example 6 - continued



$$X(j\omega) = \sum_{k=-\infty}^{+\infty} \frac{2 \sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0),$$



Sketched for $T = 4T_1$

Example 7

Find and sketch the Fourier Transform of

$$x(t) = \sin \omega_0 t$$

The Fourier series coefficients for this signal are

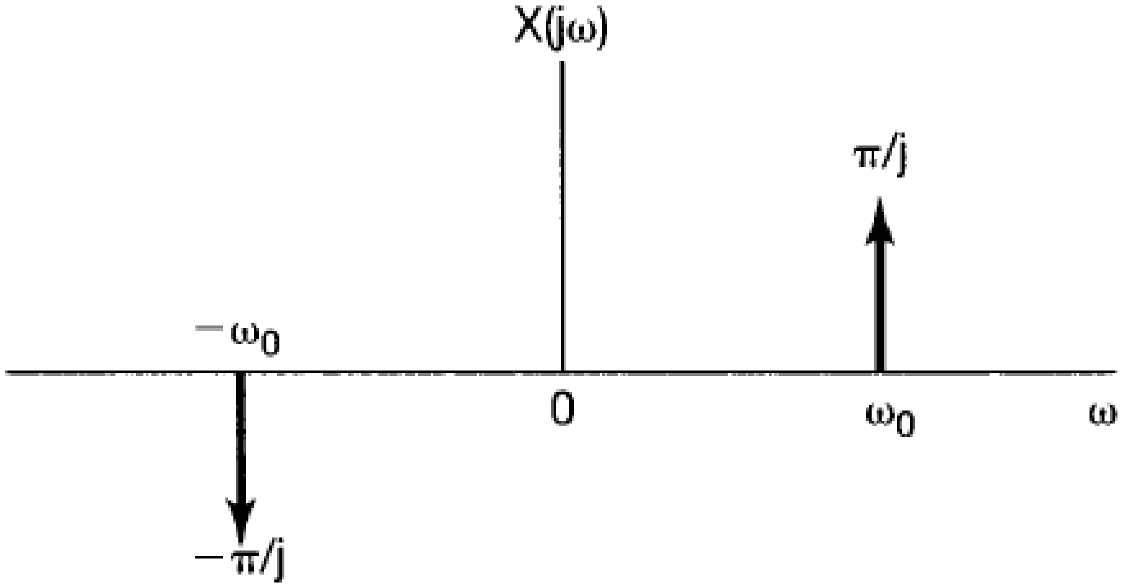
$$a_1 = \frac{1}{2j},$$

$$a_{-1} = -\frac{1}{2j},$$

$$a_k = 0, \qquad k \neq 1 \quad \text{or} \quad -1.$$

and the Fourier transform of $\sin(\omega_0 t)$ is

$$X(j\omega) = -\frac{\pi}{j}\delta(\omega + \omega_0) + \frac{\pi}{j}\delta(\omega - \omega_0)$$



Example 8

Find and sketch the Fourier Transform of

$$x(t) = \cos \omega_0 t$$

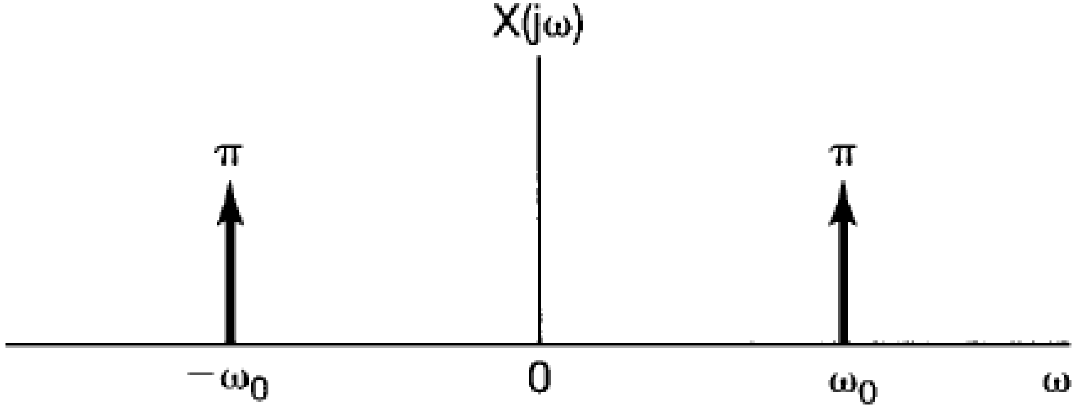
The Fourier series coefficients for this signal are

$$a_1 = a_{-1} = \frac{1}{2},$$

$$a_k = 0, \qquad k \neq 1 \quad \text{or} \quad -1$$

and the Fourier transform of $\cos(\omega_0 t)$ is

$$X(j\omega) = \pi\delta(\omega + \omega_0) + \pi\delta(\omega - \omega_0)$$



PROPERTIES OF THE CONTINUOUS-TIME FOURIER TRANSFORM

Properties of the Continuous-time Fourier Transform

Many of the properties of the continuous-time Fourier transform are useful in reducing the complexity of the evaluation of Fourier transforms or inverse transforms.

We will sometimes refer to $X(j\omega)$ with the notation $\mathcal{F}\{x(t)\}$ and to $x(t)$ with the notation $\mathcal{F}^{-1}\{X(j\omega)\}$. We will also refer to $x(t)$ and $X(j\omega)$ as a Fourier transform pair with the notation:

$$x(t) \overset{\mathfrak{F}}{\longleftrightarrow} X(j\omega)$$

For example:

$$\frac{1}{a + j\omega} = \mathfrak{F}\{e^{-at}u(t)\} \qquad e^{-at}u(t) = \mathfrak{F}^{-1}\left\{\frac{1}{a + j\omega}\right\}$$

$$e^{-at}u(t) \overset{\mathfrak{F}}{\longleftrightarrow} \frac{1}{a + j\omega}$$

Linearity

If

$$x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

and

$$y(t) \stackrel{\mathcal{F}}{\longleftrightarrow} Y(j\omega),$$

then

$$\boxed{ax(t) + by(t) \stackrel{\mathcal{F}}{\longleftrightarrow} aX(j\omega) + bY(j\omega).}$$

Time shifting

If

$$x(t) \xleftrightarrow{\mathfrak{F}} X(j\omega),$$

then

$$\boxed{x(t - t_0) \xleftrightarrow{\mathfrak{F}} e^{-j\omega t_0} X(j\omega).}$$

Time shifting

One consequence of the time-shift property is that a signal which is shifted in time does not have the magnitude of its Fourier transform altered.
That is, if we express $X(j\omega)$ in polar form as

$$\mathfrak{F}\{x(t)\} = X(j\omega) = |X(j\omega)|e^{j\sphericalangle X(j\omega)}$$

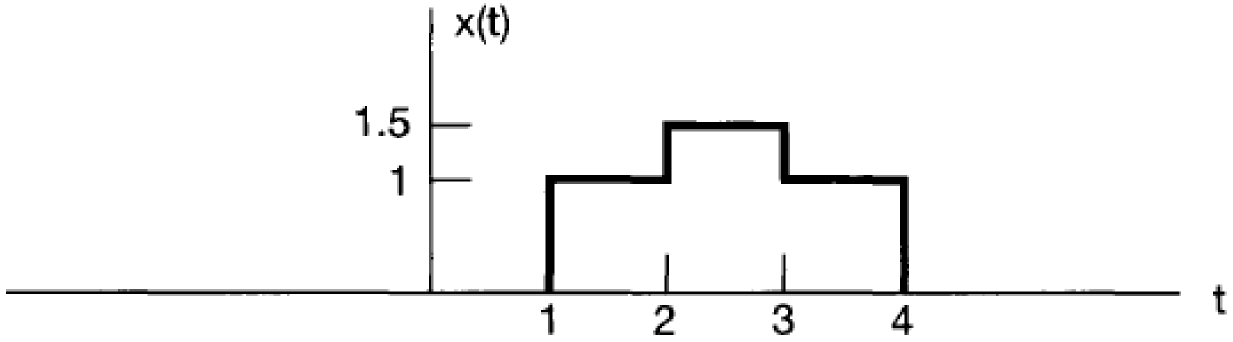
then

$$\mathfrak{F}\{x(t-t_0)\} = e^{-j\omega t_0}X(j\omega) = |X(j\omega)|e^{j[\sphericalangle X(j\omega)-\omega t_0]}$$

Thus, the effect of a time shift on a signal is to introduce into its transform a phase shift, namely, $-\omega t_0$, which is a linear function of ω.

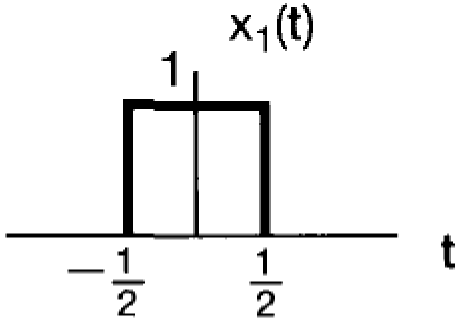
Example 9

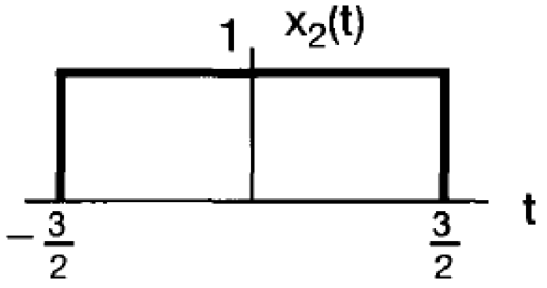
To illustrate the usefulness of the Fourier transform linearity and time-shift properties, let us consider the evaluation of the Fourier transform of the signal $x(t)$ shown in the following figure.



We observe that $x(t)$ can be expressed as the linear combination of two rectangular pulse signals $x_1(t)$ and $x_2(t)$ shown as

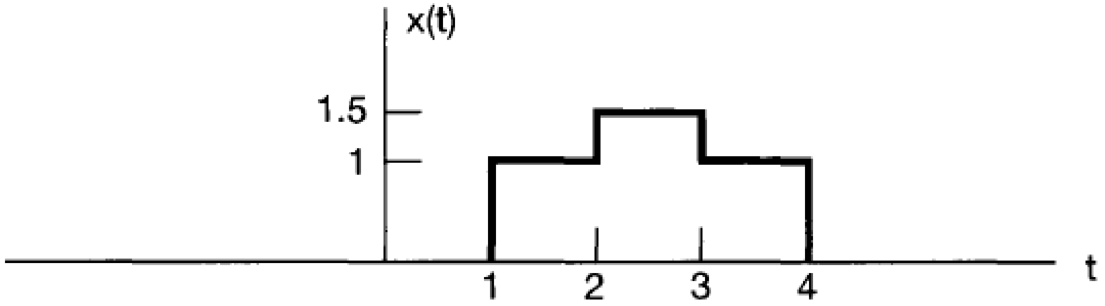
$$x(t) = \frac{1}{2}x_1(t-2.5) + x_2(t-2.5)$$

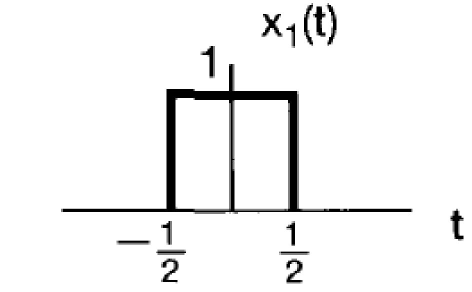




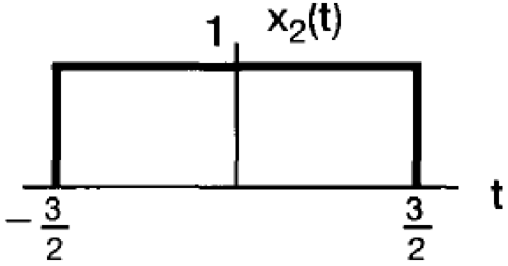
Example 9 - continued

$$x(t) = \frac{1}{2}x_1(t-2.5) + x_2(t-2.5)$$





$$X_1(j\omega) = \frac{2\sin(\omega/2)}{\omega}$$



$$X_2(j\omega) = \frac{2\sin(3\omega/2)}{\omega}$$

Using the linearity and time-shift properties of the Fourier transform yields

$$X(j\omega) = e^{-j5\omega/2}\left\{\frac{\sin(\omega/2) + 2\sin(3\omega/2)}{\omega}\right\}$$

Conjugation and Conjugate Symmetry

The conjugation property states that if

$$x(t) \overset{\mathfrak{F}}{\longleftrightarrow} X(j\omega),$$

then

$$\boxed{x^*(t) \overset{\mathfrak{F}}{\longleftrightarrow} X^*(-j\omega).}$$

One consequence of the conjugation property is that if $x(t)$ is real, then $X(j\omega)$ has conjugate

$$\boxed{X(-j\omega) = X^*(j\omega) \qquad [x(t) \text{ real}].}$$

Conjugation and Conjugate Symmetry

If $x(t)$ is real:

$$\boxed{X(-j\omega) = X^*(j\omega) \qquad [x(t) \text{ real}].}$$

For example:

$$x(t) = e^{-at}u(t),$$

$$X(j\omega) = \frac{1}{a + j\omega}$$

$$X(-j\omega) = \frac{1}{a - j\omega} = X^*(j\omega).$$

Conjugation and Conjugate Symmetry

If $x(t)$ is real:

$$\boxed{X(-j\omega) = X^*(j\omega) \qquad [x(t) \text{ real}].}$$

If we express $X(j\omega)$ in rectangular form as

$$X(j\omega) = \mathcal{R}e\{X(j\omega)\} + j\mathcal{I}m\{X(j\omega)\},$$

then if $x(t)$ is real,

$$\mathcal{R}e\{X(j\omega)\} = \mathcal{R}e\{X(-j\omega)\}$$

and

$$\mathcal{I}m\{X(j\omega)\} = -\mathcal{I}m\{X(-j\omega)\}.$$

That is, the real part of the Fourier transform is an even function of frequency, and the imaginary part is an odd function of frequency.

Conjugation and Conjugate Symmetry

If $x(t)$ is real:

$$\boxed{X(-j\omega) = X^*(j\omega) \qquad [x(t) \text{ real}].}$$

If we express $X(j\omega)$ in polar form as

$$X(j\omega) = |X(j\omega)|e^{j\sphericalangle X(j\omega)}$$

then the magnitude of the Fourier Transform $|X(j\omega)|$ is an even function of frequency and the phase of the Fourier Transform $\sphericalangle X(j\omega)$ is an odd function of frequency.

Thus, when computing or displaying the Fourier transform of a real valued signal, the real and imaginary parts or magnitude and phase of the transform need only be specified for positive frequencies, as the values for negative frequencies can be determined directly from the values for $\omega > 0$ using these relationships.

Conjugation and Conjugate Symmetry

If $x(t)$ is real:

$$\boxed{X(-j\omega) = X^*(j\omega) \qquad [x(t) \text{ real}].}$$

If we express $X(j\omega)$ in polar form as

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Thus, when computing or displaying the Fourier transform of a real valued signal, the real and imaginary parts or magnitude and phase of the transform need only be specified for positive frequencies, as the values for negative frequencies can be determined directly from the values for $\omega > 0$ using these relationships.

Conjugation and Conjugate Symmetry

If $x(t)$ is real:

$$\boxed{X(-j\omega) = X^*(j\omega) \qquad [x(t) \text{ real}].}$$

As a further consequence,

- If $x(t)$ is both real and even, then $X(j\omega)$ will also be real and even.
- If $x(t)$ is both real and odd, then $X(j\omega)$ is purely imaginary and odd.

Remember that a real function $x(t)$ can always be expressed in terms of the sum of its even and odd parts:

$$x(t) = x_e(t) + x_o(t).$$

$$\mathfrak{F}\{x(t)\} = \mathfrak{F}\{x_e(t)\} + \mathfrak{F}\{x_o(t)\}$$

$$x(t) \overset{\mathfrak{F}}{\longleftrightarrow} X(j\omega),$$

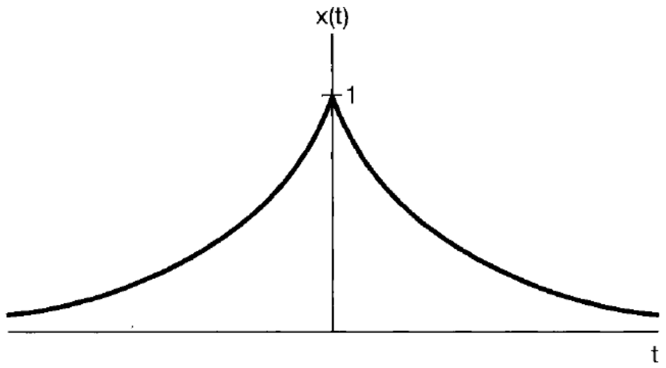
$$\mathcal{E}v\{x(t)\} \overset{\mathfrak{F}}{\longleftrightarrow} \Re e\{X(j\omega)\},$$

$$\mathcal{O}d\{x(t)\} \overset{\mathfrak{F}}{\longleftrightarrow} j\Im m\{X(j\omega)\}.$$

Example 10

Consider again the Fourier transform evaluation for the signal

$$x(t) = e^{-a|t|}, \quad a > 0.$$



This time we will utilize the symmetry properties of the Fourier transform to aid the evaluation process.

We have:

$$e^{-at}u(t) \overset{\mathfrak{F}}{\longleftrightarrow} \frac{1}{a + j\omega}$$

$$\mathcal{E}v\{e^{-at}u(t)\} \overset{\mathfrak{F}}{\longleftrightarrow} \Re e\left\{\frac{1}{a + j\omega}\right\}$$

Real valued

$$\begin{aligned} x(t) &= e^{-a|t|} = e^{-at}u(t) + e^{at}u(-t) \\ &= 2\left[\frac{e^{-at}u(t) + e^{at}u(-t)}{2}\right] \\ &= 2\mathcal{E}v\{e^{-at}u(t)\}. \end{aligned}$$

$$X(j\omega) = 2\Re e\left\{\frac{1}{a + j\omega}\right\} = \frac{2a}{a^2 + \omega^2}$$

Differentiation and Integration

Let $x(t)$ be a signal with Fourier transform $X(j\omega)$. Then,

$$\boxed{\frac{dx(t)}{dt} \stackrel{\mathfrak{F}}{\longleftrightarrow} j\omega X(j\omega).}$$

Differentiation in the time domain corresponds to multiplication by $j\omega$ in the frequency domain.
One might conclude that integration should involve division by $j\omega$ in the frequency domain. The precise relationship is

$$\boxed{\int_{-\infty}^{t} x(\tau)d\tau \stackrel{\mathfrak{F}}{\longleftrightarrow} \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega).}$$

The impulse term on the right-hand side of the equation reflects the dc or average value that can result from integration.

Example 1

Let us determine the Fourier transform $X(j\omega)$ of the unit step $x(t) = u(t)$, making use of the integration property and the knowledge that

$$g(t) = \delta(t) \stackrel{\mathfrak{F}}{\longleftrightarrow} G(j\omega) = 1$$

Note that

$$x(t) = \int_{-\infty}^{t} g(\tau)d\tau$$

Taking the Fourier transform of both sides and using the integration property

$$X(j\omega) = \frac{G(j\omega)}{j\omega} + \pi G(0)\delta(\omega)$$

$$X(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega).$$

Example 1 - continued

$$X(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega).$$

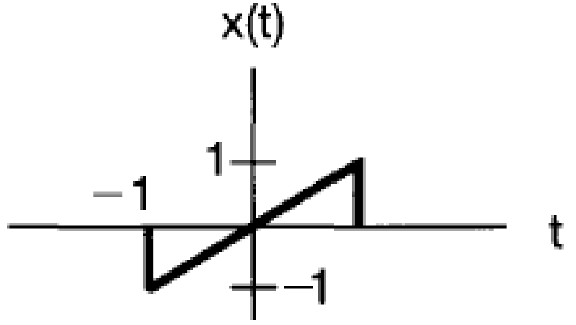
Observe that we can apply the differentiation property of Fourier Transform to recover the transform of the impulse

$$\omega\delta(\omega) = 0$$

$$\delta(t) = \frac{du(t)}{dt} \stackrel{\mathfrak{F}}{\longleftrightarrow} j\omega\left[\frac{1}{j\omega} + \pi\delta(\omega)\right] = 1;$$

Example 2

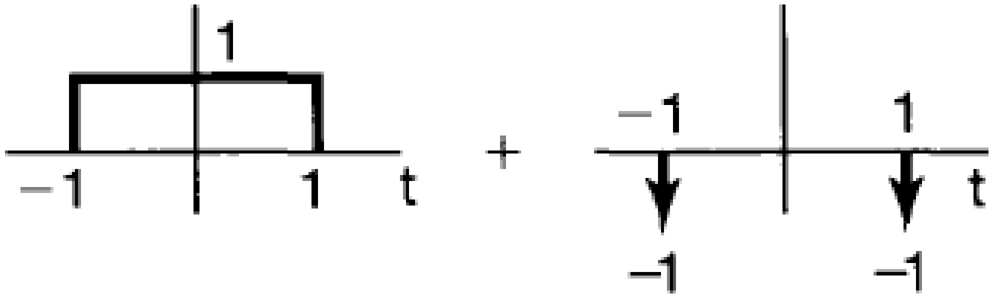
Let us calculate the Fourier transform $X(j\omega)$ for the signal $x(t)$ shown below.



Rather than applying the Fourier integral directly to $x(t)$, we instead consider the signal

$$g(t) = \frac{d}{dt}x(t).$$

$$g(t) = \frac{dx(t)}{dt} =$$

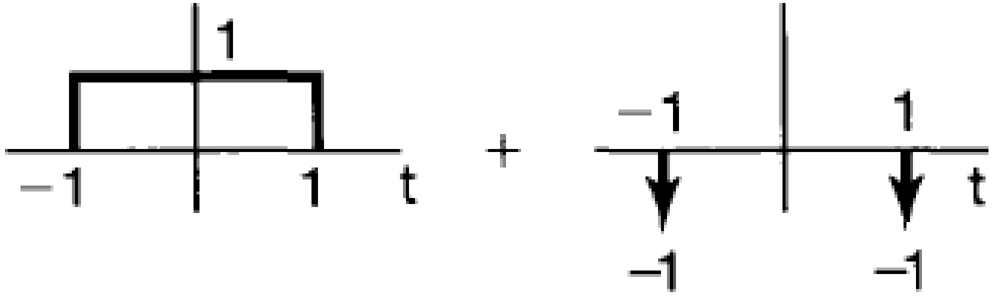


$g(t)$ is the sum of a rectangular pulse and two impulses.

Example 2 - continued

$$g(t) = \frac{d}{dt}x(t).$$

$$g(t) = \frac{dx(t)}{dt} =$$



$g(t)$ is the sum of a rectangular pulse and two impulses.

$$G(j\omega) = \left(\frac{2\sin\omega}{\omega}\right) - e^{j\omega} - e^{-j\omega}$$

Using the integration property:

$$X(j\omega) = \frac{G(j\omega)}{j\omega} + \pi G(0)\delta(\omega). \qquad G(0) = 0$$

$$X(j\omega) = \frac{2\sin\omega}{j\omega^2} - \frac{2\cos\omega}{j\omega}.$$

Time and Frequency Scaling

If

$$x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(j\omega),$$

then

$$\boxed{x(at) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|}X\left(\frac{j\omega}{a}\right),}$$

where a is a nonzero real number.
Aside from the amplitude factor $|a|$, a linear scaling in time by a factor of a corresponds to a linear scaling in frequency by a factor of $1/a$, and vice versa.

Also, letting $a = -1$, we see that

$$\boxed{x(-t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(-j\omega).}$$

That is, reversing a signal in time also reverses its Fourier transform.

Duality

We will discuss the duality property of Fourier Transform with an example. Remember that we derived two Fourier Transform pairs:

$$x_1(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases} \stackrel{\mathfrak{F}}{\longleftrightarrow} X_1(j\omega) = \frac{2\sin\omega T_1}{\omega},$$

$$x_2(t) \doteq \frac{\sin Wt}{\pi t} \stackrel{\mathfrak{F}}{\longleftrightarrow} X_2(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}.$$

The symmetry exhibited by these two examples extends to Fourier transforms in general.
For any transform pair (a signal and its Fourier transform), there is a dual pair with the time and frequency variables interchanged.

Duality

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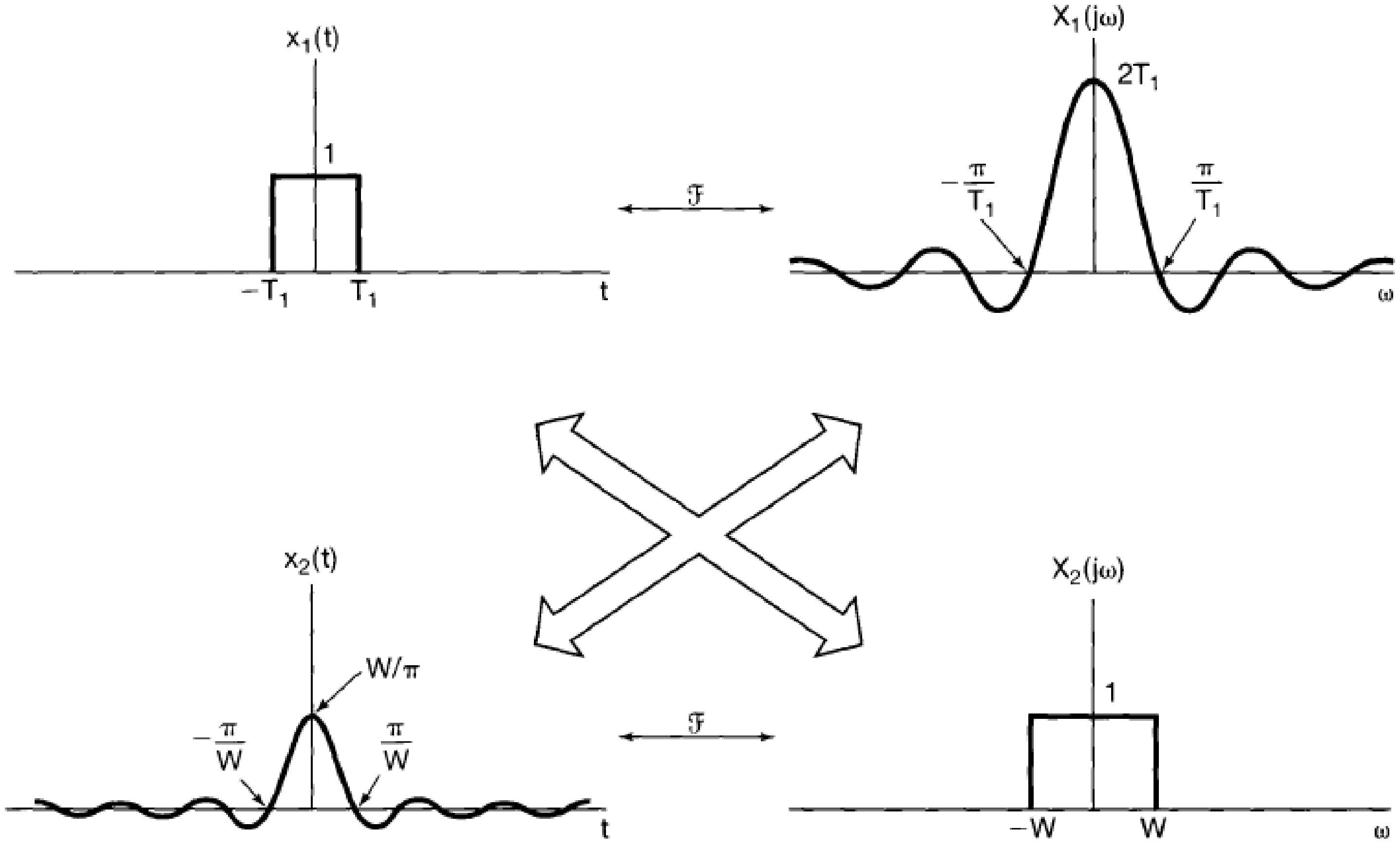


Example 3

Let us consider using duality to find the Fourier transform $G(j\omega)$ of the signal

$$g(t) = \frac{2}{1+t^2}$$

In a previous example, we encountered a Fourier transform pair in which the Fourier transform, as a function of ω, had a similar form to the signal $g(t)$.
Specifically, suppose we consider a signal $x(t)$ whose Fourier transform is

$$X(j\omega) = \frac{2}{1+\omega^2}$$

$$x(t) = e^{-|t|} \xleftrightarrow{\mathfrak{F}} X(j\omega) = \frac{2}{1+\omega^2}$$

Example 3 - continued

$$x(t) = e^{-|t|} \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega) = \frac{2}{1+\omega^2}$$

The synthesis equation for this Fourier transform pair is

$$e^{-|t|} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{2}{1+\omega^2}\right)e^{j\omega t}d\omega$$

Multiplying this equation by 2π and replacing t by $-t$, we obtain

$$2\pi e^{-|t|} = \int_{-\infty}^{\infty}\left(\frac{2}{1+\omega^2}\right)e^{-j\omega t}d\omega.$$

Now, interchanging the names of the variables t and w, we find that

$$2\pi e^{-|\omega|} = \int_{-\infty}^{\infty}\left(\frac{2}{1+t^2}\right)e^{-j\omega t}dt.$$

Example 3 - continued

$$2\pi e^{-|\omega|} = \int_{-\infty}^{\infty} \left(\frac{2}{1+t^2}\right) e^{-j\omega t}\, dt.$$

The right-hand side of equation is the Fourier transform analysis equation for $\left(\frac{2}{1+t^2}\right)$

Thus

$$\mathfrak{F}\left\{\frac{2}{1+t^2}\right\} = 2\pi e^{-|\omega|}$$

Frequency Shifting

Let $x(t)$ be a signal with Fourier transform $X(j\omega)$. Then,

$$\boxed{e^{j\omega_0 t}x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j(\omega - \omega_0))}$$

Differentiation in Frequency

Let $x(t)$ be a signal with Fourier transform $X(j\omega)$. Then,

$$\boxed{-jtx(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{dX(j\omega)}{d\omega}.}$$

Parseval's Relation

If $x(t)$ and $X(j\omega)$ are a Fourier transform pair, then

$$\boxed{\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega.}$$

This expression is referred to as Parseval's relation.

The term on the left-hand side of the equation is the total energy in the signal $x(t)$.

Parseval's relation says that this total energy may be determined either by computing the energy per unit time $|x(t)|^2$ and integrating over all time or by computing the energy per unit frequency $|X(j\omega)|^2$ and integrating over all frequencies.
For this reason, $|X(j\omega)|^2$ is often referred to as the energy-density spectrum of the signal $x(t)$.

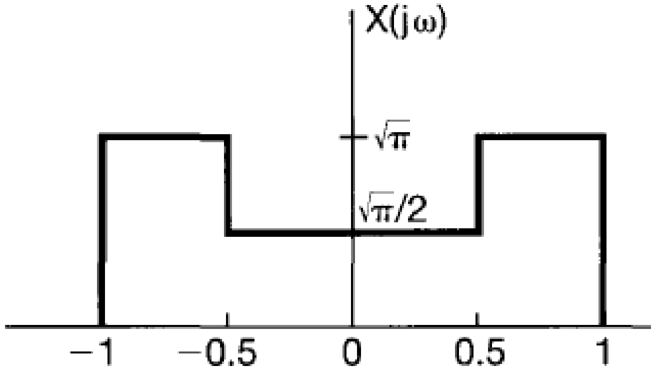
Example 4

Parseval's relation and other Fourier transform properties are often useful in determining some time domain characteristics of a signal directly from the Fourier transform.

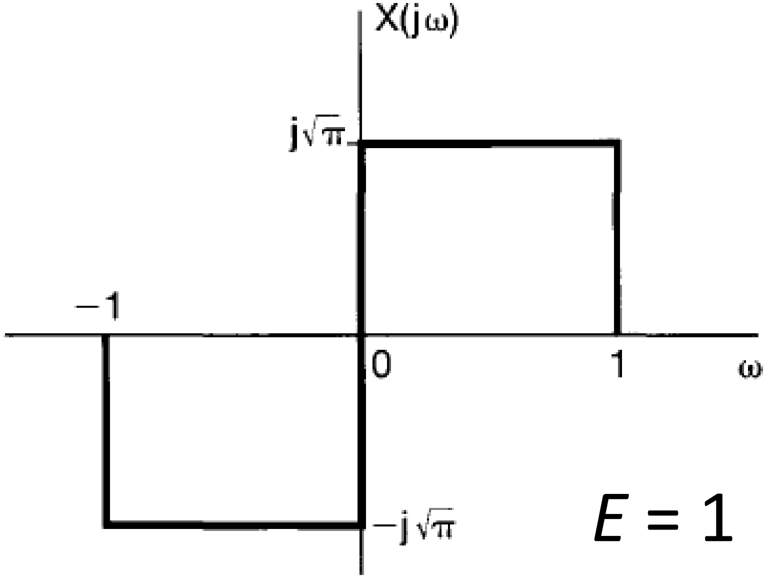
For each of the Fourier transforms shown below, we wish to evaluate the following time-domain expressions:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$D = \frac{d}{dt} x(t) \Big|_{t=0}$$



E = 5/8



E = 1

To evaluate *E* in the frequency domain, we may use Parseval's relation:

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

Example 4 - continued

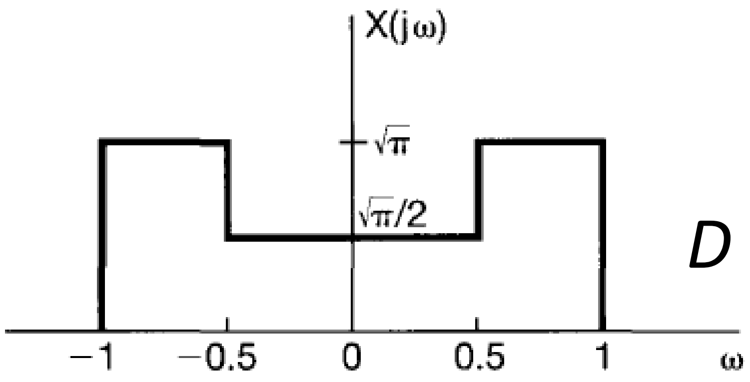
$$D = \left.\frac{d}{dt}x(t)\right|_{t=0}$$

To evaluate *D* in the frequency domain, we first use the differentiation property to observe that

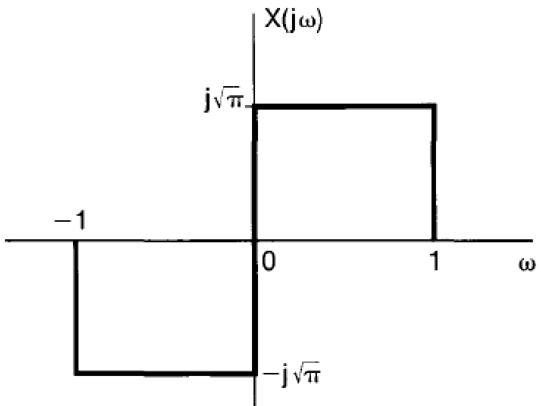
$$g(t) = \frac{d}{dt}x(t) \stackrel{\mathfrak{F}}{\longleftrightarrow} j\omega X(j\omega) = G(j\omega).$$

Evaluating the synthesis equation for $g(t)$ at $t = 0$

$$D = g(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(j\omega)d\omega \qquad D = \int_{-\infty}^{\infty} j\omega X(j\omega)\,d\omega$$



$D = \dfrac{-1}{(2\sqrt{\pi})}$



$D = 0$

Convolution Property

Consider the convolution integral of the signals $x(t)$ and $h(t)$.

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau.$$

Given the Fourier transform $X(j\omega)$ of $x(t)$ and the Fourier transform $H(j\omega)$ of $h(t)$

$$Y(j\omega) = H(j\omega)X(j\omega)$$

That is,

$$\boxed{y(t) = h(t) * x(t) \overset{\mathfrak{F}}{\longleftrightarrow} Y(j\omega) = H(j\omega)X(j\omega).}$$

The Fourier transform maps the convolution of two signals into the product of their Fourier transforms.
Convolution in the time domain corresponds to multiplication in the frequency domain.

Multiplication Property

Consider the product the signals $s(t)$ and $p(t)$.

$$r(t) = s(t)p(t)$$

Given the Fourier transform $S(j\omega)$ of $s(t)$ and the Fourier transform $P(j\omega)$ of $p(t)$

The Fourier transform $R(j\omega)$ of $r(t)$ corresponds to the convolution of $S(j\omega)$ and $P(j\omega)$ in the frequency domain:

$$\boxed{r(t) = s(t)p(t) \longleftrightarrow R(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S(j\theta)P(j(\omega - \theta))d\theta}$$

Multiplication in the time domain corresponds to convolution in the frequency domain.