Signals and Systems

Lecture 7

Analysis of LTI Systems Using Fourier Transform

Eigenfunctions of a system

A signal for which the system output is a (possibly complex) constant times the input is referred to as an eigenfunction of the system, and the amplitude factor is referred to as the system's eigenvalue.

Eigenfunctions of LTI systems

$$x(t) = e^{st} \longrightarrow \boxed{h(t)} \longrightarrow y(t) = H(s)e^{st}$$

$s = \sigma + j\omega$ is a complex number

Complex exponentials are eigenfunctions of LTI systems. The response of an LTI system to a complex exponential input is the same complex exponential with only a change in amplitude; that is

$$y(t) = H(s)e^{st}$$

where $H(s)$ is a complex constant, whose value depends on s and which is related to the system impulse response by

$$H(s) = \int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}\,d\tau.$$

The constant $H(s)$ for a specific value of s is then the eigenvalue associated with the eigenfunction e^{st}.

Eigenfunctions of LTI systems

$$x(t) = e^{st} \longrightarrow \boxed{h(t)} \longrightarrow y(t) = H(s)e^{st}$$

$s = \sigma + j\omega$ is a complex number

In general, the variable $s = \sigma + j\omega$ may be an arbitrary complex number.

Fourier analysis involves restricting our attention to a particular form of s.

In continuous time we focus on purely imaginary values of s, i.e., $s = j\omega$, and thus, we consider only complex exponentials of the form $e^{j\omega t}$.

$$e^{j\omega t} \longrightarrow \boxed{h(t)} \longrightarrow H(j\omega)e^{j\omega t}$$

Revisiting the Convolution Property

$$\boxed{y(t) = h(t) * x(t) \overset{\mathfrak{F}}{\longleftrightarrow} Y(j\omega) = H(j\omega)X(j\omega).}$$

Let $h(t)$ be the impulse response of a linear time-invariant system, and $x(t)$ be the input.
$H(j\omega)$ (the Fourier transform of the impulse response), is called **the frequency response of the system**.
The Fourier transform $Y(j\omega)$ of the output signal $y(t)$ can be calculated by multiplying $H(j\omega)$ and the Fourier transform $X(j\omega)$ of the input signal $x(t)$.

$$Y(j\omega) = H(j\omega)X(j\omega)$$

The frequency response $H(j\omega)$ plays as important a role in the analysis of LTI systems as does its inverse transform, the unit impulse response.
Since $h(t)$ completely characterizes an LTI system, then so must $H(j\omega)$.

Frequency Response

$$y(t) = h(t) * x(t) \stackrel{\mathfrak{F}}{\longleftrightarrow} Y(j\omega) = H(j\omega)X(j\omega).$$

$H(j\omega)$ (the Fourier transform of the impulse response $h(t)$), is called **the frequency response of the system**.

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt$$

$$Y(j\omega) = H(j\omega)X(j\omega)$$

Example 1

Consider a continuous-time LTI system with impulse response

$$h(t) = \delta(t - t_0)$$

The frequency response of this system is the Fourier transform of $h(t)$ and is given by

$$H(j\omega) = e^{-j\omega t_0}$$

Thus, for any input $x(t)$ with Fourier transform $X(j\omega)$, the Fourier transform of the output is

$$\begin{aligned} Y(j\omega) &= H(j\omega)X(j\omega) \\ &= e^{-j\omega t_0}X(j\omega). \end{aligned}$$

Using the time-shifting property of Fourier Transform:

$$y(t) = x(t - t_0)$$

Example 2

Let us examine a differentiator-that is, an LTI system for whichthe input x(t) and the output y(t) are related by

$$y(t) = \frac{dx(t)}{dt}.$$

From the differentiation property of the Fourier Transform:

$$Y(j\omega) = j\omega X(j\omega)$$

It follows that the frequency response of a differentiator is

$$H(j\omega) = j\omega.$$

Example 3

Consider an integrator -that is, an LTI system specified by the equation

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau.$$

The impulse response for this system is the unit step $u(t)$.

$$\mathrm{h(t)} = u(t)$$

Then, the frequency response of this system is

$$H(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega).$$

For any input $x(t)$ with Fourier transform $X(j\omega)$, the Fourier transform of the output is

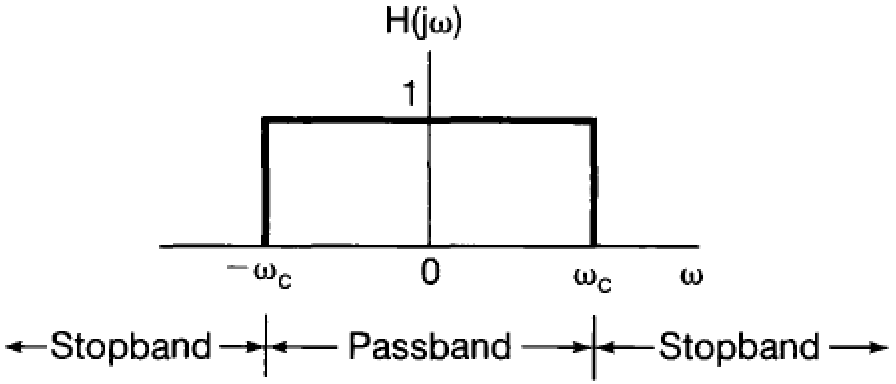
$$\begin{aligned} Y(j\omega) &= H(j\omega)X(j\omega) \\ &= \frac{1}{j\omega}X(j\omega) + \pi X(j\omega)\delta(\omega) \\ &= \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega), \end{aligned}$$

Example 4

Frequency-selective filtering can be accomplished with an LTI system whose frequency response $H(j\omega)$ passes the desired range of frequencies and significantly attenuates frequencies outside that range.

In this example, we will consider the ideal lowpass filter, which has the frequency response

$$H(j\omega) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & |\omega| > \omega_c \end{cases}$$



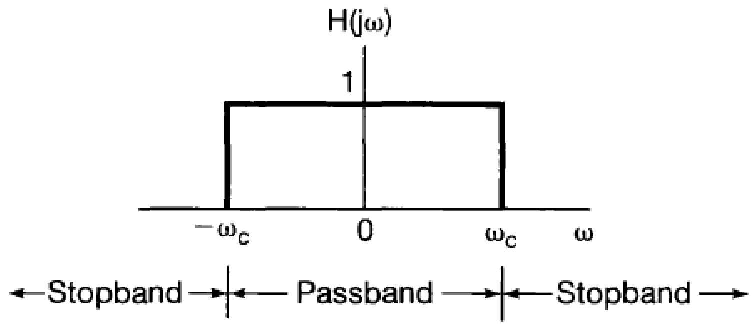
The impulse response $h(t)$ of this ideal filter is the inverse transform of $H(j\omega)$:

$$h(t) = \frac{\sin \omega_c t}{\pi t}$$

Example 4 -continued

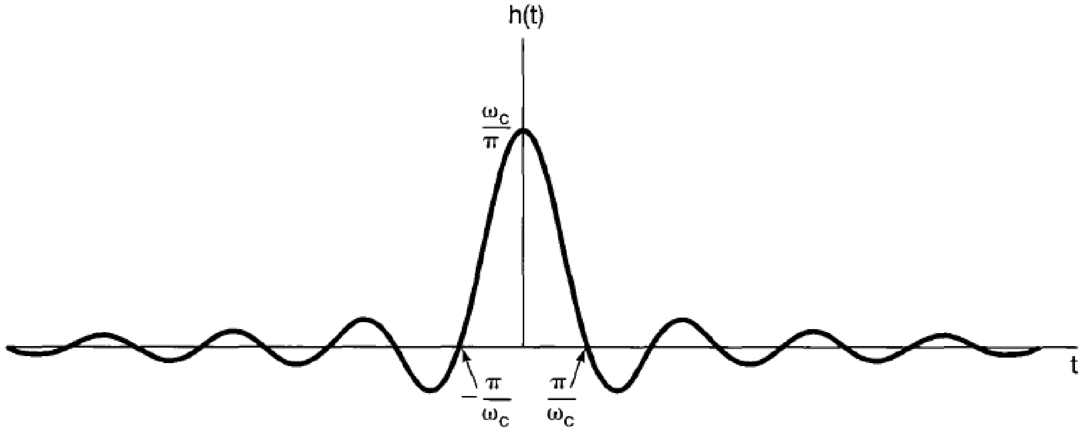
The ideal lowpass filter

$$H(j\omega) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & |\omega| > \omega_c \end{cases}$$



The impulse response $h(t)$ of this ideal filter is the inverse transform of $H(j\omega)$:

$$h(t) = \frac{\sin \omega_c t}{\pi t}$$



The ideal filter is not causal and not easy to approximate closely.
Furthermore, in some applications (such as the automobile suspension system), oscillatory behavior in the impulse response of a lowpass filter may be undesirable.

Example 5

Nonideal frequency-selective filters that are more easily implemented are typically preferred.

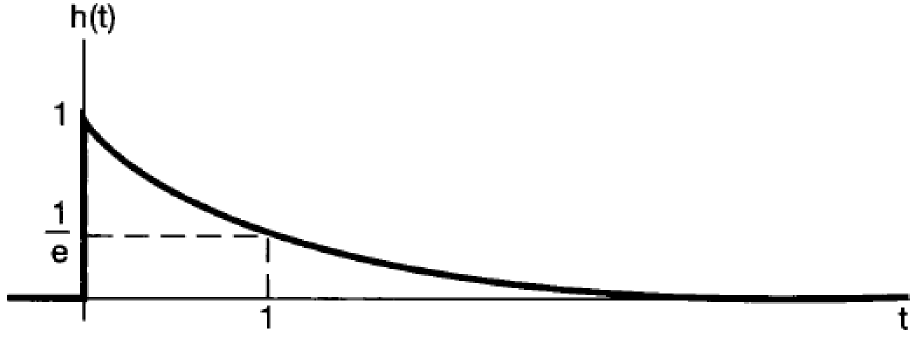
For example, consider the LTI system with impulse response

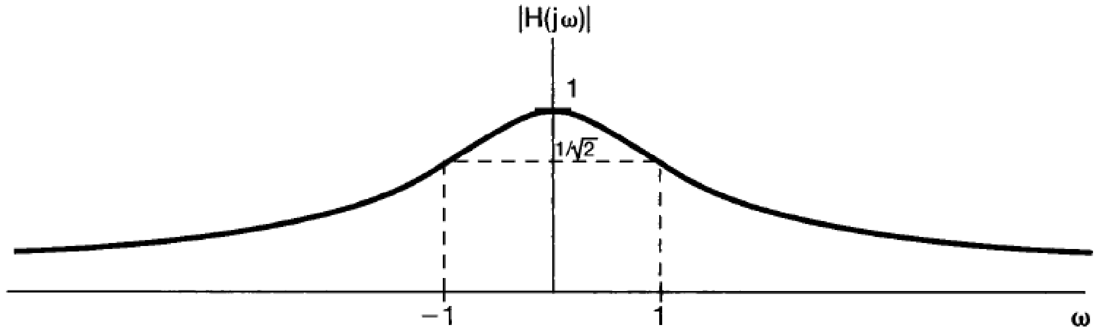
$$h(t) = e^{-t}u(t)$$

The frequency response of this system is

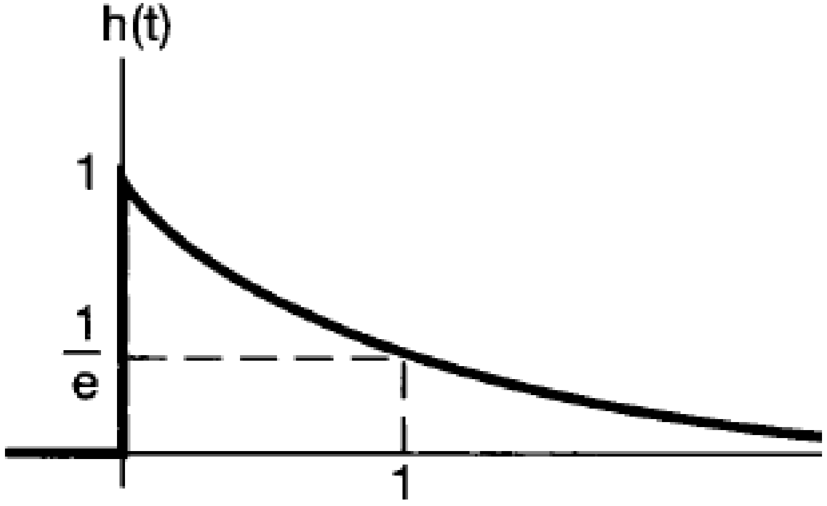
$$H(j\omega) = \frac{1}{j\omega + 1}$$

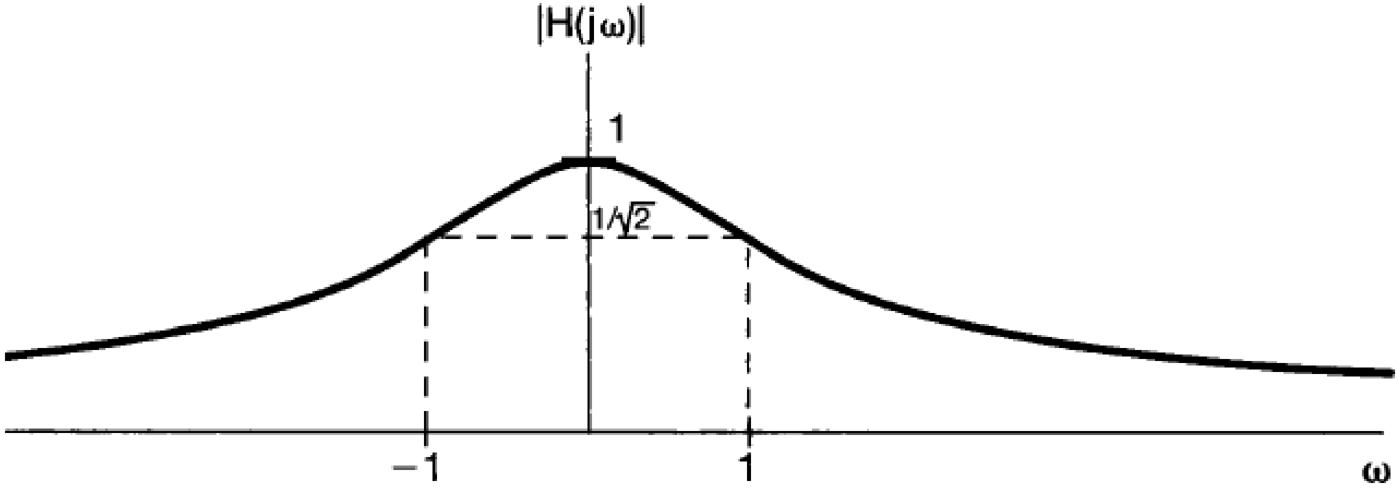
This system can be implemented with a simple RC circuit.





Example 5 - continued





This system does not have the strong frequency selectivity of the ideal lowpass filter,. But it is causal and has an impulse response that decays monotonically, i.e., without oscillations.

Nonideal filters are quite frequently preferred to ideal filters because of

- their causality,
- ease of implementation,
- and flexibility in allowing trade-offs, among other design considerations such as frequency selectivity and oscillatory behavior in the time domain.

Example 6

Find the output of an LTI sytem with impulse response

$$h(t) = e^{-at}u(t), \quad a > 0,$$

when the input signal is

$$x(t) = e^{-bt}u(t), \quad b > 0.$$

Rather than computing $y(t) = x(t) * h(t)$ directly, let us transform the problem into the frequency domain.
The Fourier transforms of $x(t)$ and $h(t)$ are

$$X(j\omega) = \frac{1}{b + j\omega} \qquad H(j\omega) = \frac{1}{a + j\omega}.$$

Then, the Fourier transform of the output signal is

$$Y(j\omega) = H(j\omega)X(j\omega). \qquad Y(j\omega) = \frac{1}{(a + j\omega)(b + j\omega)}$$

Example 6 - continued

The Fourier transform of the output signal is

$$Y(j\omega) = \frac{1}{(a + j\omega)(b + j\omega)}$$

To determine the output $y(t)$, we wish to obtain the inverse transform of $Y(j\omega)$. This is most simply done by expanding $Y(j\omega)$ in a partial-fraction expansion. Assuming that $b \neq a$, the partial fraction expansion for $Y(j\omega)$ takes the form

$$Y(j\omega) = \frac{A}{a + j\omega} + \frac{B}{b + j\omega} = \frac{1}{(a + j\omega)(b + j\omega)}$$

where A and B are constants to be determined.

$$A = \frac{1}{b - a} = -B,$$

$$Y(j\omega) = \frac{1}{b - a}\left[\frac{1}{a + j\omega} - \frac{1}{b + j\omega}\right]$$

Example 6 - continued

$$Y(j\omega) = \frac{1}{b-a}\left[\frac{1}{a+j\omega} - \frac{1}{b+j\omega}\right]$$

The inverse transform for each of the two terms in can be recognized by inspection. Using the linearity property of Fourier Transform, we have

$$\boxed{y(t) = \frac{1}{b-a}[e^{-at}u(t) - e^{-bt}u(t)]}$$

When $b = a$, the partial fraction expansion is not valid.
With $b = a$, Y($j\omega$) becomes

$$Y(j\omega) = \frac{1}{(a+j\omega)^2}$$

Example 6 - continued

With $b = a$, $Y(j\omega)$ becomes

$$Y(j\omega) = \frac{1}{(a + j\omega)^2}$$

Recognizing this as

$$\frac{1}{(a + j\omega)^2} = j\frac{d}{d\omega}\left[\frac{1}{a + j\omega}\right]$$

we can use the property for differentiation in frequency

$$\boxed{-jtx(t) \overset{\mathfrak{F}}{\longleftrightarrow} \frac{dX(j\omega)}{d\omega}.}$$

$$e^{-at}u(t) \overset{\mathfrak{F}}{\longleftrightarrow} \frac{1}{a + j\omega}$$

$$te^{-at}u(t) \overset{\mathfrak{F}}{\longleftrightarrow} j\frac{d}{d\omega}\left[\frac{1}{a + j\omega}\right] = \frac{1}{(a + j\omega)^2}$$

$$\boxed{y(t) = te^{-at}u(t)}$$

Multiplication Property

- Modulation
- Frequency-Selective Filtering with Variable Center Frequency

Multiplication Property - revisited

Consider the product the signals $s(t)$ and $p(t)$.

$$r(t) = s(t)p(t)$$

$$\boxed{r(t) = s(t)p(t) \longleftrightarrow R(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S(j\theta)P(j(\omega-\theta))d\theta}$$

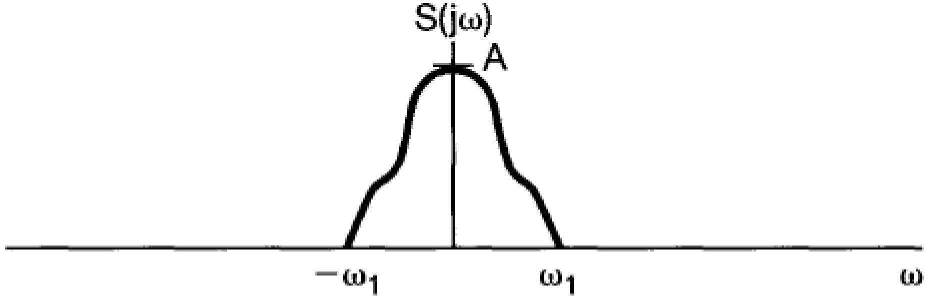
Multiplication in the time domain corresponds to convolution in the frequency domain.

Multiplication of one signal by another can be thought of as using one signal to scale or modulate the amplitude of the other, and consequently, the multiplication of two signals is often referred to as amplitude modulation.

The multiplication property is sometimes referred to as the modulation property.

Example 1

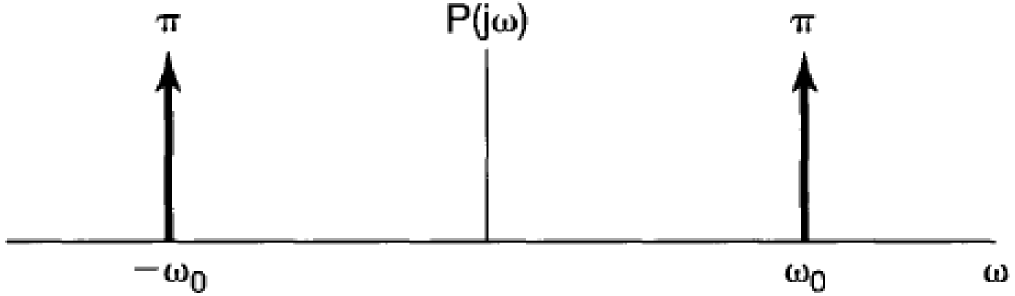
Let $s(t)$ be a signal whose spectrum $S(j\omega)$ is depicted below



Also, consider the signal

$$p(t) = \cos \omega_0 t.$$

$$P(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$



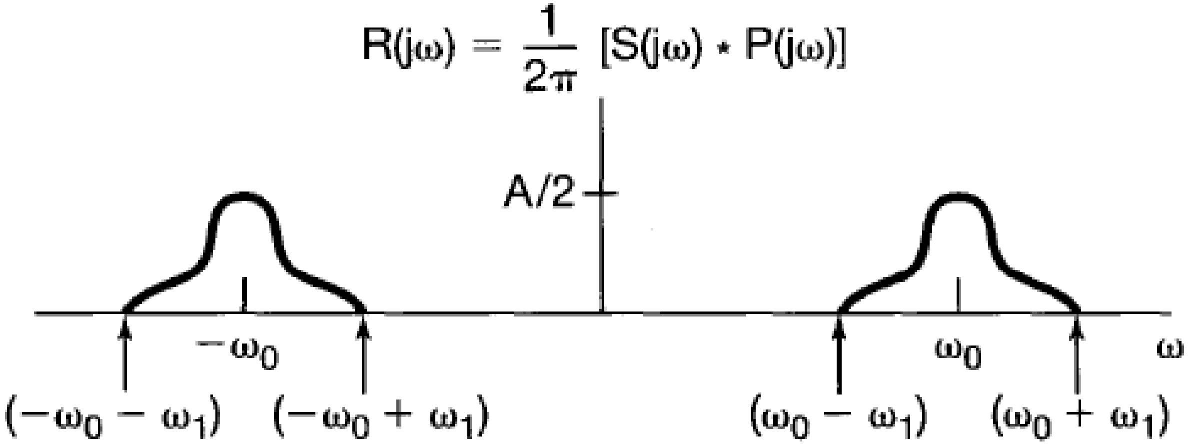
The spectrum $R(j\omega)$ of $r(t) = s(t)p(t)$ is obtained by

$$R(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S(j\theta)P(j(\omega - \theta))d\theta = \frac{1}{2}S(j(\omega - \omega_0)) + \frac{1}{2}S(j(\omega + \omega_0))$$

Example 1 - continued

The spectrum $R(j\omega)$ of $r(t) = s(t)p(t)$ is obtained by

$$R(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S(j\theta)P(j(\omega-\theta))d\theta = \frac{1}{2}S(j(\omega-\omega_0)) + \frac{1}{2}S(j(\omega+\omega_0))$$



Here we have assumed that $\omega_0 > \omega_1$, so that the two nonzero portions of $R(j\omega)$ do not overlap.
The spectrum of $r(t)$ consists of the sum of two shifted and scaled versions of $S(j\omega)$.
The information in the signal $s(t)$ is preserved when we multiply this signal by a sinusoidal signal, although the information has been shifted to higher frequencies. This fact forms the basis for sinusoidal **amplitude modulation systems for communications.**

Example 2

Determine the Fourier transform of the signal

$$x(t) = \frac{\sin(t)\sin(t/2)}{\pi t^2}$$

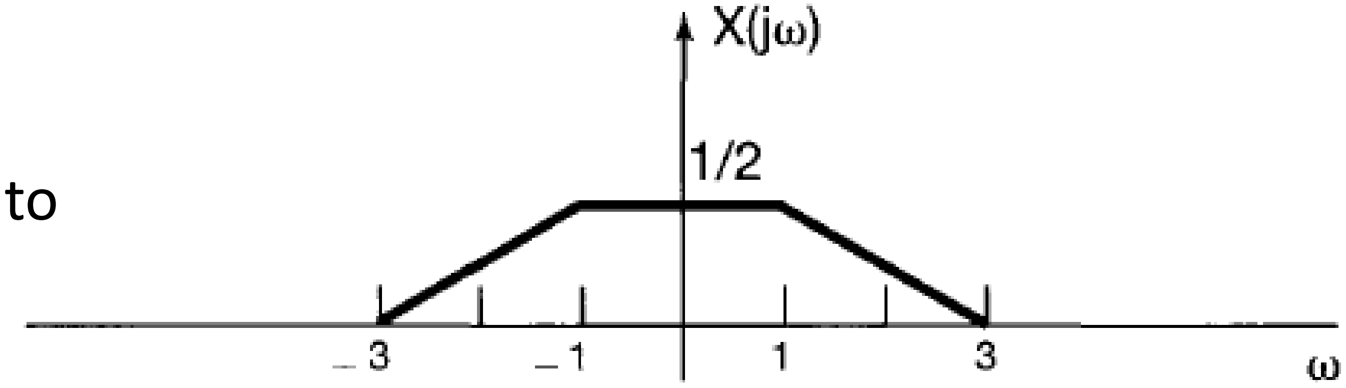
The key here is to recognize $x(t)$ as the product of two sinc functions:

$$x(t) = \pi\left(\frac{\sin(t)}{\pi t}\right)\left(\frac{\sin(t/2)}{\pi t}\right)$$

Applying the multiplication property of the Fourier transform, we obtain

$$X(j\omega) = \frac{1}{2}\mathfrak{F}\left\{\frac{\sin(t)}{\pi t}\right\} * \mathfrak{F}\left\{\frac{\sin(t/2)}{\pi t}\right\}$$

Noting that the Fourier transform of each sinc function is a rectangular pulse, we can convolve those pulses to obtain the function $X(j\omega)$ displayed here.



Frequency-Selective Filtering with Variable Center Frequency

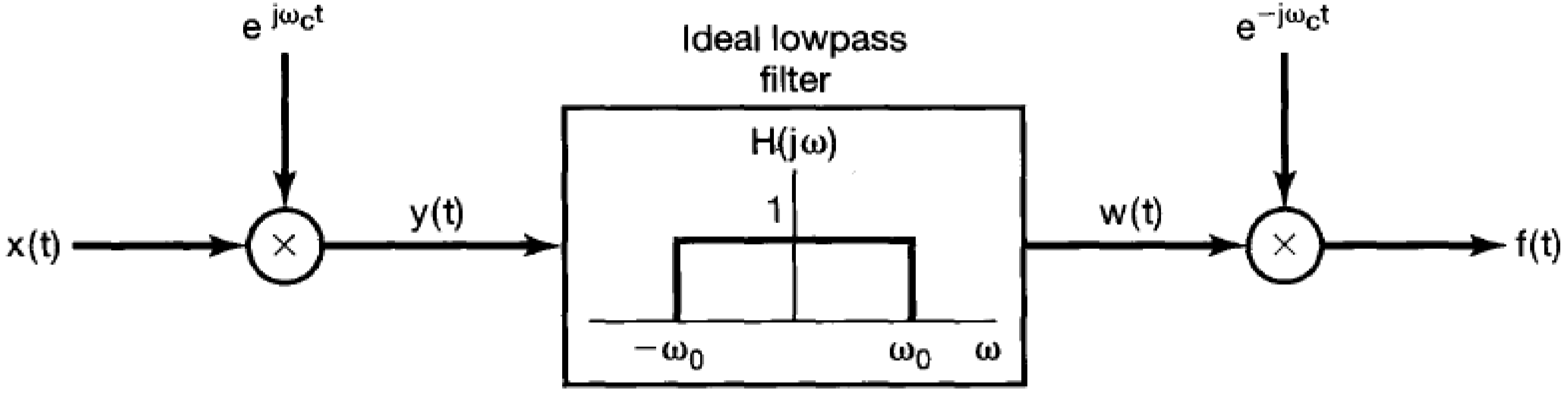
One of the important applications of the multiplication property is amplitude modulation in communication systems.

Another important application is in the implementation of frequency-selective bandpass filters with tunable center frequencies that can be adjusted by the simple turn of a dial.

This is usually done by implementing a fixed frequency-selective filter and shifting the spectrum of the signal appropriately, using the principles of sinusoidal amplitude modulation.

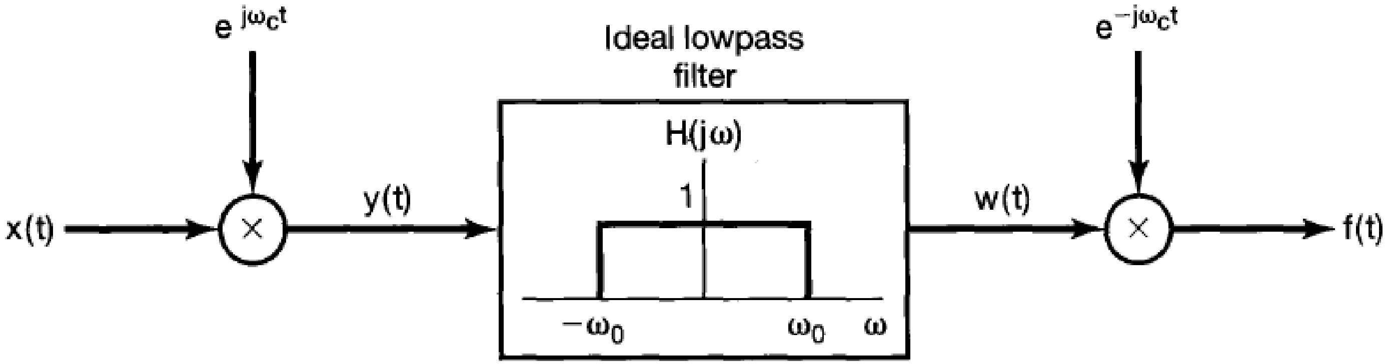
Frequency-Selective Filtering with Variable Center Frequency

As an example consider the following system:



Implementation of a bandpass filter using amplitude modulation with a complex exponential carrier.

Frequency-Selective Filtering with Variable Center Frequency

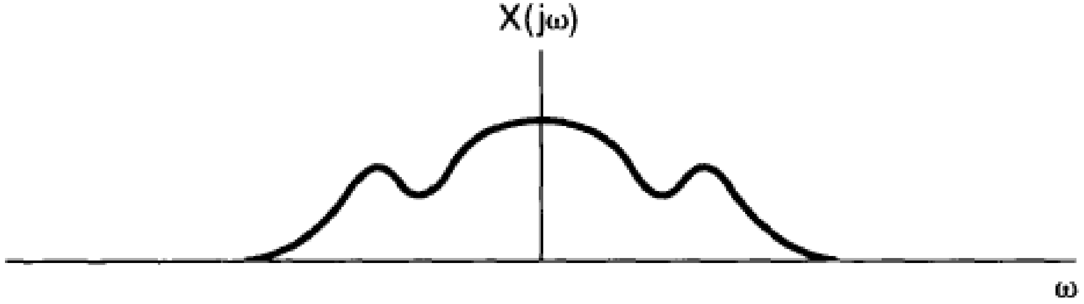


Implementation of a bandpass filter using amplitude modulation with a complex exponential carrier.

From either the multiplication property or the frequency-shifting property, the Fourier transform of $y(t) = e^{j\omega_c t}x(t)$ is

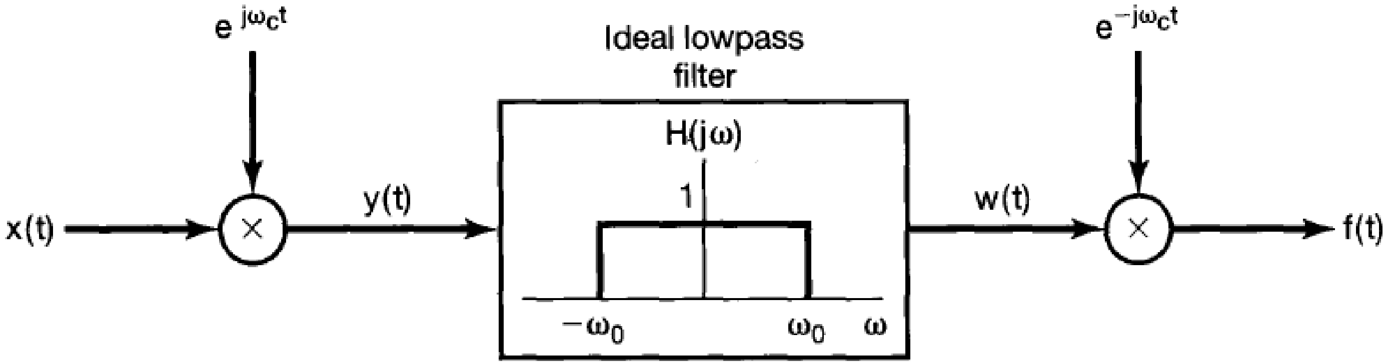
$$Y(j\omega) = \int_{-\infty}^{+\infty} \delta(\theta - \omega_c) X(\omega - \theta) d\theta$$

$$Y(j\omega) = X(j(\omega - \omega_c))$$

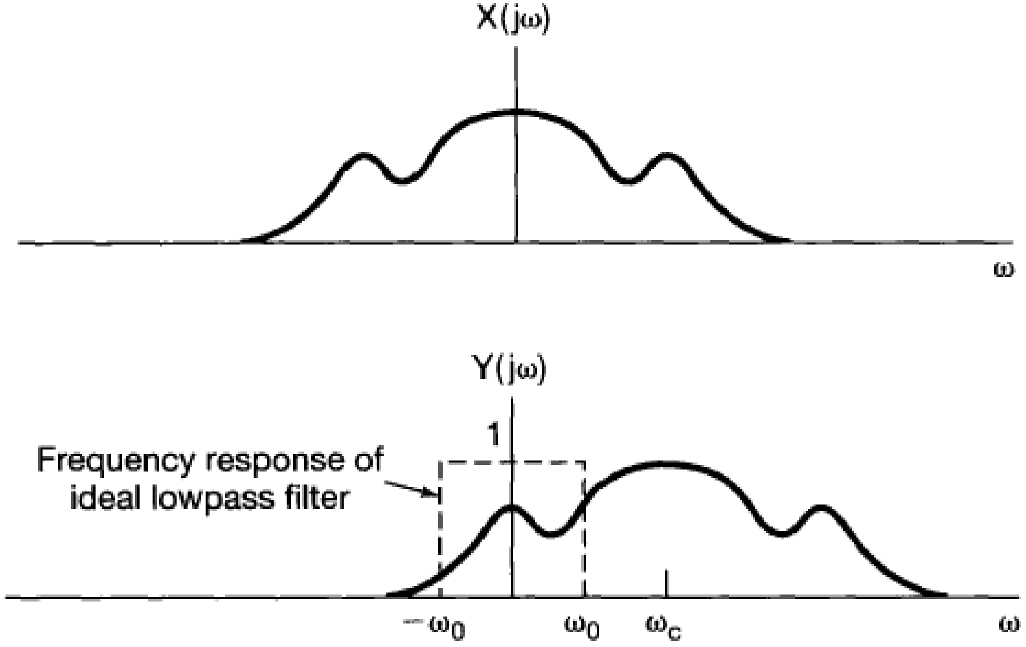


$Y(j\omega)$ equals $X(j\omega)$ shifted to the right by ω_c and frequencies in $X(j\omega)$ near $\omega = \omega_c$ have been shifted into the passband of the lowpass filter.

Frequency-Selective Filtering with Variable Center Frequency

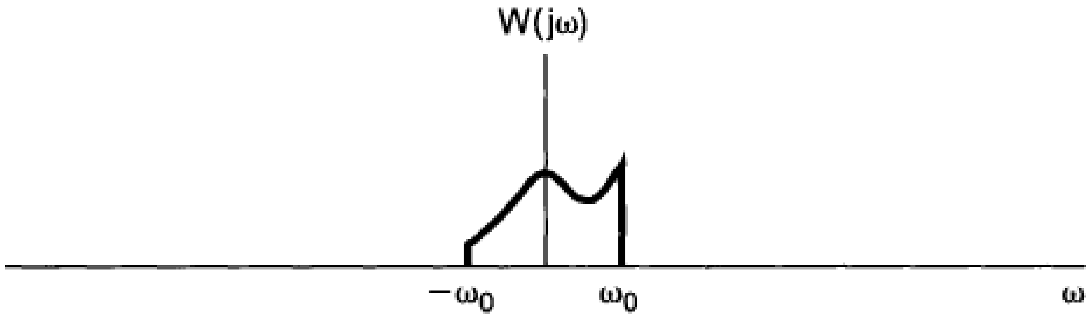


Implementation of a bandpass filter using amplitude modulation with a complex exponential carrier.

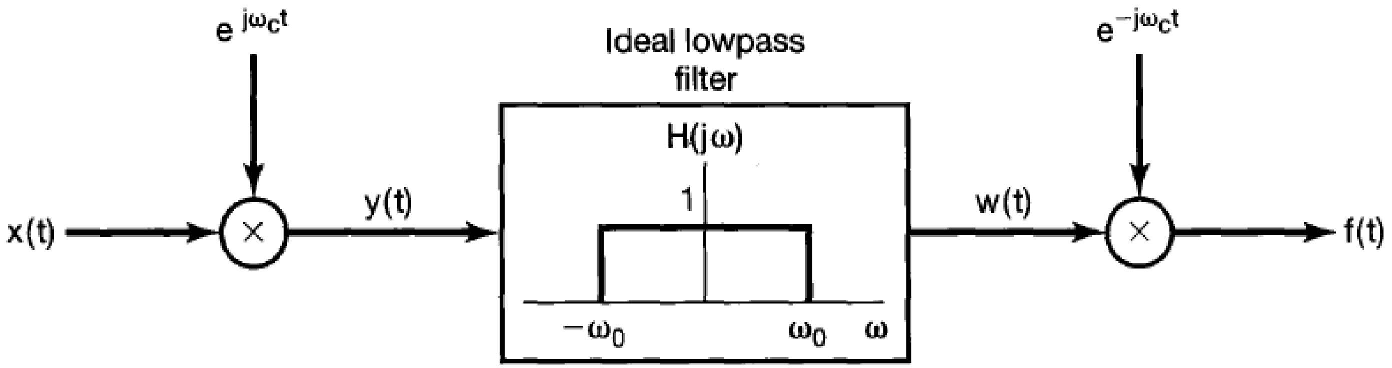


$$Y(j\omega) = X(j(\omega - \omega_c))$$

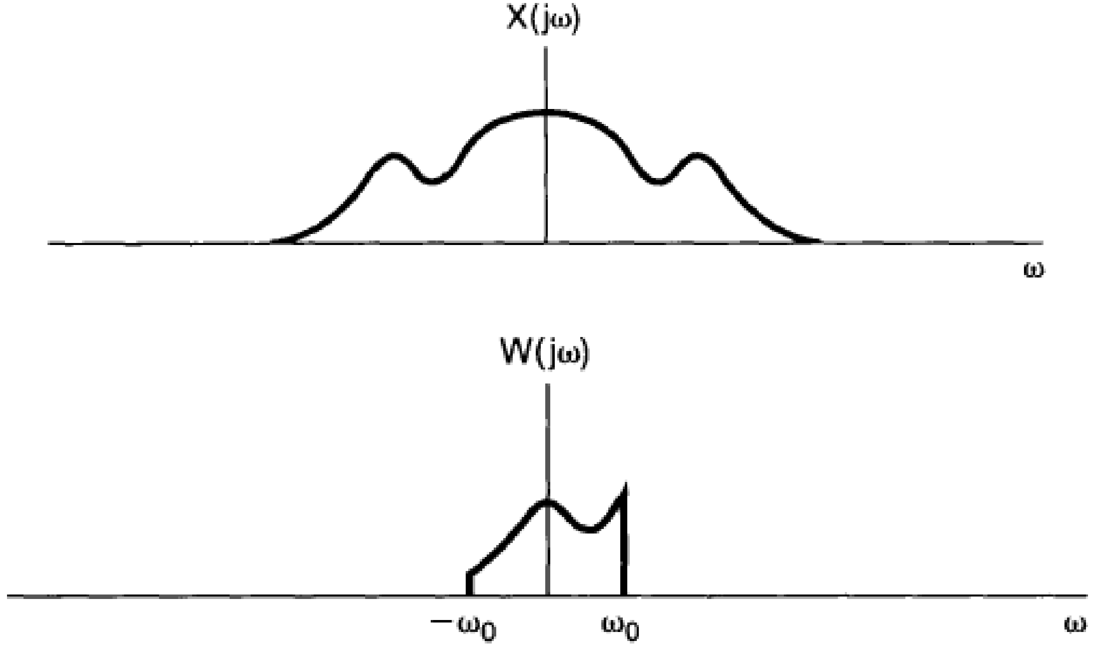
$Y(j\omega)$ equals $X(j\omega)$ shifted to the right by ω_c and frequencies in $X(j\omega)$ near $\omega = \omega_c$ have been shifted into the passband of the lowpass filter.



Frequency-Selective Filtering with Variable Center Frequency

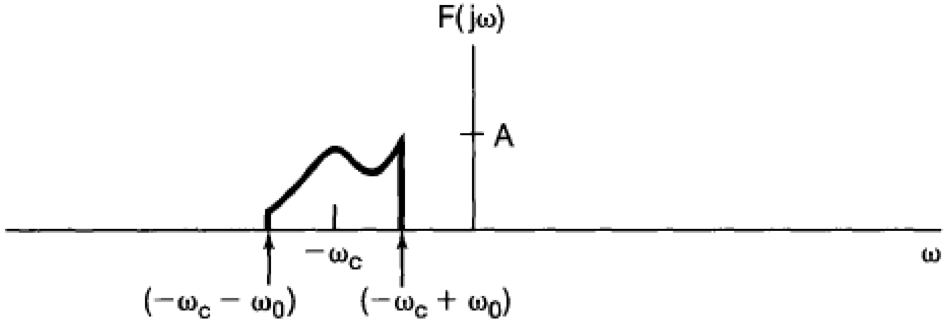


Implementation of a bandpass filter using amplitude modulation with a complex exponential carrier.

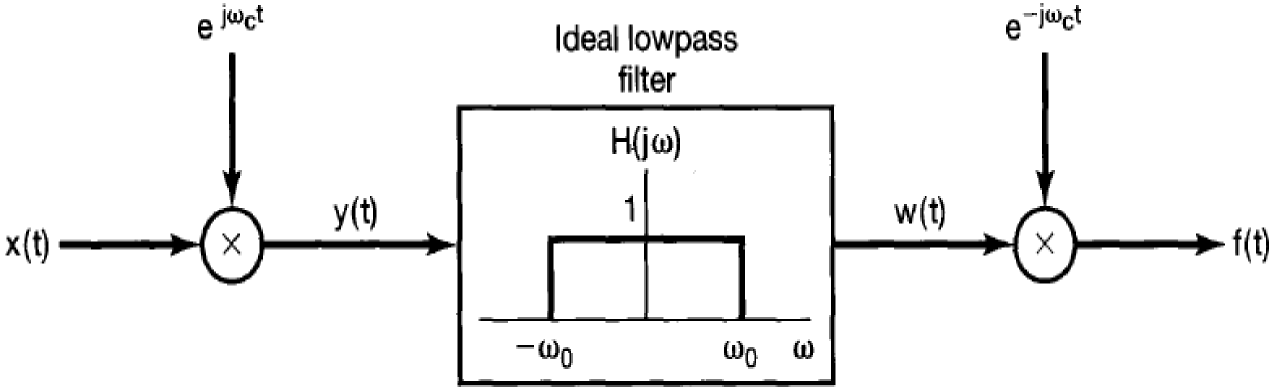


The Fourier Transform of $f(t) = e^{-j\omega_c t}w(t)$ is

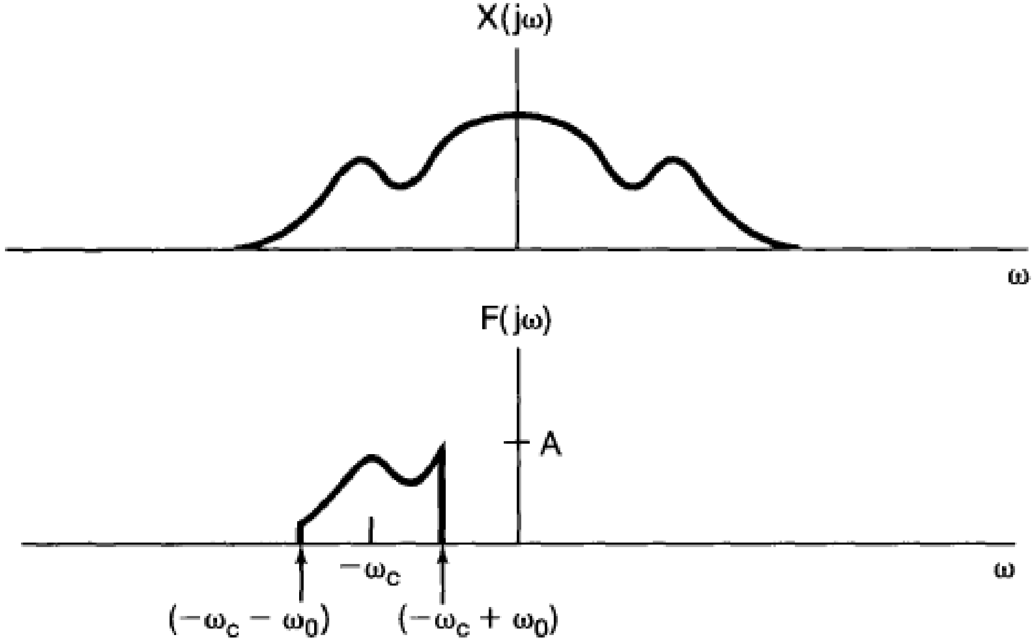
$$F(j\omega) = W(j(\omega + \omega_c))$$



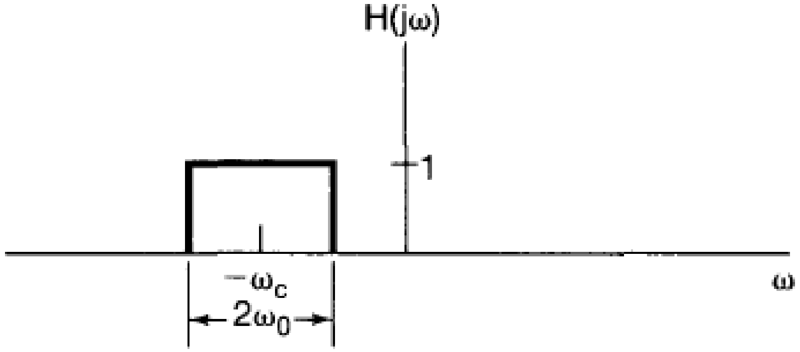
Frequency-Selective Filtering with Variable Center Frequency



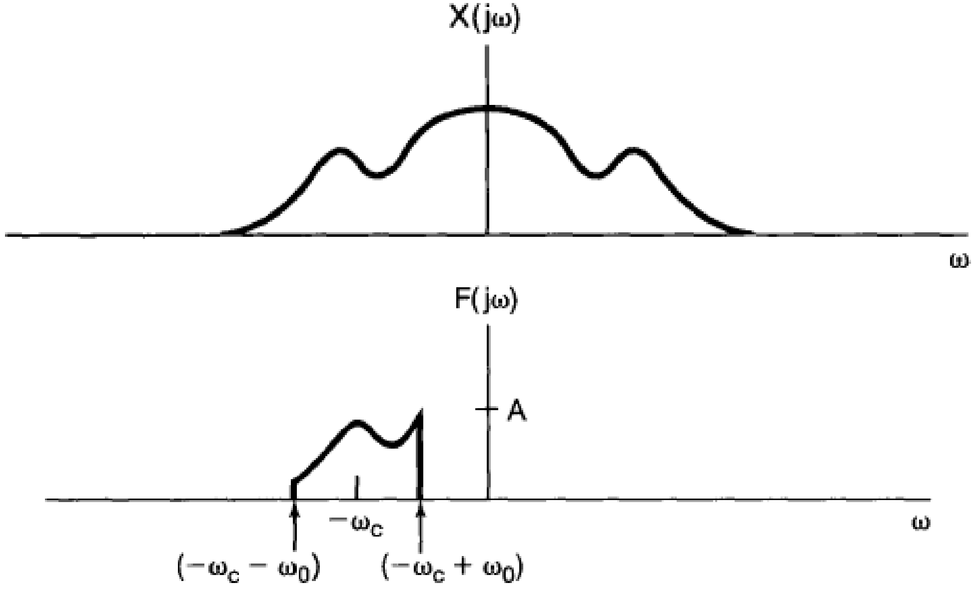
Implementation of a bandpass filter using amplitude modulation with a complex exponential carrier.

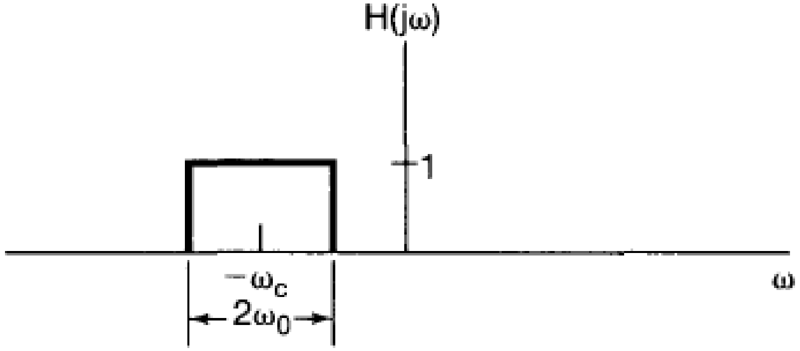


We observe that the overall system is equivalent to an ideal bandpass filter with center frequency ω_c and bandwidth $2\omega_0$, as shown below.



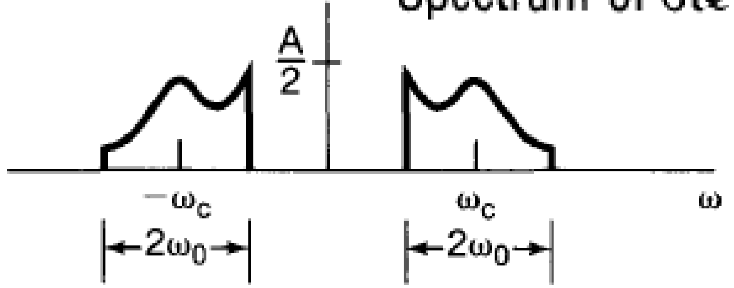
Frequency-Selective Filtering with Variable Center Frequency



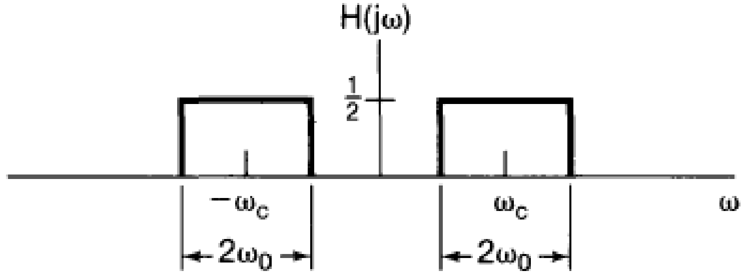


If we retain only the real part of f(t), the resulting spectrum is as shown below, and the equivalent bandpass filter passes bands of frequencies centered around ω_c and $-\omega_c$.

Spectrum of $\mathcal{R}e\{f(t)\}$



Equivalent bandpass filter for $\mathcal{R}e\{f(t)\}$



SYSTEMS CHARACTERIZED BY LINEAR CONSTANT-COEFFICIENT DIFFERENTIAL EQUATIONS

Systems Characterized by Linear Constant-Coefficient Differential Equations

An important and useful class of continuous-time LTI systems is those for which the input and output satisfy a linear constant-coefficient differential equation of the form

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

We will consider the question of determining the frequency response of such an LTI system.

Systems Characterized by Linear Constant-Coefficient Differential Equations

Consider an LTI system characterized by

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

If $X(j\omega)$, $Y(j\omega)$, and $H(j\omega)$ are the Fourier transforms of the input $x(t)$, output $y(t)$, and impulse response $h(t)$, respectively, from the convolution property of Fourier Transform:

$$Y(j\omega) = H(j\omega)X(j\omega) \qquad H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$$

Consider applying the Fourier transform to both sides of the differential equation:

$$\mathfrak{F}\left\{\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k}\right\} = \mathfrak{F}\left\{\sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}\right\}$$

Systems Characterized by Linear Constant-Coefficient Differential Equations

$$\mathfrak{F}\left\{\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k}\right\} = \mathfrak{F}\left\{\sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}\right\}$$

From the linearity property of Fourier Transform:

$$\sum_{k=0}^{N} a_k \mathfrak{F}\left\{\frac{d^k y(t)}{dt^k}\right\} = \sum_{k=0}^{M} b_k \mathfrak{F}\left\{\frac{d^k x(t)}{dt^k}\right\}$$

From the differentiation property:

$$\sum_{k=0}^{N} a_k (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(j\omega),$$

$$Y(j\omega)\left[\sum_{k=0}^{N} a_k (j\omega)^k\right] = X(j\omega)\left[\sum_{k=0}^{M} b_k (j\omega)^k\right]$$

Systems Characterized by Linear Constant-Coefficient Differential Equations

$$Y(j\omega)\left[\sum_{k=0}^{N} a_k (j\omega)^k\right] = X(j\omega)\left[\sum_{k=0}^{M} b_k (j\omega)^k\right]$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}.$$

Observe that the frequency response $H(j\omega)$ of the system is a rational function; that is, it is a ratio of polynomials in $(j\omega)$.

Example 1

Consider a stable LTI system characterized by the differential equation

$$\frac{dy(t)}{dt} + ay(t) = x(t), \qquad \text{with } a > 0.$$

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

Here $a_0 = a$, $a_1 = 1$ and $b_0 = 1$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}.$$

Then, the frequency response for this systems is

$$H(j\omega) = \frac{1}{j\omega + a}.$$

The impulse response of the system is the inverse Fourier transform of $H(j\omega)$:

$$h(t) = e^{-at} u(t).$$

Example 2

Consider a stable LTI system characterized by the differential equation

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t).$$

Find the impulse response of this system.

Here $a_0 = 3, a_1 = 4\ , a_2 = 1$ and $b_0 = 2, b_1 = 1$

Then, the frequency response for this systems is

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}. \qquad H(j\omega) = \frac{(j\omega) + 2}{(j\omega)^2 + 4(j\omega) + 3}$$

To determine the corresponding impulse response, we require the inverse Fourier transform of $H(j\omega)$. This can be found using the technique of partial-fraction expansion.

Example 2 - continued

$$H(j\omega) = \frac{(j\omega) + 2}{(j\omega)^2 + 4(j\omega) + 3}$$

To determine the corresponding impulse response, we require the inverse Fourier transform of $H(j\omega)$. This can be found using the technique of partial-fraction expansion.

$$H(j\omega) = \frac{j\omega + 2}{(j\omega + 1)(j\omega + 3)} \qquad H(j\omega) = \frac{\frac{1}{2}}{j\omega + 1} + \frac{\frac{1}{2}}{j\omega + 3}$$

The impulse response $h(t)$ of the system is the inverse Fourier transform of $H(j\omega)$:

$$h(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t)$$

Example 2 - continued

Find the output of this system if the input is

$$x(t) = e^{-t}u(t).$$

The Fourier Transform of the output is

$$Y(j\omega) = H(j\omega)X(j\omega) = \left[\frac{j\omega + 2}{(j\omega + 1)(j\omega + 3)}\right]\left[\frac{1}{j\omega + 1}\right]$$

$$= \frac{j\omega + 2}{(j\omega + 1)^2(j\omega + 3)}.$$

The partial-fraction expansion takes the form

$$Y(j\omega) = \frac{A_{11}}{j\omega + 1} + \frac{A_{12}}{(j\omega + 1)^2} + \frac{A_{21}}{j\omega + 3}$$

The values obtained are

$$A_{11} = \frac{1}{4}, \quad A_{12} = \frac{1}{2}, \quad A_{21} = -\frac{1}{4},$$

$$Y(j\omega) = \frac{\frac{1}{4}}{j\omega + 1} + \frac{\frac{1}{2}}{(j\omega + 1)^2} - \frac{\frac{1}{4}}{j\omega + 3}$$

Example 2 - continued

The Fourier Transform of the output is

$$Y(j\omega) = \frac{\frac{1}{4}}{j\omega + 1} + \frac{\frac{1}{2}}{(j\omega + 1)^2} - \frac{\frac{1}{4}}{j\omega + 3}$$

The output $y(t)$ of the system is the inverse Fourier transform of $Y(j\omega)$:

$$y(t) = \left[\frac{1}{4}e^{-t} + \frac{1}{2}te^{-t} - \frac{1}{4}e^{-3t}\right]u(t)$$