Signals and Systems

Lecture 8

THE DISCRETE-TIME FOURIER TRANSFORM

REPRESENTATION OF APERIODIC SIGNALS: THE DISCRETE-TIME FOURIER TRANSFORM

The discrete-time Fourier Transform pair can be written as

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega, \quad \rightarrow \text{Synthesis equation}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}. \quad \rightarrow \text{Analysis equation}$$

The function $X(e^{j\omega})$ is referred to as the discrete-time Fourier transform of the signal $x[n]$. $X(e^{j\omega})$ is the spectrum of $x[n]$.

THE DISCRETE-TIME FOURIER TRANSFORM

$X(e^{j\omega})$ is a continuous function of ω and **is periodic with $\boldsymbol{2\pi}$.**

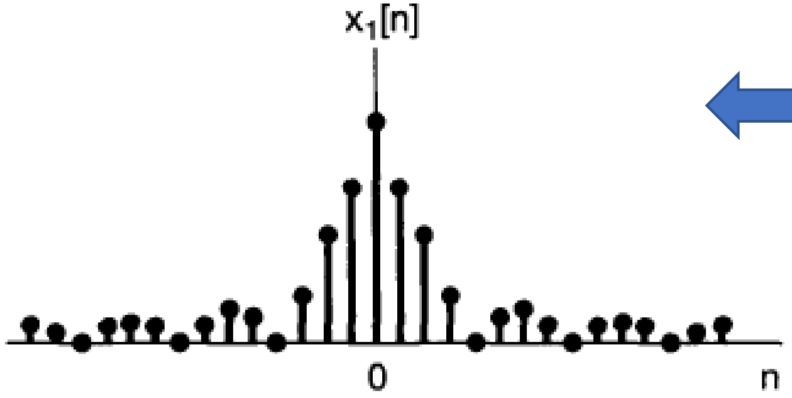
The integral for calculating the inverse discrete-time Fourier Transform of $X\left(e^{j\omega}\right)$:

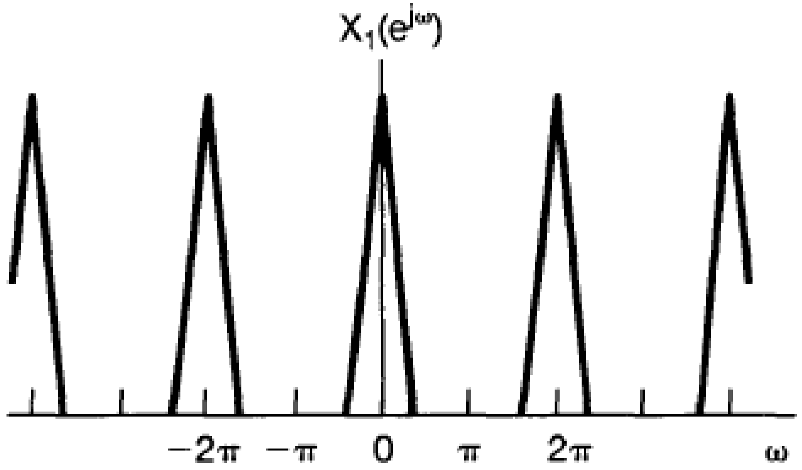
Synthesis equation

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega$$

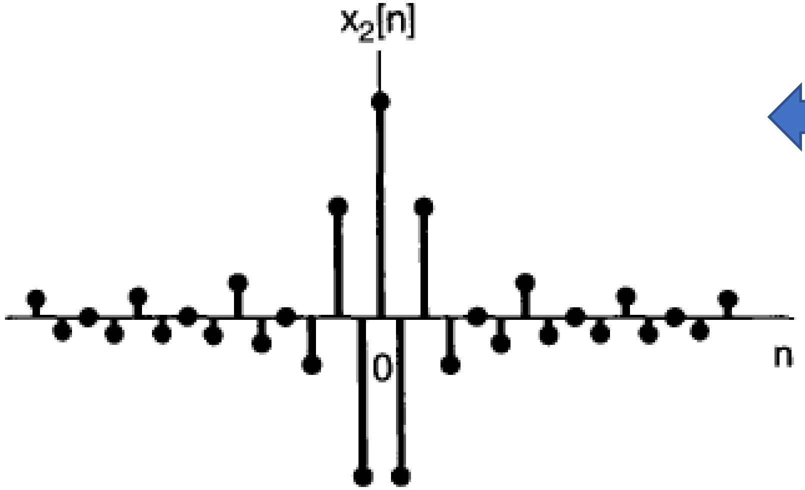
The interval of integration is $\boldsymbol{2\pi}$.

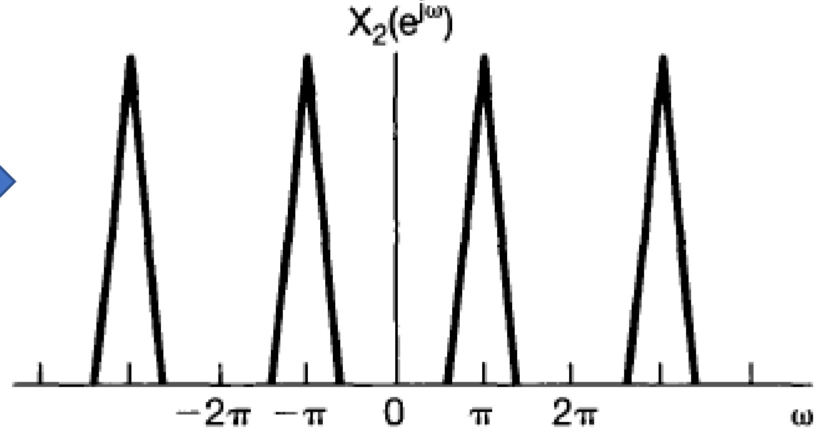
$X(e^{j\omega})$ is a continuous function of ω and **is periodic with 2π.**





Signals at frequencies near even multiples of π, such as $0, \pm 2\pi, \pm 4\pi, \ldots$ are slowly varying and therefore are all thought of as low-frequency signals.





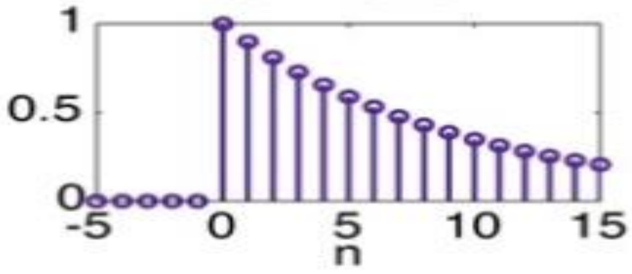
The high frequencies in discrete time are the values of ω near odd multiples of π, such as $\pm\pi, \pm 3\pi$, ...

Example 1

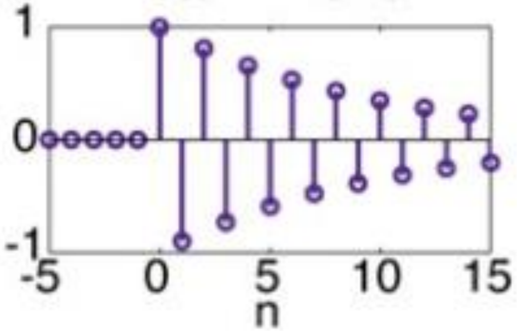
Find the Discrete-time Fourier Transform (DTFT) of the following signal.

$$x[n] = a^n u[n], \qquad |a| < 1$$

$0 < a < 1$



$-1 < a < 0$



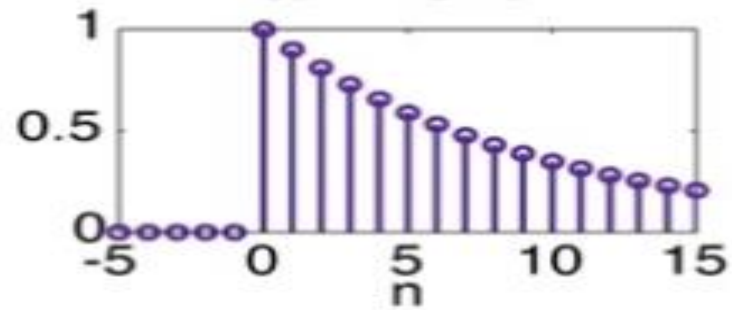
$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} a^n u[n] e^{-j\omega n}$$

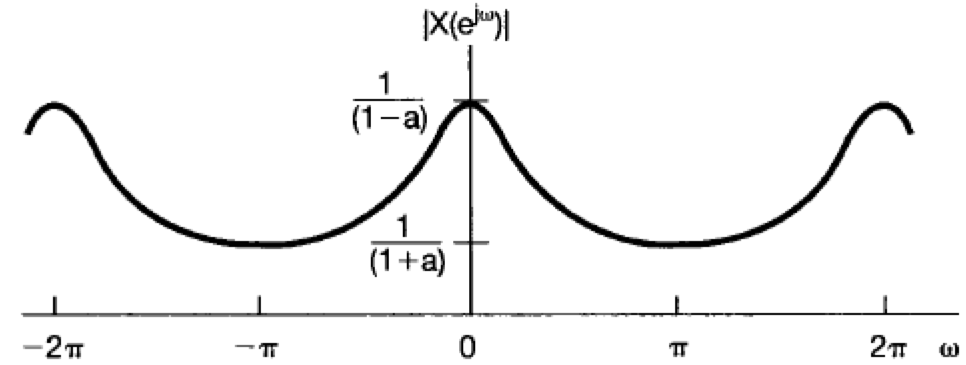
$$= \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1 - ae^{-j\omega}}$$

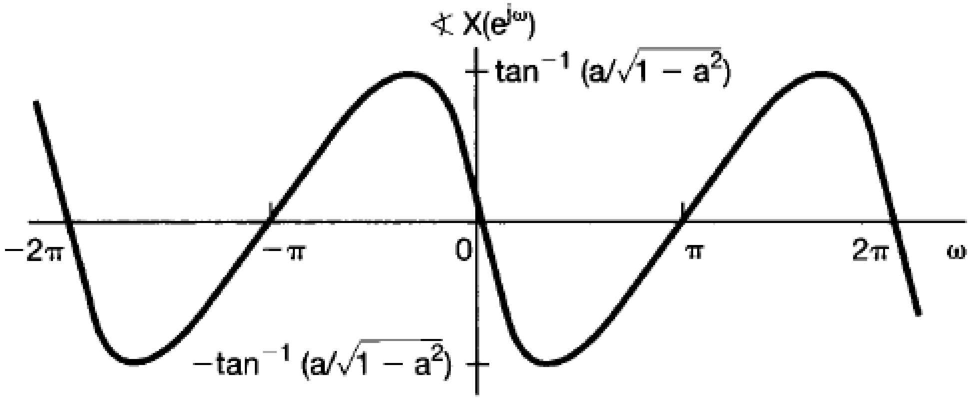
Example 1 - continued

$$x[n] = a^n u[n], \qquad |a| < 1 \qquad \Longleftrightarrow \qquad X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

For $0 < a < 1$





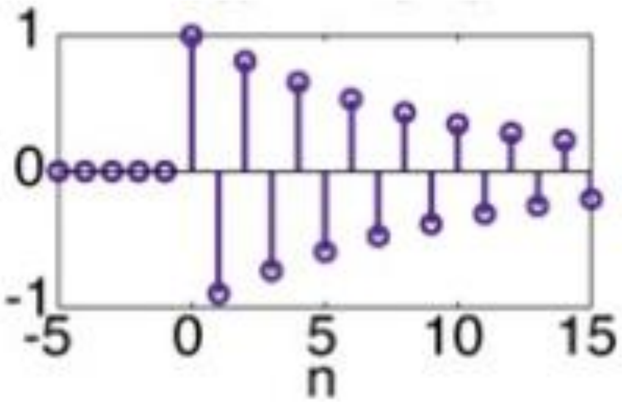


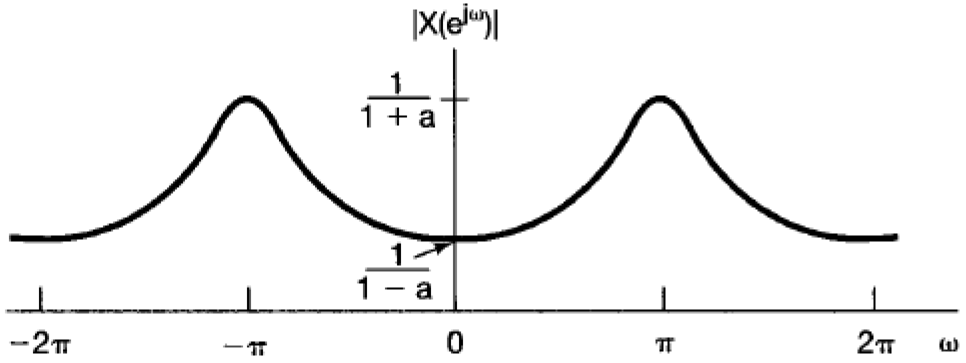
Note that $X(e^{j\omega})$ is periodic in ω with period 2π.
The magnitude spectrum is higher around frequencies $0, \pm 2\pi, \pm 4\pi, ...$,
meaning that the signal is a low-frequency (slowly varying signal).

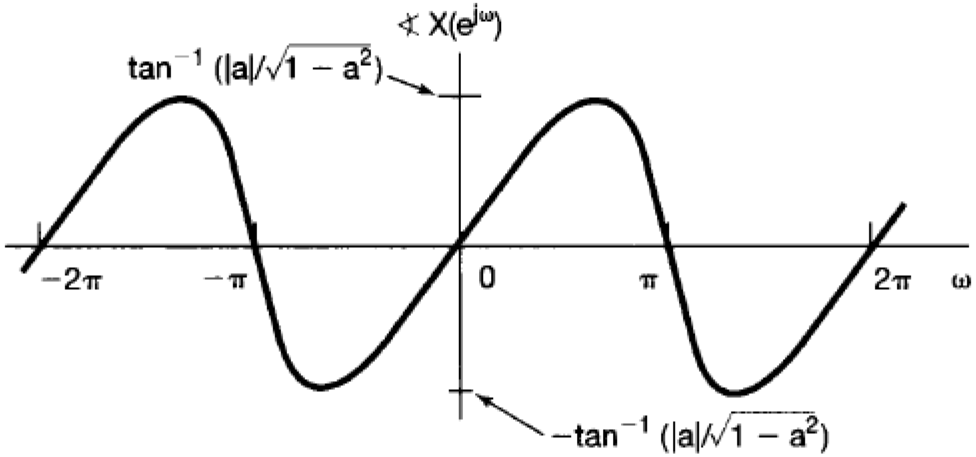
Example 1 - continued

$$x[n] = a^n u[n], \qquad |a| < 1 \quad \Longleftrightarrow \quad X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

For $-1 < a < 0$



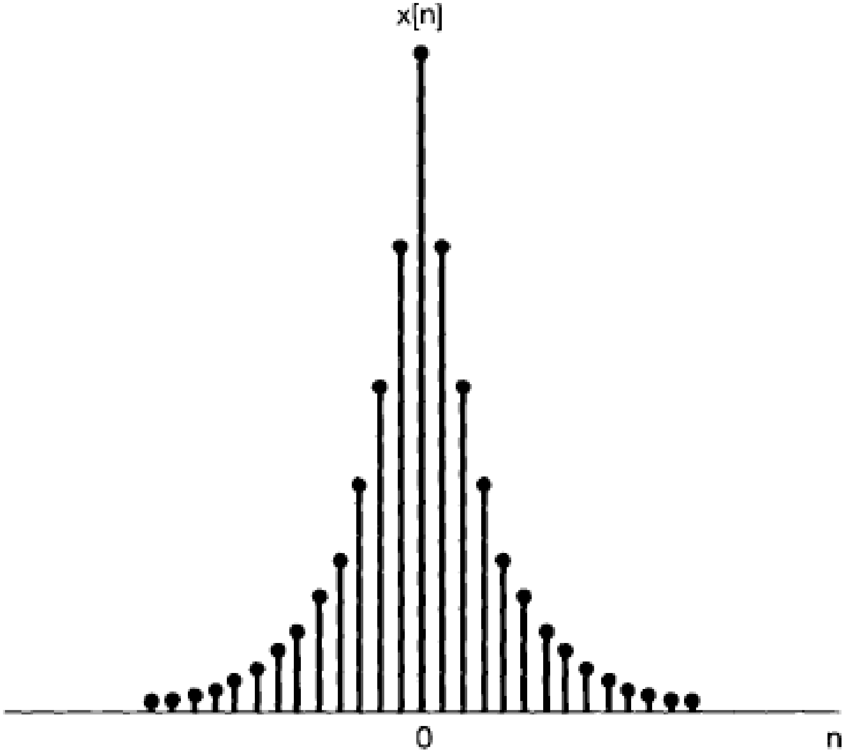




Note that $X(e^{j\omega})$ is periodic in ω with period 2π.
The magnitude spectrum is higher around frequencies $\pm\pi, \pm 3\pi, \ldots$,
meaning that the signal is a high-frequency.

Example 2

Find the Discrete-time Fourier transform (DTFT) of the following signal.



$$x[n] = a^{|n|} \quad 0 < a < 1$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} a^{|n|} e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} a^{n} e^{-j\omega n} + \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n}$$

Making the substitution of variables $m = -n$ in the second summation, we obtain

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} (ae^{-j\omega})^{n} + \sum_{m=1}^{\infty} (ae^{j\omega})^{m}$$

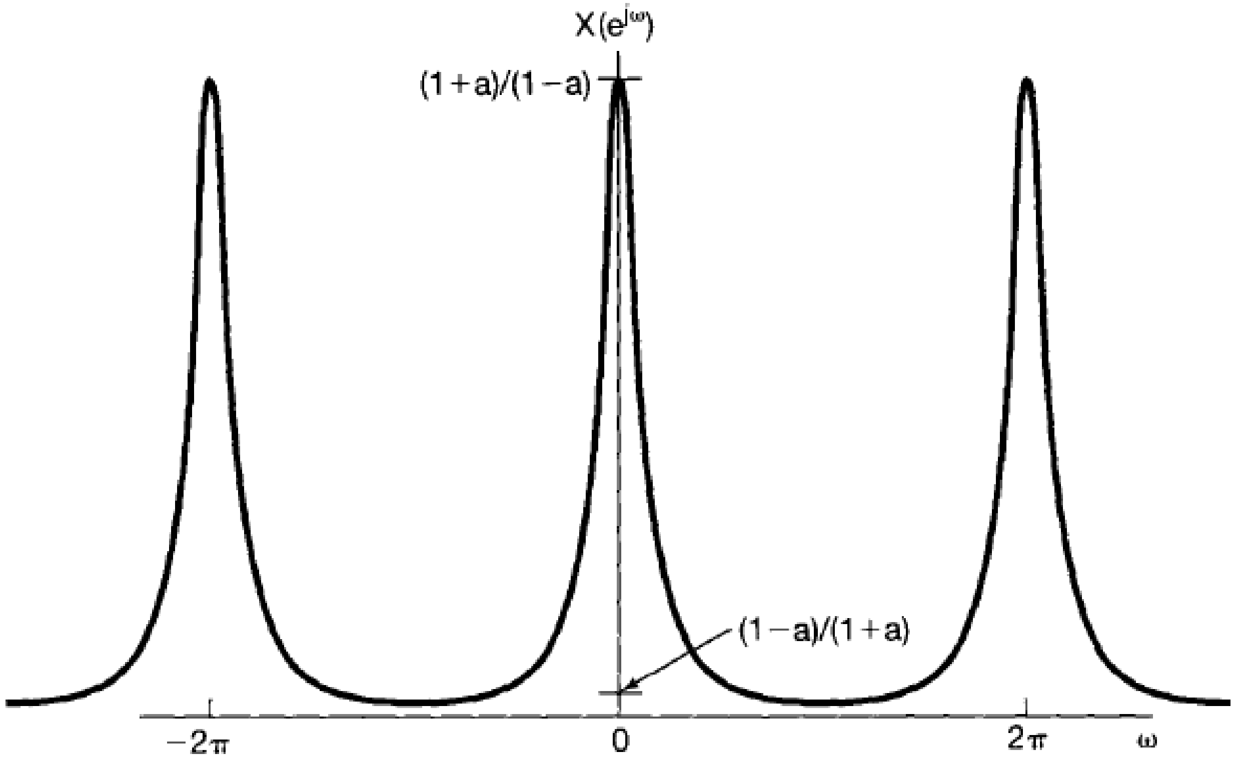
Example 2 - continued

$$X(e^{j\omega}) = \sum_{n=0}^{\infty}(ae^{-j\omega})^n + \sum_{m=1}^{\infty}(ae^{j\omega})^m$$

Both summations are infinite geometric series that we can evaluate in closed form:

$$X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}} + \frac{ae^{j\omega}}{1 - ae^{j\omega}}$$

$$= \frac{1 - a^2}{1 - 2a\cos\omega + a^2}.$$

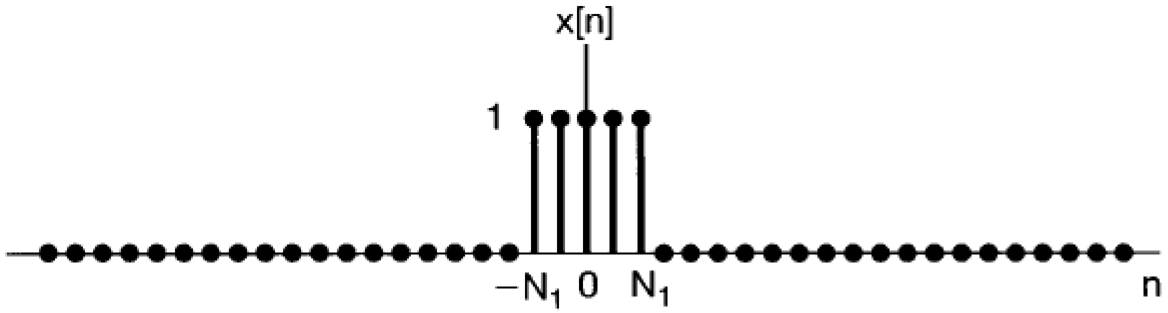


In this case, $X(e^{j\omega})$ is real.

Example 3

Find the Discrete-time Fourier Transform (DTFT) of the following rectangular pulse.

$$x[n] = \begin{cases} 1, & |n| \leq N_1 \\ 0, & |n| > N_1 \end{cases}$$

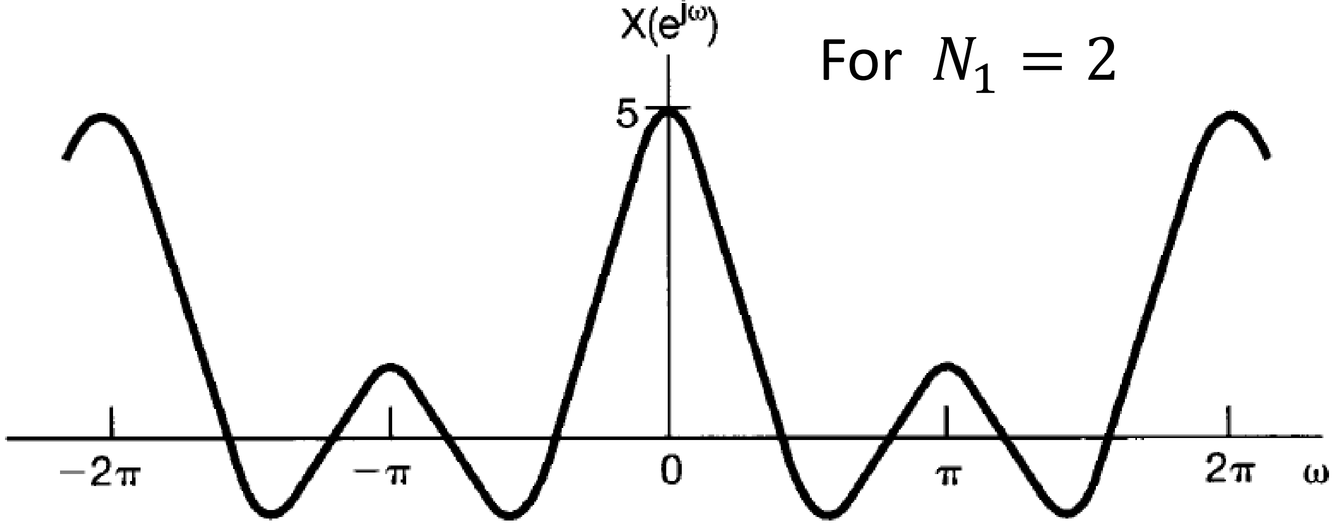


$$X(e^{j\omega}) = \sum_{n=-N_1}^{N_1} e^{-j\omega n}$$

You can show that :

$$X(e^{j\omega}) = \frac{\sin \omega\left(N_1 + \frac{1}{2}\right)}{\sin(\omega/2)}$$

For $N_1 = 2$



The Discrete-Time Fourier Transform of Periodic Signals

Consider the discrete-time signal

$$x[n] = e^{j\omega_0 n}$$

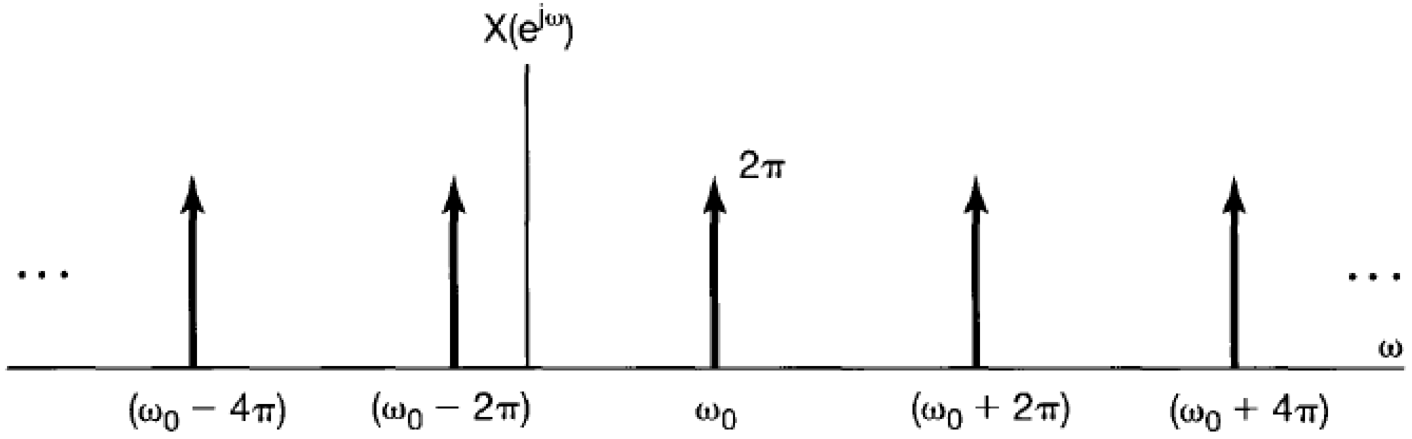
Remember that the Continuous-time Fourier Transform of the continuous signal $e^{j\omega_0 t}$ is an impulse located at $\omega = \omega_0$.

We might expect the same type of transform to result for the discrete-time complex exponential.
However, the discrete-time Fourier transform must be periodic in ω with period 2π.
This then suggests that the Fourier transform of x[n] in should have impulses at
$\omega_0, \omega_0 \pm 2\pi, \omega_0 \pm 4\pi$, and so on.
In fact, the Fourier transform of x[n] is the impulse train

$$X(e^{j\omega}) = \sum_{l=-\infty}^{+\infty} 2\pi\,\delta(\omega - \omega_0 - 2\pi l)$$

The Discrete-Time Fourier Transform of Periodic Signals

$$x[n] = e^{j\omega_0 n} \quad \Leftrightarrow \quad X(e^{j\omega}) = \sum_{l=-\infty}^{+\infty} 2\pi\,\delta(\omega - \omega_0 - 2\pi l)$$

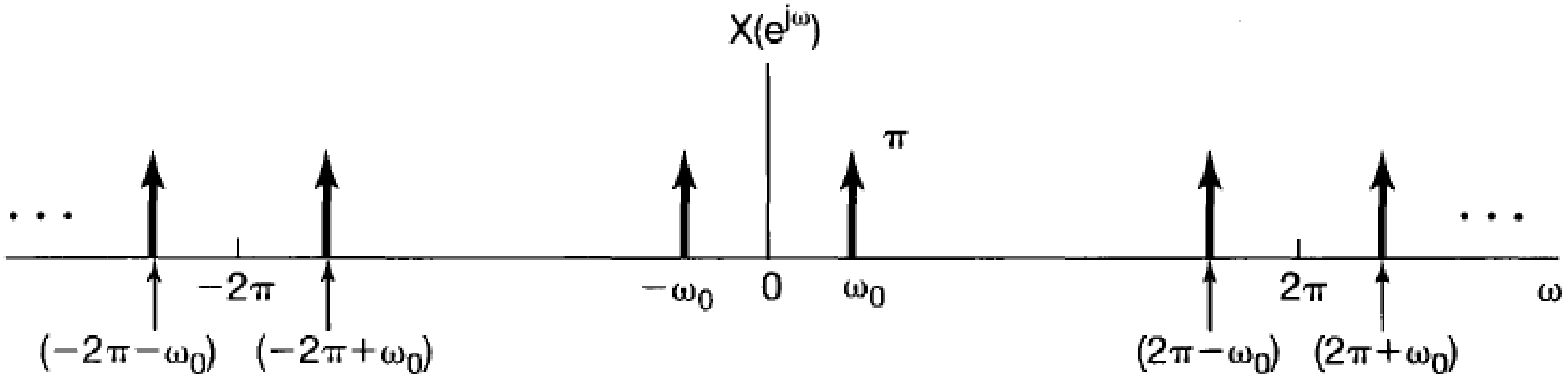


Example 4

Consider the periodic discrete-time signal

$$x[n] = \cos \omega_0 n = \frac{1}{2} e^{j\omega_0 n} + \frac{1}{2} e^{-j\omega_0 n}, \qquad \text{with} \quad \omega_0 = \frac{2\pi}{5}.$$

$$X(e^{j\omega}) = \sum_{l=-\infty}^{+\infty} \pi \delta\left(\omega - \frac{2\pi}{5} - 2\pi l\right) + \sum_{l=-\infty}^{+\infty} \pi \delta\left(\omega + \frac{2\pi}{5} - 2\pi l\right)$$



The Fourier Series Representation of Discrete-time Periodic Signals

A periodic discrete-time signal with period N can be represented as

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}.$$

$$a_k = \frac{1}{N} \sum_{n=\langle N\rangle} x[n] e^{-jk(2\pi/N)n}$$

where a_k are called the Fourier Series coefficients of $x[n]$. The Fourier Series coefficients for discrete-time periodic signals are also periodic in k with period N.

Both summations are evaluated over one period.

The Discrete-Time Fourier Transform of Periodic Signals

The Discrete-time Fourier Transform of a periodic discrete-time signal with period N is

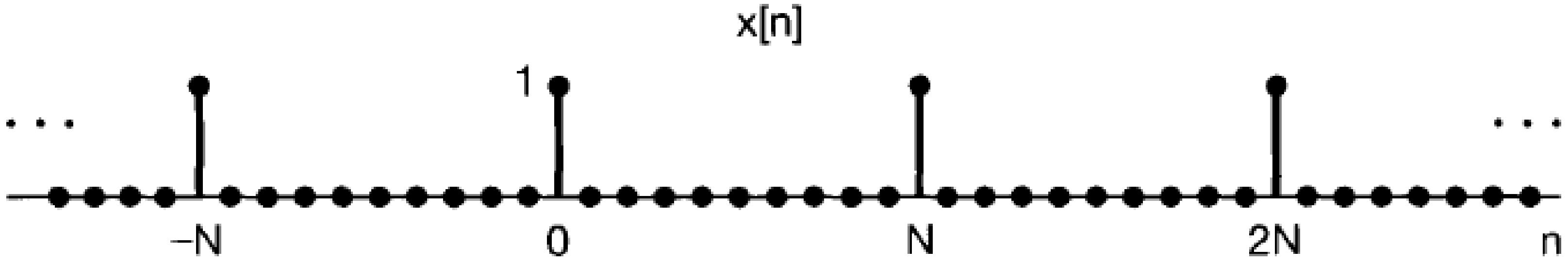
$$X(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$$

where a_k are the Fourier Series coefficients of $x[n]$.

Example 5

The discrete-time counterpart of the periodic impulse train is the sequence

$$x[n] = \sum_{k=-\infty}^{+\infty} \delta[n - kN]$$

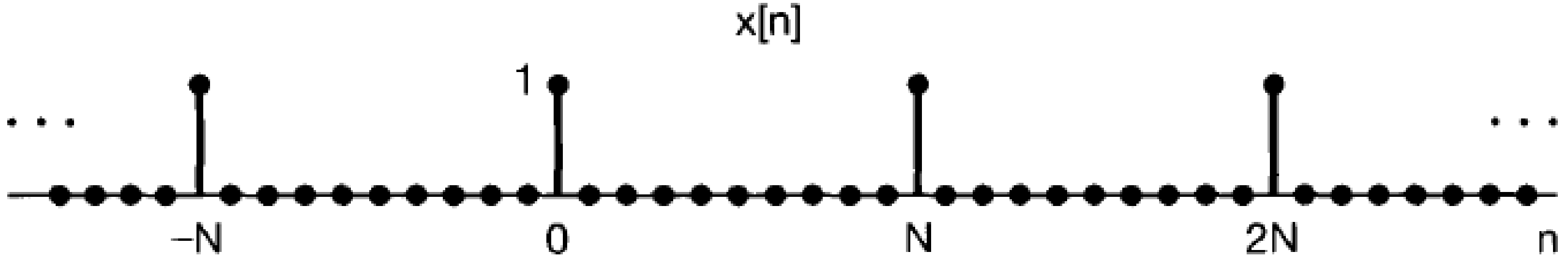


$$X(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \, \delta\left(\omega - \frac{2\pi k}{N}\right)$$

We need to calculate the Fourier Coefficients of this signal using:

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk(2\pi/N)n}.$$

Example 5 - continued



$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk(2\pi/N)n}.$$

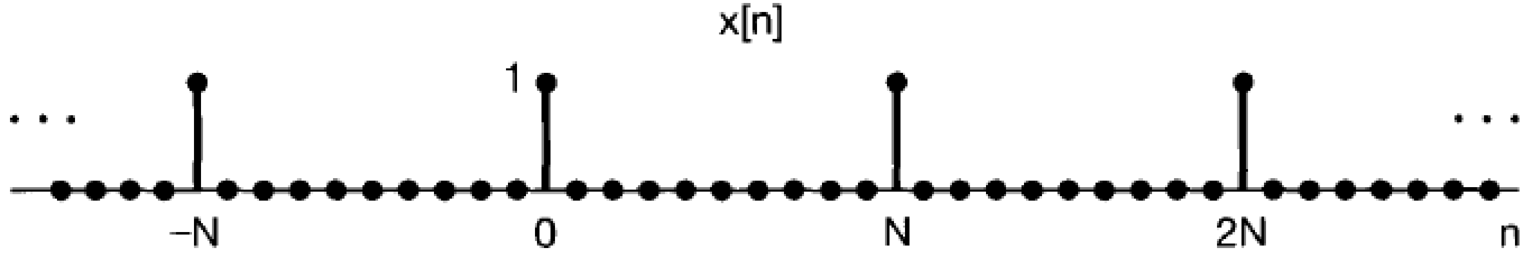
Choosing the interval of summation as $0 \leq n \leq N-1$, we have

$$a_k = \frac{1}{N}$$

Then, the Fourier Transform of $x[n]$ is

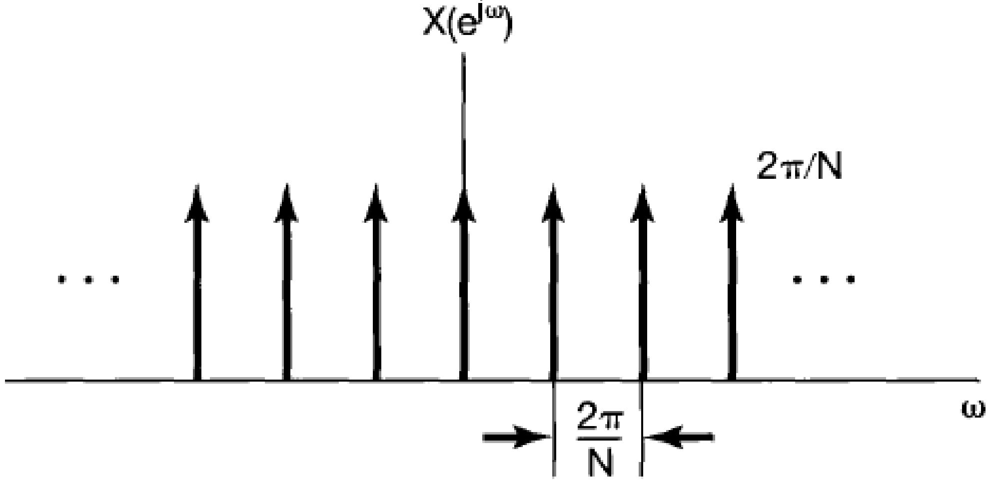
$$X(e^{j\omega}) = \frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$$

Example 5 - continued



The Fourier Transform of $x[n]$ is

$$X(e^{j\omega}) = \frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$$



PROPERTIES OF THE DISCRETE-TIME FOURIER TRANSFORM

Properties of the Discrete-time Fourier Transform

Properties of the discrete-time Fourier transform provide further insight into the transform.
They are also useful in reducing the complexity in the evaluation of transforms and inverse transforms.

We will use the following notation to indicate the pairing of a signal and its transform.

$$X(e^{j\omega}) = \mathcal{F}\{x[n]\} \qquad x[n] = \mathcal{F}^{-1}\{X(e^{j\omega})\}$$

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega})$$

Periodicity of the Discrete-time Fourier Transform

The discrete-time Fourier transform is always periodic in ω with period 2π.

$$\boxed{X(e^{j(\omega+2\pi)}) = X(e^{j\omega}).}$$

This is in contrast to the continuous-time Fourier transform, which in general is not periodic.

Linearity

If

$$x_1[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X_1(e^{j\omega})$$

and

$$x_2[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X_2(e^{j\omega}),$$

then

$$\boxed{a x_1[n] + b x_2[n] \stackrel{\mathcal{F}}{\longleftrightarrow} a X_1(e^{j\omega}) + b X_2(e^{j\omega}).}$$

Time Shifting

If

$$x[n] \overset{\mathfrak{F}}{\longleftrightarrow} X(e^{j\omega}),$$

then

$$\boxed{x[n-n_0] \overset{\mathfrak{F}}{\longleftrightarrow} e^{-j\omega n_0} X(e^{j\omega})}$$

Frequency Shifting

If

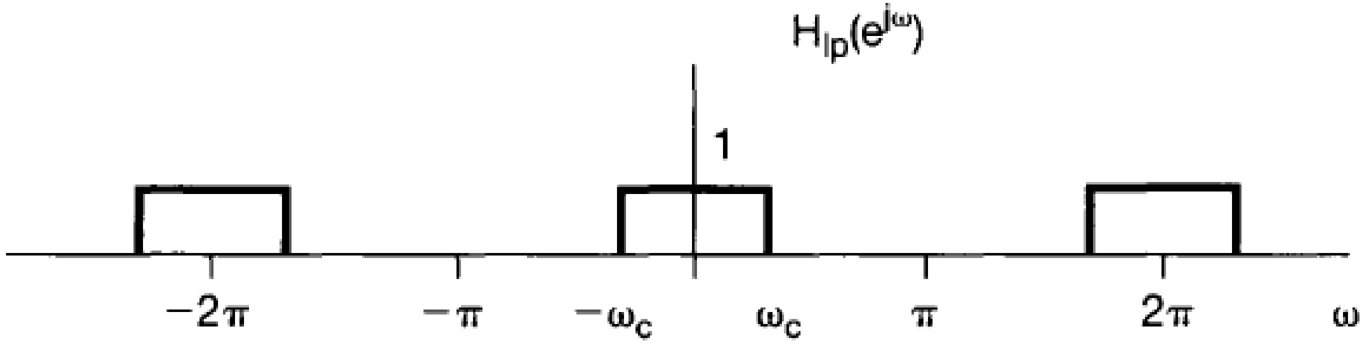
$$x[n] \overset{\mathfrak{F}}{\longleftrightarrow} X(e^{j\omega}),$$

then

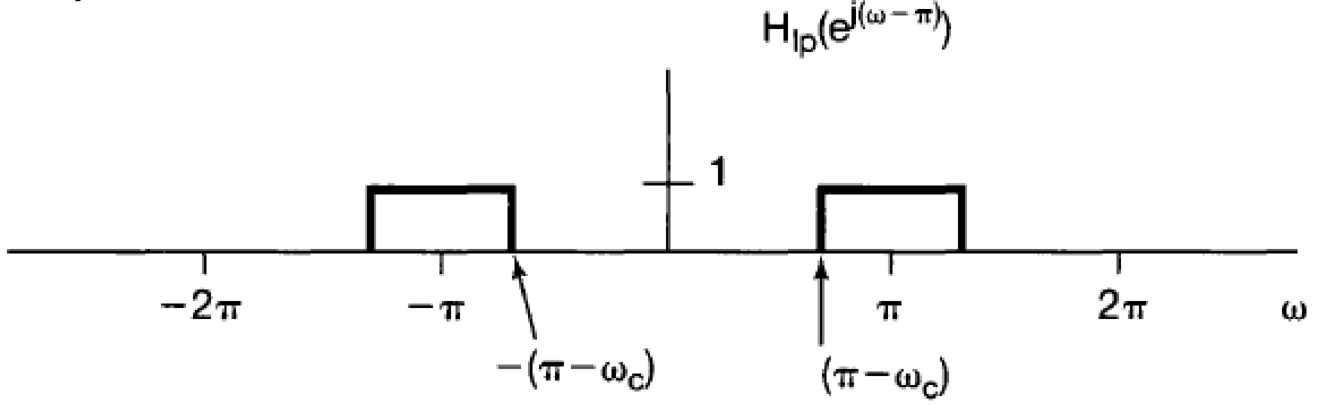
$$\boxed{e^{j\omega_0 n} x[n] \overset{\mathfrak{F}}{\longleftrightarrow} X(e^{j(\omega-\omega_0)}).}$$

Example 1

Below is the frequency response $H_{lp}(e^{j\omega})$ of a lowpass filter with cutoff frequency ω_C.



$H_{lp}(e^{j(\omega-\pi)})$, that is, the frequency response $H_{lp}(e^{j\omega})$ shifted by one-half period, i.e., by π is shown below.



The new filter is an ideal high pass filter with cutoff frequency $\pi - \omega_C$.

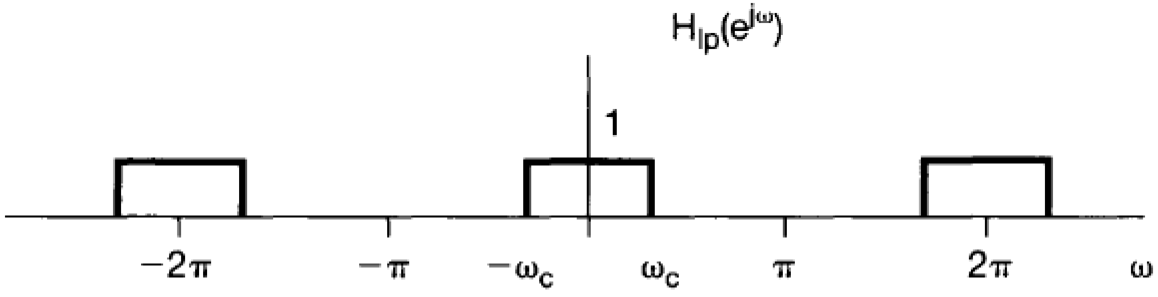
$$H_{\text{hp}}(e^{j\omega}) = H_{\text{lp}}(e^{j(\omega-\pi)})$$

High frequencies in discrete time are concentrated near π (and other odd multiples of π).

Example 1 - continued

The frequency response of an LTI system is the Fourier transform of the impulse response of the system.

The frequency response $H_{lp}(e^{j\omega})$ of the ideal lowpass filter with cutoff frequency ω_C:

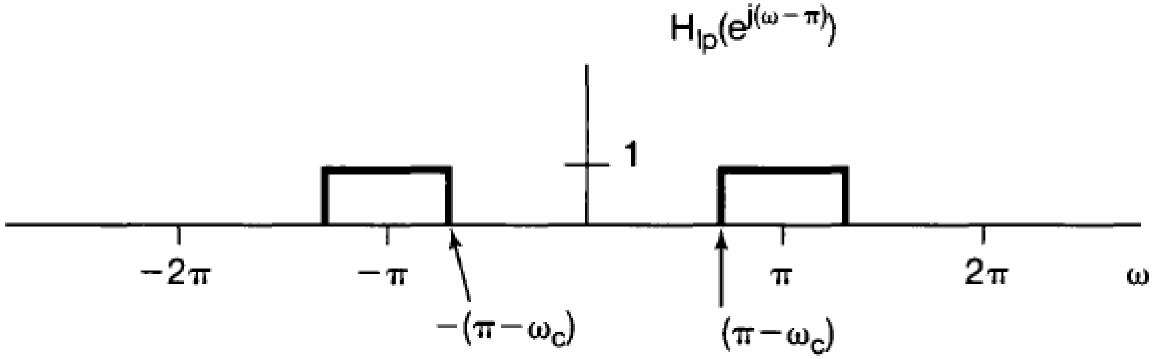


$h_{lp}[n]$: The impulse response of the low-pass filter

$$H_{hp}(e^{j\omega}) = H_{lp}(e^{j(\omega-\pi)}).$$

$h_{hp}[n]$: The impulse response of the high-pass filter

The frequncey response $H_{hp}(e^{j\omega})$ of the ideal high pass filter with cutoff frequency $\pi - \omega_C$:



From the frequency-shifting property:

$$h_{hp}[n] = e^{j\pi n} h_{lp}[n]$$

$$= (-1)^n h_{lp}[n].$$

Conjugation and Conjugate Symmetry

If

$$x[n] \stackrel{\mathfrak{F}}{\longleftrightarrow} X(e^{j\omega}),$$

then

$$\boxed{x^*[n] \stackrel{\mathfrak{F}}{\longleftrightarrow} X^*(e^{-j\omega}).}$$

Also, if $x[n]$ is real valued, its transform $X(e^{j\omega})$ is conjugate symmetric. That is,

$$\boxed{X(e^{j\omega}) = X^*(e^{-j\omega}) \quad [x[n] \text{real}].}$$

Conjugation and Conjugate Symmetry

$$\boxed{X(e^{j\omega}) = X^*(e^{-j\omega}) \quad [x[n] \text{real}].}$$

If $x[n]$ is real

$\Re e\{X(e^{j\omega})\}$: The real part of $X(e^{j\omega})$ is an even function of ω.
$\mathcal{I}m\{X(e^{j\omega})\}$: The imaginary part of $X(e^{j\omega})$ is an odd function of ω.

$|X(e^{j\omega})|$: The magnitude of $X(e^{j\omega})$ is an even function of ω.
$\sphericalangle X(e^{j\omega})$: The phase angle of $X(e^{j\omega})$ is an odd function of ω.

Conjugation and Conjugate Symmetry

If $x[n]$ is real

$$\mathcal{E}v\{x[n]\} \overset{\mathfrak{F}}{\longleftrightarrow} \mathfrak{Re}\{X(e^{j\omega})\}$$

$$\mathcal{O}d\{x[n]\} \overset{\mathfrak{F}}{\longleftrightarrow} j\mathfrak{Im}\{X(e^{j\omega})\}$$

$\mathfrak{Re}\{X(e^{j\omega})\}$ is the Fourier transform of the even part of $x[n]$.
$j\mathfrak{Im}\{X(e^{j\omega})\}$ is the Fourier transform of the odd part of $x[n]$.

If $x[n]$ is real and even then $X(e^{j\omega})$ is real and even.

If $x[n]$ is real and odd then $X(e^{j\omega})$ is purely imaginary and odd.

Differencing

Let $x[n]$ be a signal with Fourier transform $X\left(e^{j\omega}\right)$.

Consider the first-difference signal $x[n] - x[n-1]$.

From the linearity and time-shifting properties, the Fourier transform pair for this signal:

$$\boxed{x[n] - x[n-1] \stackrel{\mathcal{F}}{\longleftrightarrow} (1 - e^{-j\omega})X(e^{j\omega}).}$$

Accumulation

Let $x[n]$ be a signal with Fourier transform $X\left(e^{j\omega}\right)$.

Consider the signal

$$y[n] = \sum_{m=-\infty}^{n} x[m]$$

$$\boxed{\sum_{m=-\infty}^{n} x[m] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{1-e^{-j\omega}} X(e^{j\omega}) + \pi X(e^{j0}) \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k).}$$

The impulse train on the right-hand side of the equation reflects the dc or average value that can result from summation.

Example 2

Find the Fourier Transform of the unit step $x[n] = u[n]$ using the accumulation property and the knowledge that

$$g[n] = \delta[n] \stackrel{\mathfrak{F}}{\longleftrightarrow} G(e^{j\omega}) = 1$$

We know that:

$$x[n] = \sum_{m=-\infty}^{n} g[m]$$

Taking the Fourier transform of both sides and using accumulation yields

$$X(e^{j\omega}) = \frac{1}{(1 - e^{-j\omega})} G(e^{j\omega}) + \pi G(e^{j0}) \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

$$= \frac{1}{1 - e^{-j\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k).$$

Time reversal

Let $x[n]$ be a signal with spectrum $X(e^{j\omega})$.

Then

$$\boxed{x[-n] \stackrel{\mathfrak{F}}{\longleftrightarrow} X(e^{-j\omega}).}$$

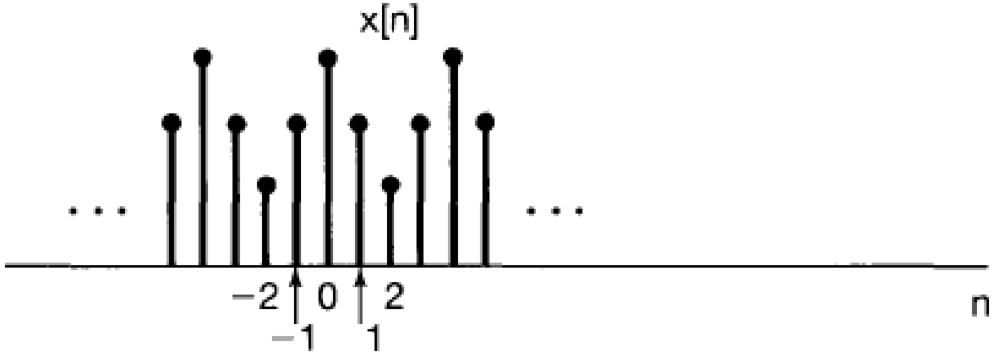
Time expansion

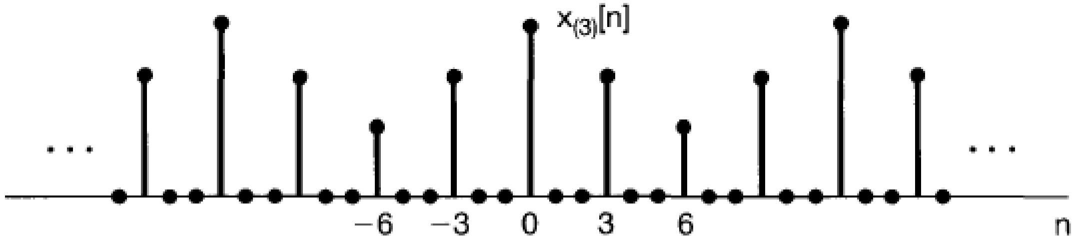
Time expansion in discrete-time is different from continuous-time.
If we try to define the signal $x[an]$, we run into difficulties if a is not an integer.

Let's define the following signal:

$$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n \text{ is a multiple of } k \\ 0, & \text{if } n \text{ is not a multiple of } k. \end{cases}$$

For example;
Take $k = 3$. We insert two zeros between successive values of the original signal.





Time expansion

Let $x[n]$ be a signal with spectrum $X(e^{j\omega})$.

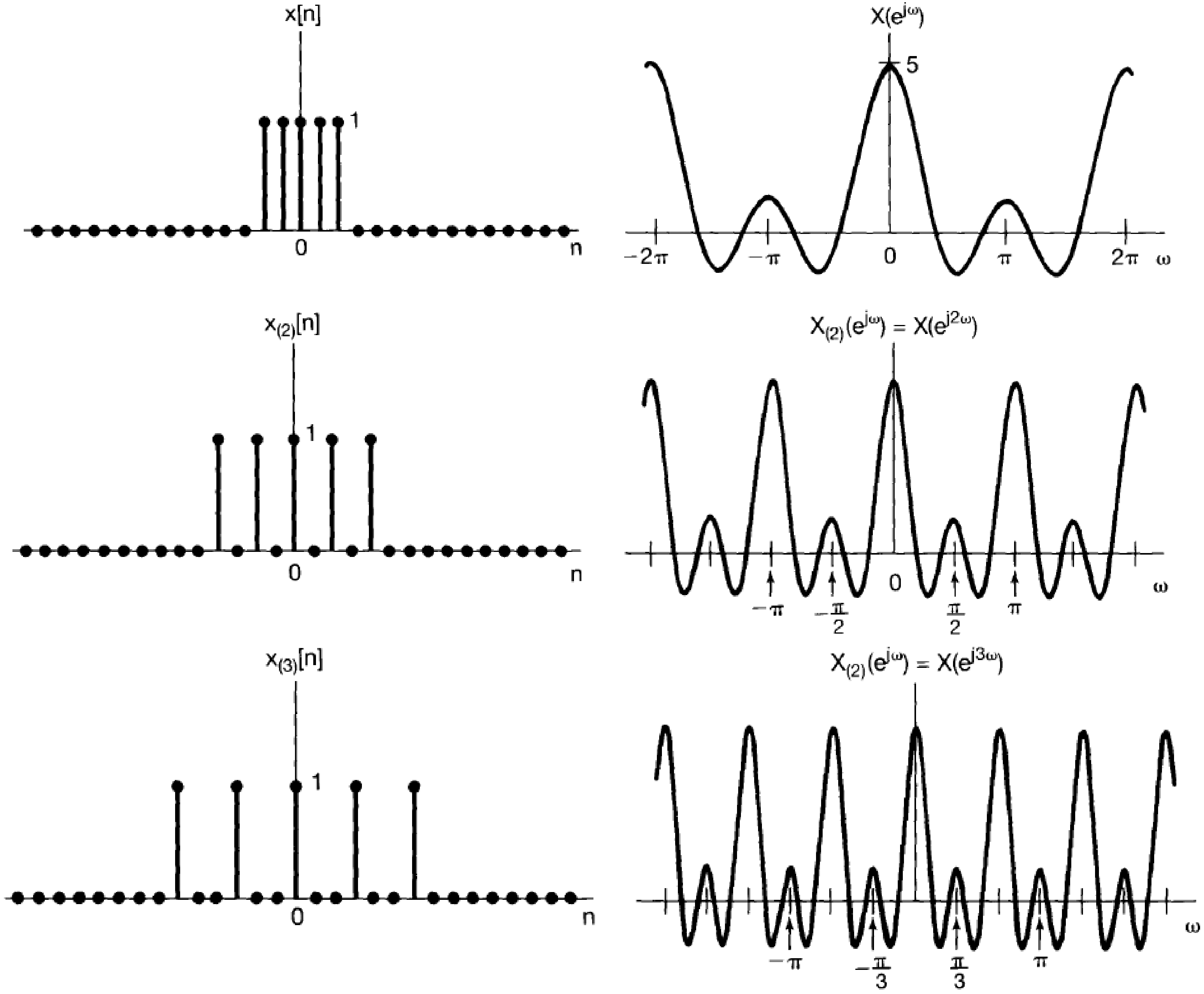
$$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n \text{ is a multiple of } k \\ 0, & \text{if } n \text{ is not a multiple of } k. \end{cases}$$

The Fourier Transform pair of the new signal:

$$\boxed{x_{(k)}[n] \overset{\mathfrak{F}}{\longleftrightarrow} X(e^{jk\omega}).}$$

As the signal is spread out and slowed down in time by taking $k > 1$, its Fourier transform is compressed.
Since $X(e^{j\omega})$ is periodic with period 2π, $X(e^{jk\omega})$ is periodic with period $2\pi/k$.

Time expansion

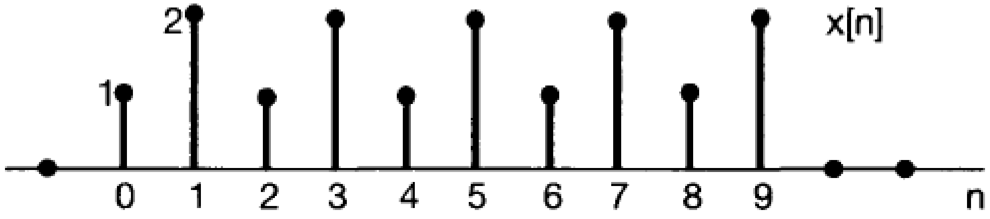


$$x_{(k)}[n] \overset{\mathfrak{F}}{\longleftrightarrow} X(e^{jk\omega}).$$

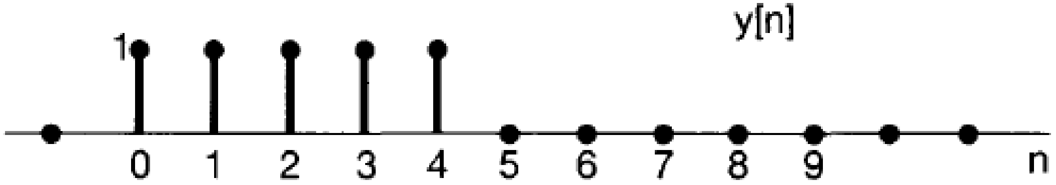
Inverse relationship between the time and frequency domains: As k increases, $x_{(k)}[n]$ spreads out while its transform is compressed.

Example 3

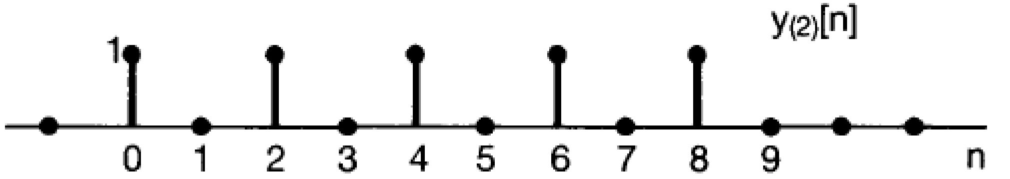
Find the Fourier Transform of the following signal $x[n]$.



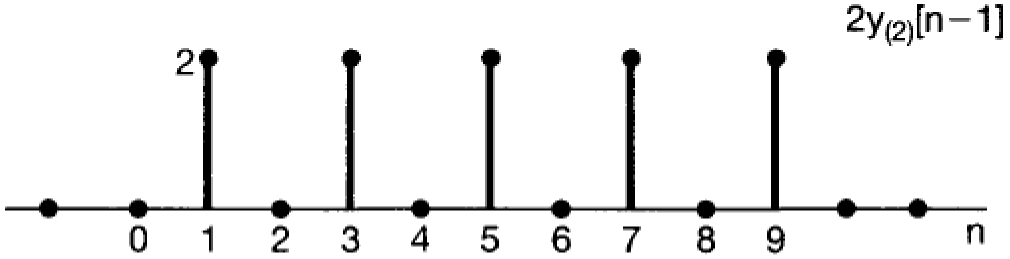
This sequence can be related to the simpler sequence y[n]



$$x[n] = y_{(2)}[n] + 2y_{(2)}[n-1]$$



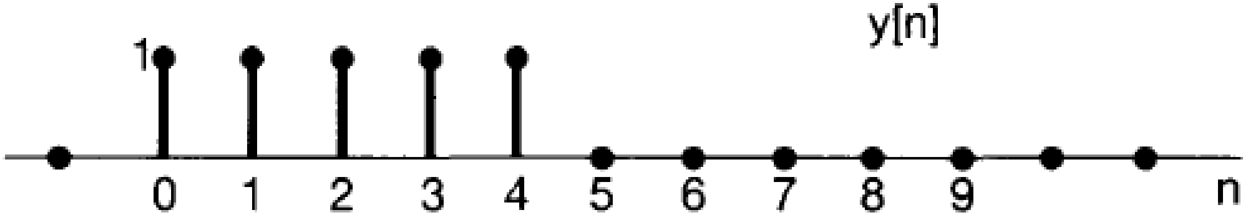
$$y_{(2)}[n] = \begin{cases} y[n/2], & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases}$$



Example 3 - continued

$$x[n] = y_{(2)}[n] + 2y_{(2)}[n-1]$$

$$y_{(2)}[n] = \begin{cases} y[n/2], & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases}$$



$g[n]$

Note that $y[n] = g[n-2]$, where $g[n]$ is a rectangular pulse as considered in a previous example (with $N1 = 2$).

$$G(e^{j\omega}) = \frac{\sin \omega \left(N_1 + \frac{1}{2}\right)}{\sin(\omega/2)}.$$

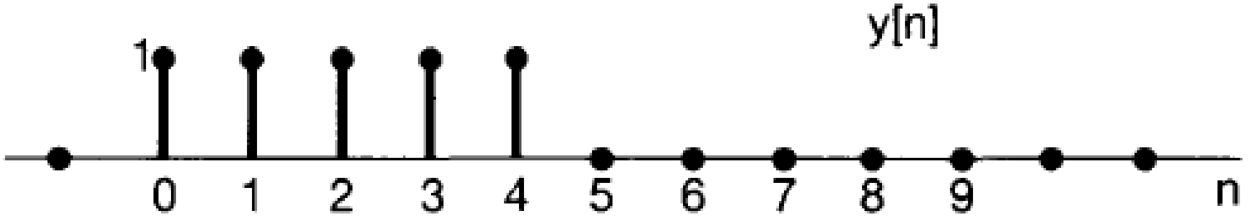
From the time-shifting property

$$Y(e^{j\omega}) = e^{-j2\omega} \frac{\sin(5\omega/2)}{\sin(\omega/2)}$$

Example 3 - continued

$$x[n] = y_{(2)}[n] + 2y_{(2)}[n-1]$$

$$y_{(2)}[n] = \begin{cases} y[n/2], & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases}$$



$$Y(e^{j\omega}) = e^{-j2\omega}\frac{\sin(5\omega/2)}{\sin(\omega/2)}$$

Using the time-expansion property, we then obtain

$$y_{(2)}[n] \overset{\mathfrak{F}}{\longleftrightarrow} e^{-j4\omega}\frac{\sin(5\omega)}{\sin(\omega)}$$

Using the linearity and time-shifting properties

$$2y_{(2)}[n-1] \overset{\mathfrak{F}}{\longleftrightarrow} 2e^{-j5\omega}\frac{\sin(5\omega)}{\sin(\omega)}$$

Combining these two results we get the Fourier Transform of $x[n]$ as

$$X(e^{j\omega}) = e^{-j4\omega}(1 + 2e^{-j\omega})\left(\frac{\sin(5\omega)}{\sin(\omega)}\right)$$

Differentiation in Frequency

Let $x[n]$ be a signal with spectrum $X(e^{j\omega})$.

$$x[n] \overset{\mathfrak{F}}{\longleftrightarrow} X(e^{j\omega})$$

Then

$$\boxed{nx[n] \overset{\mathfrak{F}}{\longleftrightarrow} j\frac{dX(e^{j\omega})}{d\omega}.}$$

Parseval's Relation

If $x[n]$ and $X(e^{j\omega})$ are a Fourier transform pair, then

$$\boxed{\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega.}$$

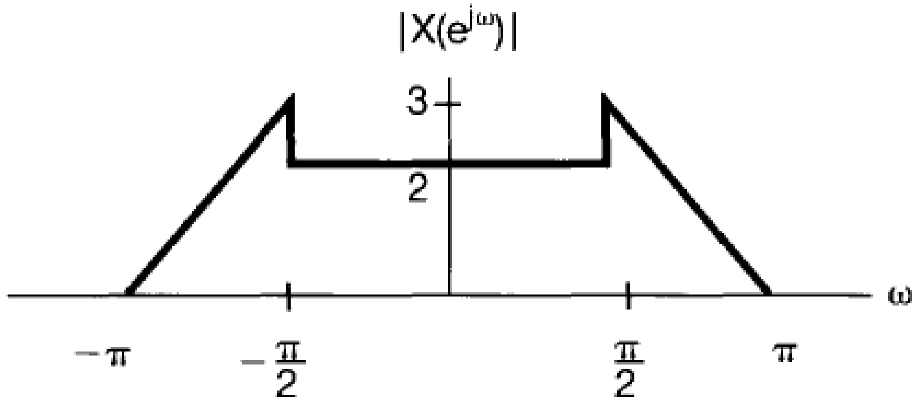
The quantity on the left-hand side of the equation is the total energy in the signal $x[n]$.

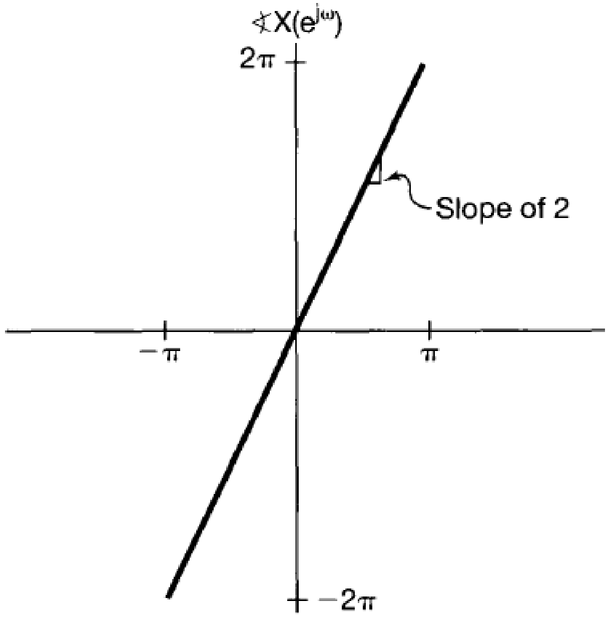
Parseval's relation states that this energy can also be determined by integrating the energy per unit frequency, $\left|X(e^{j\omega})\right|^2/2\pi$, over a full 2π interval of distinct discrete-time frequencies.

$\left|X(e^{j\omega})\right|^2$ is referred to as the energy-density spectrum of the signal $x[n]$.

Example 4

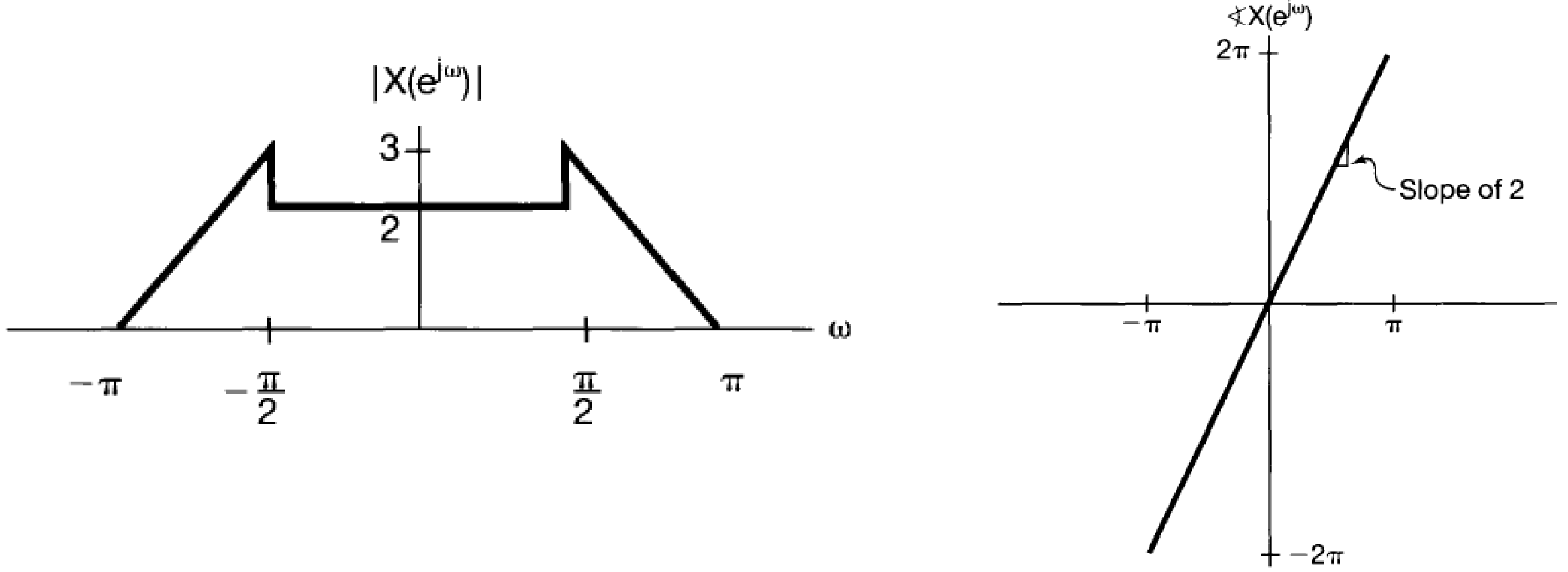
Consider the sequence $x[n]$ whose Fourier transform $X(e^{j\omega})$ is depicted for $-\pi \leq \omega \leq \pi$ in the Figure below. We wish to determine whether or not, in the time domain, $x[n]$ is periodic, real, even, and/or of finite energy.





Periodicity in the time domain implies that the Fourier transform is zero, except possibly for impulses located at various integer multiples of the fundamental frequency.
This is not true for $X(e^{j\omega})$. Then $x[n]$ is not periodic.

Example 4 - continued



From the symmetry properties for Fourier transforms, we know that a real valued sequence must have a Fourier transform of even magnitude and a phase function that is odd.

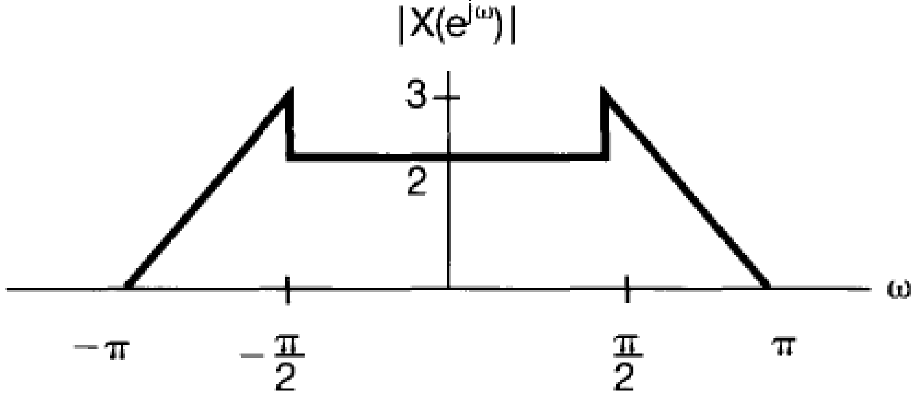
This is true for the given $\left|X(e^{j\omega})\right|$ and $\sphericalangle X(e^{j\omega})$.

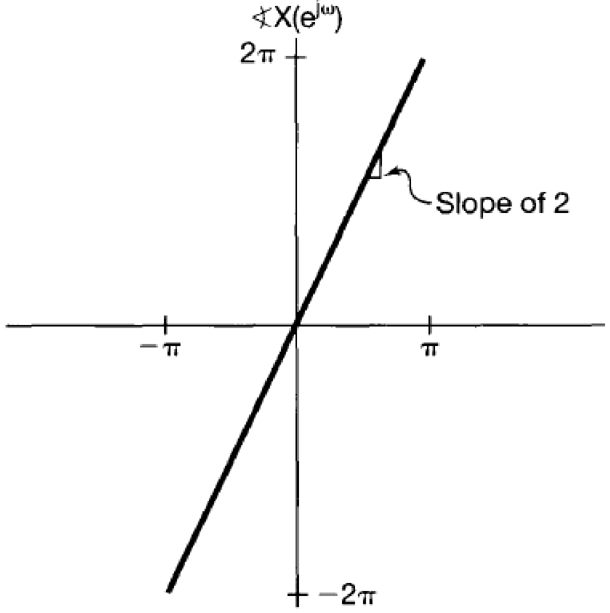
$\left|X(e^{j\omega})\right|$: The magnitude of $X(e^{j\omega})$ is an even function of ω.

$\sphericalangle X(e^{j\omega})$: The phase angle of $X(e^{j\omega})$ is an odd function of ω.

We thus conclude that $x[n]$ is real.

Example 4 - continued





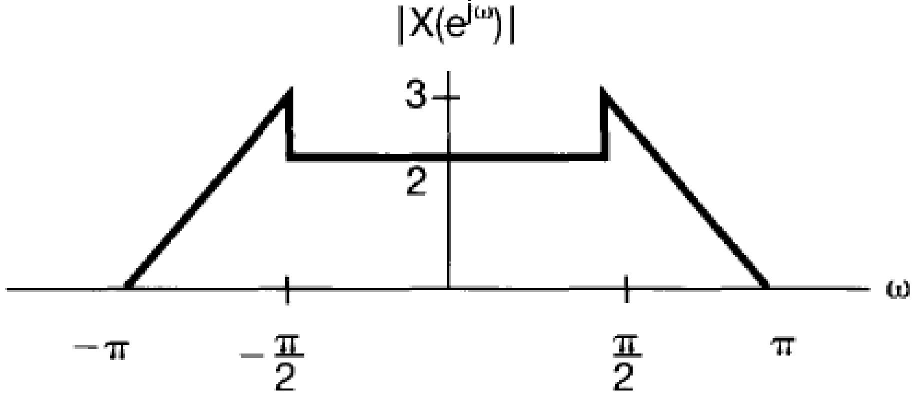
If $x[n]$ is an even function, then, by the symmetry properties for real signals, $X(e^{j\omega})$ must be real and even.

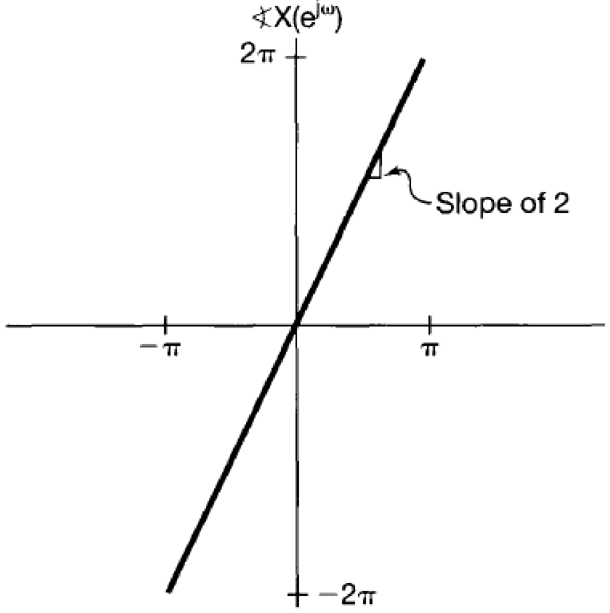
However, since $X\left(e^{j\omega}\right) = \left|X(e^{j\omega})\right|e^{j2\omega}$

$X\left(e^{j\omega}\right)$ is not a real-valued function.

Consequently, x[n] is not even.

Example 4 - continued





To test for the finite-energy property, we use Parseval's relation

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

The integral on the right-hand side of the equation is finite.
Thus $x[n]$ has finite energy.

The multiplication property

Consider $y[n]$ equal to the product of $x_1[n]$ and $x_2[n]$, with $Y(e^{j\omega})$, $X_1(e^{j\omega})$ and $X_2(e^{j\omega})$ denoting the corresponding Fourier transforms.

$$y[n] = x_1[n]x_2[n]$$

Then

$$\boxed{Y(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta.}$$

This equation corresponds to a periodic convolution of $X_1(e^{j\omega})$ and $X_2(e^{j\omega})$, and the integral in this equation can be evaluated over any interval of length 2π.

Example 5

Find the Fourier transform $X(e^{j\omega})$ of a signal $x[n]$ which is the product of two other signals:

$$x[n] = x_1[n]x_2[n]$$

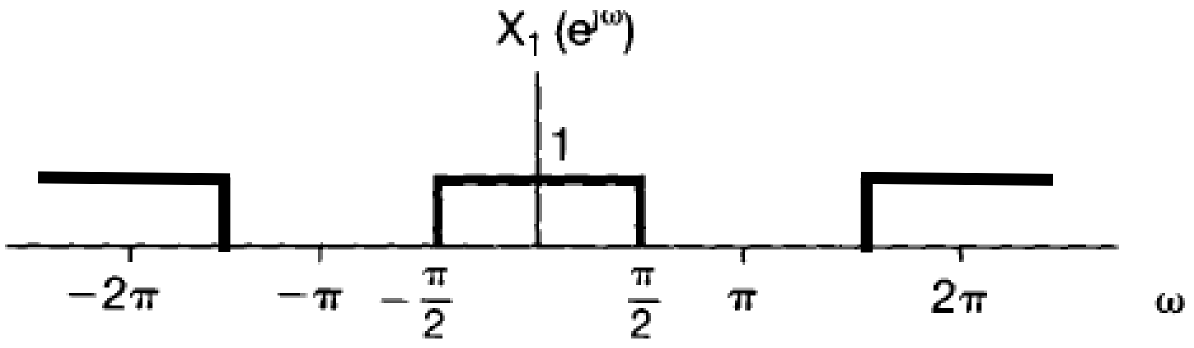
$$x_1[n] = \frac{\sin(3\pi n/4)}{\pi n} \qquad x_2[n] = \frac{\sin(\pi n/2)}{\pi n}.$$

From the multiplication property, we know that $X(e^{j\omega})$ is the periodic convolution of $X_1(e^{j\omega})$ and $X_2(e^{j\omega})$.
The integral can be taken over any interval of length 2π.
Choosing the interval $-\pi < \theta < \pi$, we obtain

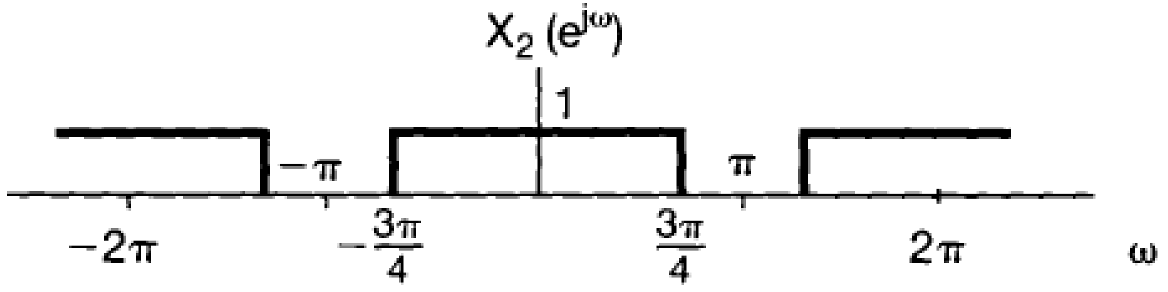
$$X(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta$$

Example 5 - continued

$$x_1[n] = \frac{\sin(3\pi n/4)}{\pi n} \quad \Longleftrightarrow$$



$$x_2[n] = \frac{\sin(\pi n/2)}{\pi n} \quad \Longleftrightarrow$$



$$x[n] = x_1[n]x_2[n] \quad \Longleftrightarrow$$

Periodic convolution

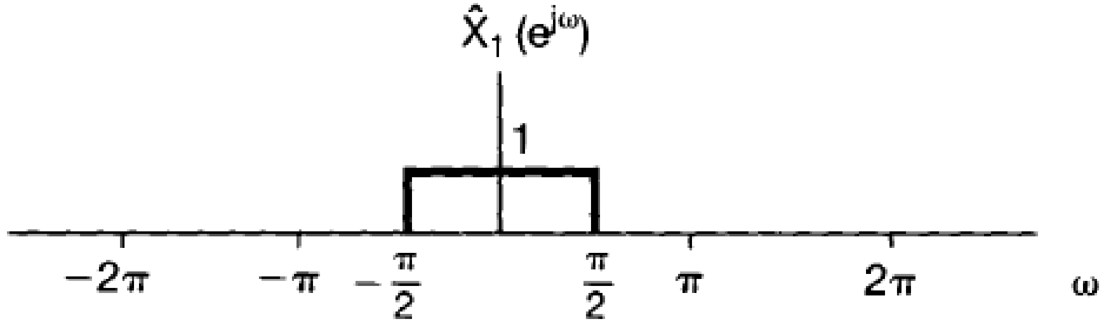
$$X(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta$$

Example 5 - continued

$$X(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta$$

We can convert the equation into an ordinary convolution by defining

$$\hat{X}_1(e^{j\omega}) = \begin{cases} X_1(e^{j\omega}) & \text{for } -\pi < \omega \leq \pi \\ 0 & \text{otherwise} \end{cases}$$



Then

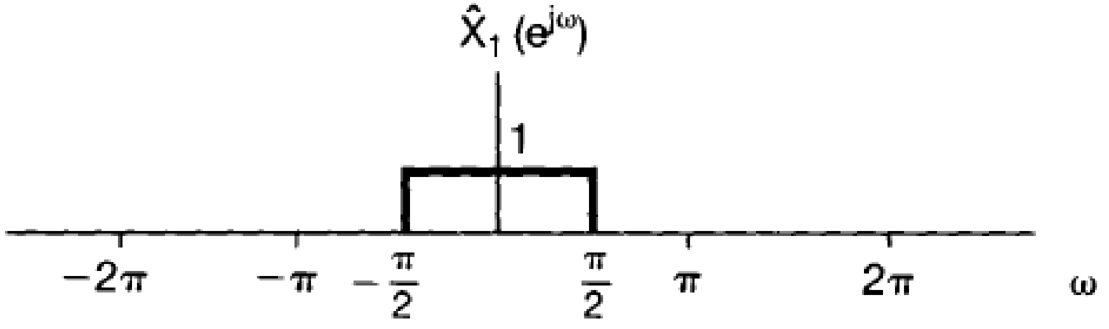
$$X(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{X}_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta$$

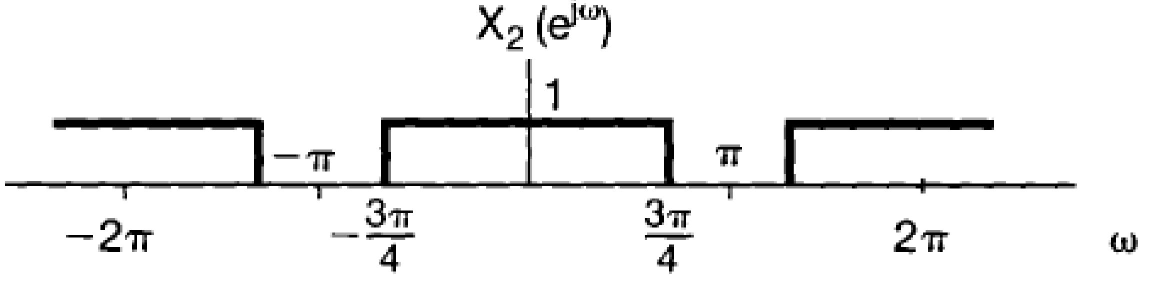
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{X}_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta.$$

Example 5 - continued

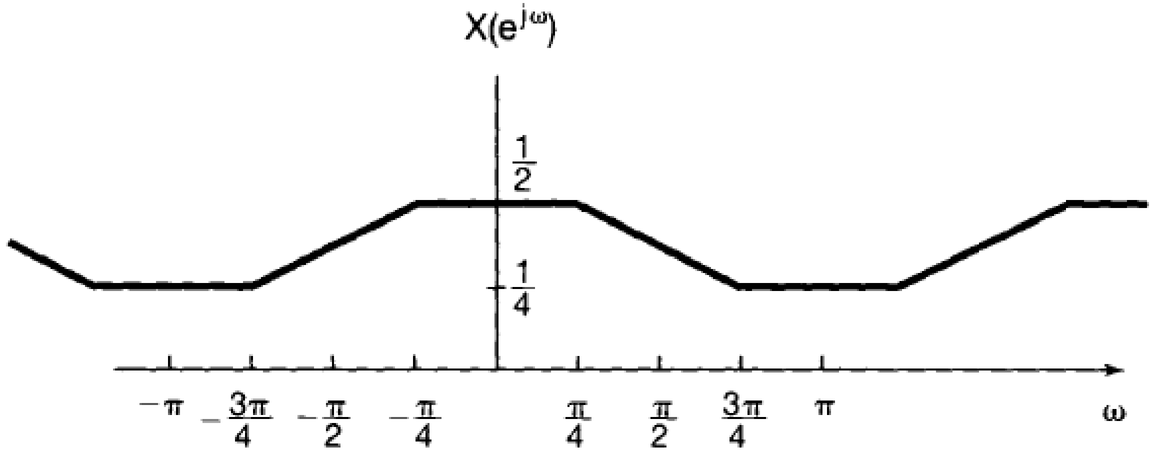
$$X(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \widehat{X}_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{X}_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta.$$





The linear convolution of $\hat{X}_1(e^{j\omega})$ and $X_2(e^{j\omega})$ corresponds to the periodic convolution of $X_1(e^{j\omega})$ and $X_2(e^{j\omega})$.



THE CONVOLUTION PROPERTY OF THE DISCRETE-TIME FOURIER TRANSFORM

The Convolution Property

If $x[n]$, $h[n]$, and $y[n]$ are the input, impulse response, and output, respectively, of an LTI system

$$y[n] = x[n] * h[n],$$

Then

$$\boxed{Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}),}$$

where $X(e^{j\omega})$, $H(e^{j\omega})$, and $Y(e^{j\omega})$ are the Fourier Transforms of $x[n]$, $h[n]$, and $y[n]$, respectively.

The frequency response $H(e^{j\omega})$ of a discrete-time LTI system is the Fourier transform of the impulse response $h[n]$ of the system.

The Convolution Property

$$y[n] = x[n] * h[n],$$

$$\boxed{Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}),}$$

- The convolution of two signals in time domain corresponds to multiplication of their Fourier transforms.
- This makes the analysis of signals and systems easier.
- It also adds significantly to our understanding of the way in which an LTI system responds to the input signals that are applied to it.
- The frequency response $H(e^{j\omega})$ captures the change in complex amplitude of the Fourier transform of the input at each frequency ω.
- For example, in frequency selective filtering, we want $H(e^{j\omega}) \approx 1$ over the range of frequencies corresponding to the desired passband and $H(e^{j\omega}) \approx 0$ over the band of frequencies to be eliminated or significantly attenuated.

Example 1

Consider a discrete-time LTI system with impulse response

$$h[n] = \delta[n - n_0]$$

The frequency response is

$$H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} \delta[n - n_0] e^{-j\omega n} = e^{-j\omega n_0}$$

For any input $x[n]$ with Fourier transform $X(e^{j\omega})$, the Fourier transform of the output is

$$Y(e^{j\omega}) = e^{-j\omega n_0} X(e^{j\omega})$$

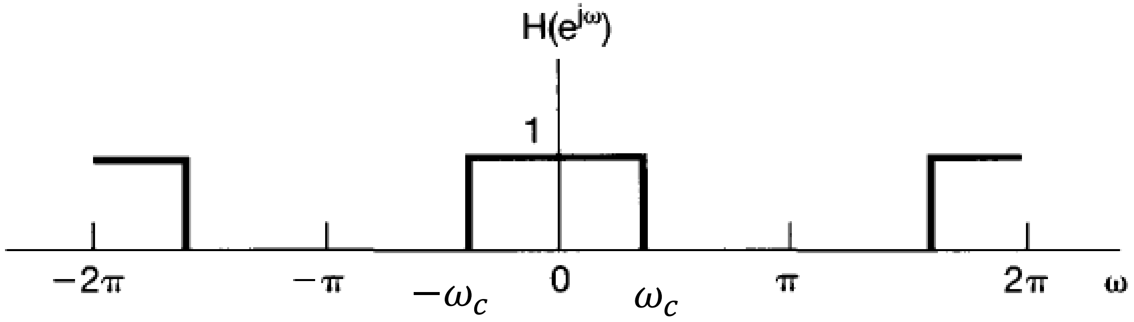
From time-shifting property

$$y[n] = x[n - n_0]$$

The frequency response $H(e^{j\omega}) = e^{-j\omega n_0}$ of a pure time shift has unity magnitude at all frequencies and a phase characteristic $-\omega n_0$ that is linear with frequency.

Example 2

Consider a discrete-time ideal lowpass filter with the frequency response $H(e^{j\omega})$ illustrated below.



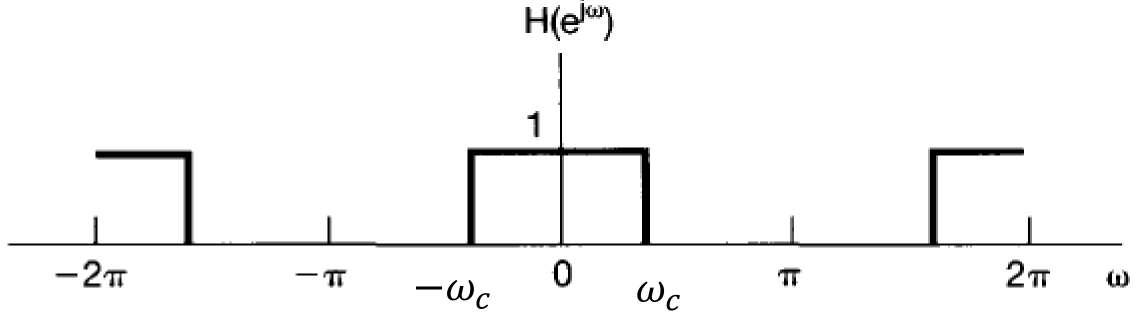
We can determine the impulse response of the ideal lowpass filter from the frequency response using the inverse Fourier transform (synthesis equation).

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}d\omega$$

$$= \frac{\sin \omega_c n}{\pi n},$$

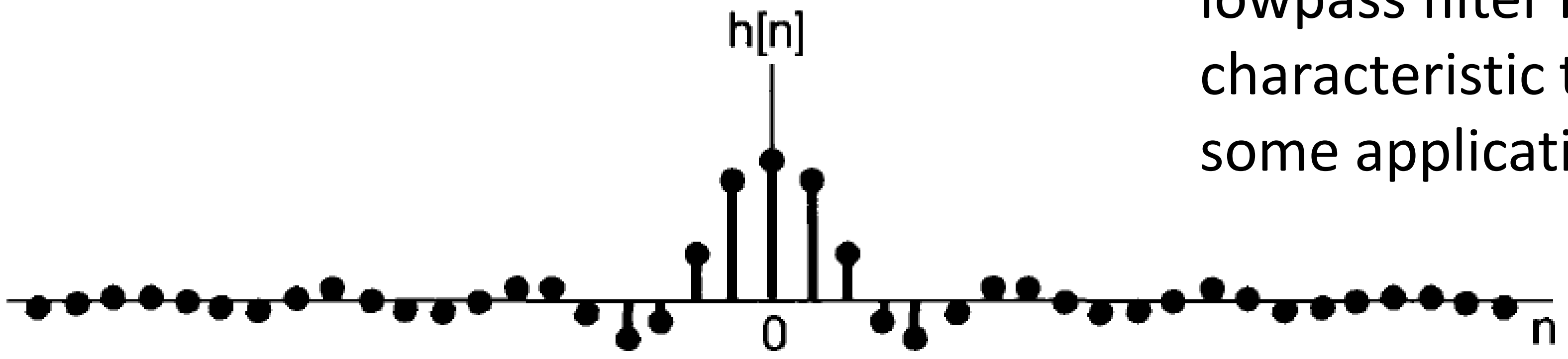
Example 2 - continued

Frequency response of the discrete-time ideal lowpass filter



Impulse response of the discrete-time ideal lowpass filter

$$h[n] = \frac{\sin \omega_c n}{\pi n}$$



The impulse response of the ideal lowpass filter is oscillatory, a characteristic that is undesirable in some applications.

Example 3

Consider an LTI system with impulse response

$$h[n] = \alpha^n u[n] \qquad |\alpha| < 1$$

Suppose that the input to this system is

$$x[n] = \beta^n u[n] \qquad |\beta| < 1$$

Evaluating the Fourier transforms of $h[n]$ and $x[n]$, we have

$$H(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}} \qquad X(e^{j\omega}) = \frac{1}{1 - \beta e^{-j\omega}}$$

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) = \frac{1}{(1 - \alpha e^{-j\omega})(1 - \beta e^{-j\omega})}$$

Example 3 – continued

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) = \frac{1}{(1-\alpha e^{-j\omega})(1-\beta e^{-j\omega})}$$

$Y(e^{j\omega})$ is a ratio of polynomials in powers of $e^{-j\omega}$.
Determining the inverse transform of $Y(e^{j\omega})$ is most easily done by expanding $Y(e^{j\omega})$ by the method of partial fractions.
If $\alpha \neq \beta$, the partial fraction expansion of $Y(e^{j\omega})$ is of the form

$$Y(e^{j\omega}) = \frac{A}{1-\alpha e^{-j\omega}} + \frac{B}{1-\beta e^{-j\omega}}$$

Equating the right-hand sides of the two equations:

$$A = \frac{\alpha}{\alpha-\beta}, \quad B = -\frac{\beta}{\alpha-\beta}$$

Example 3 – continued

$$Y(e^{j\omega}) = \frac{A}{1-\alpha e^{-j\omega}} + \frac{B}{1-\beta e^{-j\omega}} \qquad A = \frac{\alpha}{\alpha-\beta}, \quad B = -\frac{\beta}{\alpha-\beta}$$

The inverse Fourier transform of $Y\left(e^{j\omega}\right)$ is

$$\begin{aligned} y[n] &= \frac{\alpha}{\alpha-\beta}\alpha^n u[n] - \frac{\beta}{\alpha-\beta}\beta^n u[n] \\ &= \frac{1}{\alpha-\beta}[\alpha^{n+1}u[n] - \beta^{n+1}u[n]]. \end{aligned}$$

Example 3 – continued

For $\alpha = \beta$, the previous form of partial fraction expansion is not valid. In this case:

$$Y(e^{j\omega}) = \left(\frac{1}{1 - \alpha e^{-j\omega}}\right)^2$$

$$Y(e^{j\omega}) = \frac{j}{\alpha} e^{j\omega} \frac{d}{d\omega}\left(\frac{1}{1 - \alpha e^{-j\omega}}\right)$$

We can use the frequency differentiation property, together with the Fourier transform pair

$$\alpha^n u[n] \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{1 - \alpha e^{-j\omega}}$$

$$n\alpha^n u[n] \stackrel{\mathcal{F}}{\longleftrightarrow} j \frac{d}{d\omega}\left(\frac{1}{1 - \alpha e^{-j\omega}}\right)$$

Example 3 – continued

For $\alpha = \beta$:

$$Y(e^{j\omega}) = \frac{j}{\alpha} e^{j\omega} \frac{d}{d\omega}\left(\frac{1}{1 - \alpha e^{-j\omega}}\right)$$

$$n\alpha^n u[n] \stackrel{\mathfrak{F}}{\longleftrightarrow} j\frac{d}{d\omega}\left(\frac{1}{1 - \alpha e^{-j\omega}}\right)$$

To account for the factor $e^{j\omega}$, we use the time-shifting property to obtain

$$(n+1)\alpha^{n+1} u[n+1] \stackrel{\mathfrak{F}}{\longleftrightarrow} je^{j\omega} \frac{d}{d\omega}\left(\frac{1}{1 - \alpha e^{-j\omega}}\right)$$

accounting for the factor $1/\alpha$, we obtain

$$y[n] = (n+1)\alpha^n u[n+1]$$

The sequence $(n + 1)\alpha^n u[n + 1]$ is still zero prior to $n = 0$, since the factor $n + 1$ is zero at $n = -1$. Thus, we can alternatively express $y[n]$ as

$$y[n] = (n+1)\alpha^n u[n]$$